# Phys 7221 Homework #2

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1. **Derivation 1-9: Gauge transformations for electromagnetic potential**

   If two Lagrangians differ by a total derivative of a function of coordinates and time, they lead to the same equation of motion:

$$L'(q_i, \dot{q}_i, t) = L(q_i, \dot{q}_i, t) + \frac{dF(q_i, t)}{dt} = L(q_i, \dot{q}_i, t) + \frac{\partial F}{\partial t} + \sum_j \frac{\partial F}{\partial q_j}\dot{q}_j$$

   as proven in the following lines:

$$\frac{\partial L'}{\partial \dot{q}_i} = \frac{\partial L'}{\partial \dot{q}_i} + \frac{\partial F}{\partial q_i}$$

$$\frac{\partial L'}{\partial q_i} = \frac{\partial L}{\partial q_i} + \frac{\partial}{\partial q_i}\frac{dF}{dt}$$

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_i} - \frac{\partial L'}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \frac{d}{dt}\frac{\partial F}{\partial q_i} - \frac{\partial}{\partial q_i}\frac{dF}{dt}$$

$$\begin{aligned}
\frac{d}{dt}\frac{\partial F}{\partial q_i} &= \left(\frac{\partial}{\partial t} + \sum_j \dot{q}_j \frac{\partial}{\partial q_j}\right)\frac{\partial F}{\partial q_i} \\
&= \frac{\partial}{\partial q_i}\left(\frac{\partial F}{\partial t} + \sum_j \dot{q}_j \frac{\partial F}{\partial q_j}\right) \\
&= \frac{\partial}{\partial q_i}\frac{dF}{dt}
\end{aligned}$$

   and thus the equations of motion are the same, derived from either Lagrangian:

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_i} - \frac{\partial L'}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}$$

(What goes wrong if $F = F(q, \dot{q}, t)$?)

A particle moving in an electromagnetic field, with scalar potential $\phi$ and vector potential $\mathbf{A}$, has a generalized potential function $U = \phi - \mathbf{A} \cdot \mathbf{v}$ and a Lagrangian

$$L = T - U = \frac{1}{2} m v^2 - q\left(\phi - \mathbf{A} \cdot \mathbf{v}\right)$$

Consider a gauge transformation for the scalar and vector potentials of the form

$$\phi' = \phi - \frac{\partial \psi(\mathbf{r}, t)}{\partial t}$$
$$\mathbf{A}' = \mathbf{A} + \nabla \psi(\mathbf{r}, t)$$

The change in potential energy under this transformation is

$$\begin{aligned}
U' &= \phi' - \mathbf{A}' \cdot \mathbf{v} \\
&= U - q\left(\frac{\partial \psi}{\partial t} + \nabla \psi \cdot \mathbf{v}\right) \\
&= U - q\left(\frac{\partial \psi}{\partial t} + \nabla \psi \cdot \frac{d\mathbf{r}}{dt}\right) \\
&= U - q\left(\frac{\partial \psi}{\partial t} + \sum \frac{\partial \psi}{\partial x_i} \frac{d x_i}{dt}\right) \\
&= U - q\frac{d\psi}{dt}
\end{aligned}$$

and thus, the new Lagrangian differs from the original one by a total derivative:

$$L' = T - U' = T - U + q\frac{d\psi}{dt} = L + \frac{d(q\psi)}{dt}$$

and thus we know the equations of motion are the same.

We can also see that the electromagnetic forces (and thus Newton's equations) are invariant. The forces are derived from the potential, following (1.58):

$$Q_i = -\frac{\partial U}{\partial q_i} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_i}$$

If $U = q(\phi(\mathbf{r}, t) - \mathbf{A}(\mathbf{r}, t) \cdot \mathbf{v})$, then the force is

$$\begin{aligned}
Q_i &= -q\frac{\partial}{\partial x_i}(\phi - \mathbf{A} \cdot \mathbf{v}) + q\frac{d}{dt}\frac{\partial}{\partial v_i}(\phi - \mathbf{A} \cdot \mathbf{v}) \\
&= -q\frac{\partial \phi}{\partial x_i} + q\frac{\partial}{\partial x_i}(\mathbf{A} \cdot \mathbf{v}) - q\frac{dA_i}{dt}
\end{aligned}$$

$$
\begin{aligned}
&= -q\frac{\partial \phi}{\partial x_i} + q\frac{\partial}{\partial x_i}(\mathbf{A}\cdot\mathbf{v}) - q\left(\frac{\partial}{\partial t} + \sum_j \dot{x}_j \frac{\partial}{\partial x_j}\right)\mathbf{A} \\
\mathbf{F} &= -q\left(\nabla\phi + \frac{\partial \mathbf{A}}{\partial t}\right) + q\nabla(\mathbf{A}\cdot\mathbf{v}) - q(\mathbf{v}\cdot\nabla)\mathbf{A} \\
&= -q\left(\nabla\phi + \frac{\partial \mathbf{A}}{\partial t}\right) + q\mathbf{v}\times(\nabla\times\mathbf{A}) \\
&= q(\mathbf{E} + v\times\mathbf{B})
\end{aligned}
$$

and, as we know, the electric and magnetic fields are invariant under the gauge transformation:

$$
\begin{aligned}
\mathbf{E}' &= -\nabla\phi' - \frac{\partial \mathbf{A}'}{\partial t} \\
&= -\nabla(\phi - \frac{\partial \psi}{\partial t}) - \frac{\partial}{\partial t}(\mathbf{A} + \nabla\psi) \\
&= -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \nabla\frac{\partial \psi}{\partial t} - \frac{\partial}{\partial t}\nabla\psi \\
&= -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \\
&= \mathbf{E}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{B}' &= \nabla\times\mathbf{A}' = \nabla\times(\mathbf{A} + \nabla\psi) \\
&= \nabla\times\mathbf{A} + \nabla\times\nabla\psi = \nabla\times\mathbf{A} \\
&= \mathbf{B}
\end{aligned}
$$

2. **Problem 1-15: A potential for spin interactions**

Consider a point particle moving in space under the influence of a force derivable from a generalized potential:

$$U(\mathbf{r},\mathbf{v}) = V(r) + \sigma\cdot\mathbf{L}$$

We can derive the generalized forces using (1.58):

$$Q_i = -\frac{\partial U}{\partial q_i} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_i}$$

using either Cartesian coordinates $(x, y, z)$ or spherical polar coordinates $(r, \psi, \theta)$ for generalized coordinates $q_i$. The first choice will produce the components of the force vector $\mathbf{F}_i$; the generalized forces for any other choice of generalized coordinates (such as spherical coordinates) will be related to the force vector as in (1.49):

$$Q_i = \sum_j \mathbf{F}_j \cdot \frac{\partial \mathbf{r}_j}{\partial q_i}$$

This problem is a marathon in vector calculus in different coordinate systems. We first calculate the force vector using Cartesian coordinates:

$$
\begin{aligned}
U &= V(r) + \sigma \cdot \mathbf{L} \\
&= V(r) + \sigma \cdot (\mathbf{r} \times m\mathbf{v}) \\
&= V(r) + m \sum_l \sigma_l (\mathbf{r} \times m\mathbf{v})_l \\
&= V(r) + m \sum_l \sigma_l \sum_{m,n} \epsilon_{lmn} x_m \dot{x}_n \\
&= V(r) + m \sum_{l,m,n} \epsilon_{lmn} \sigma_l x_m \dot{x}_n \\
\frac{\partial U}{\partial \dot{x}_i} &= m \sum_{l,m} \epsilon_{lmi} \sigma_l x_m \\
\frac{d}{dt} \frac{\partial U}{\partial \dot{x}_i} &= m \sum_{l,m} \epsilon_{lmi} \sigma_l \dot{x}_m = (\sigma \times \mathbf{p})_i \\
\frac{\partial U}{\partial x_i} &= \frac{\partial V(r)}{\partial x_i} + m \sum_{ln} \epsilon_{lin} \sigma_l \dot{x}_n \\
&= \frac{dV(r)}{dr} \frac{\partial r}{\partial x_i} - m \sum_{ln} \epsilon_{iln} \sigma_l \dot{x}_n \\
&= \frac{dV(r)}{dr} \frac{x_i}{r} - (\sigma \times \mathbf{p})_i \\
\mathbf{F}_i &= -\frac{\partial U}{\partial q_i} + \frac{d}{dt} \frac{\partial U}{\partial \dot{q}_i} \\
\mathbf{F} &= -\frac{dV(r)}{dr} \hat{\mathbf{e}_r} + 2(\sigma \times \mathbf{p})
\end{aligned}
$$

Now we want to calculate the generalized forces in spherical polar coordinates $r, \psi, \theta$. We first need to express the potential in terms of the coordinates and their derivatives. We will use unit vectors $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\psi, \hat{\mathbf{e}}_\theta$, which are themselves functions of the coordinates and time:

$$\hat{\mathbf{e}}_r = \sin\psi \cos\theta \hat{i} + \sin\psi \sin\theta \hat{j} + \cos\psi \hat{k}$$

$$\hat{\mathbf{e}}_\psi = \cos\psi \cos\theta \hat{i} + \cos\psi \sin\theta \hat{j} - \sin\psi \hat{k}$$

$$\hat{\mathbf{e}}_\theta = -\sin\theta \hat{i} + \cos\theta \hat{j}$$

They are orthogonal and cyclical: $\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \epsilon_{ijk} \hat{\mathbf{e}}_k$ but they are not constant: in general $\partial \hat{\mathbf{e}}_i / \partial q_j \neq 0$, and also $d\hat{\mathbf{e}}_i / dt \neq 0$.

We first calculate the angular momentum in spherical coordinates:

$$\begin{aligned}
\mathbf{r} &= r\hat{\mathbf{e}}_r \\
\mathbf{v} &= \frac{d}{dt}(r\hat{\mathbf{e}}_r) = \dot{r}\hat{\mathbf{e}}_r + r\dot{\psi}\hat{\mathbf{e}}_\psi + r\sin\psi\dot{\theta}\hat{\mathbf{e}}_\theta \\
\mathbf{L} &= \mathbf{r}\times\mathbf{p} = \mathbf{r}\times m\mathbf{v} \\
&= mr\hat{\mathbf{e}}_r \times (\dot{r}\hat{\mathbf{e}}_r + r\dot{\psi}\hat{\mathbf{e}}_\psi + r\sin\psi\dot{\theta}\hat{\mathbf{e}}_\theta) \\
&= mr^2(\dot{\psi}\hat{\mathbf{e}}_\theta - \sin\psi\dot{\theta}\hat{\mathbf{e}}_\psi)
\end{aligned}$$

so the potential, in terms of generalized coordinates $r, \psi, \theta$ is

$$U = V(r) + \sigma\cdot\mathbf{L} = V(r) + mr^2\sigma\cdot(\dot{\psi}\hat{\mathbf{e}}_\theta - \sin\psi\dot{\theta}\hat{\mathbf{e}}_\psi)$$

We can now calculate the generalized forces:

$$\begin{aligned}
Q_r &= -\frac{\partial U}{\partial r} + \frac{d}{dt}\frac{\partial U}{\partial \dot{r}} = -\frac{\partial U}{\partial r} \\
&= -\frac{dV}{dr} - 2mr\sigma\cdot(\dot{\psi}\hat{\mathbf{e}}_\theta - \sin\psi\dot{\theta}\hat{\mathbf{e}}_\psi) \\
&= -\frac{dV}{dr} - \frac{2}{r}\sigma\cdot\mathbf{L} = -\frac{dV}{dr} + 2\hat{\mathbf{e}}_r\cdot(\sigma\times\mathbf{p})
\end{aligned}$$

$$\begin{aligned}
Q_\psi &= -\frac{\partial U}{\partial \psi} + \frac{d}{dt}\frac{\partial U}{\partial \dot{\psi}} \\
&= -mr^2\sigma\cdot\frac{\partial}{\partial\psi}(\dot{\psi}\hat{\mathbf{e}}_\theta - \sin\psi\dot{\theta}\hat{\mathbf{e}}_\psi) + \frac{d}{dt}(mr^2\sigma\cdot\hat{\mathbf{e}}_\theta) \\
&= -mr^2\sigma\cdot(-\cos\psi\dot{\theta}\hat{\mathbf{e}}_\psi + \sin\psi\dot{\theta}\hat{\mathbf{e}}_r) + 2mr\dot{r}\sigma\cdot\hat{\mathbf{e}}_\theta - mr^2\sigma\cdot\dot{\theta}(\sin\psi\hat{\mathbf{e}}_r + \cos\psi\hat{\mathbf{e}}_\psi) \\
&= -2mr^2\sin\psi\dot{\theta}\sigma\cdot\hat{\mathbf{e}}_r + 2mr\dot{r}\sigma\cdot\hat{\mathbf{e}}_\theta \\
&= 2mr\sigma\cdot(\dot{r}\hat{\mathbf{e}}_\theta - r\sin\psi\dot{\theta}\hat{\mathbf{e}}_r) \\
&= -2mr\sigma\cdot(\hat{\mathbf{e}}_\psi\times\mathbf{v}) \\
&= 2r\hat{\mathbf{e}}_\psi\cdot(\sigma\times\mathbf{p})
\end{aligned}$$

$$\begin{aligned}
Q_\theta &= -\frac{\partial U}{\partial \theta} + \frac{d}{dt}\frac{\partial U}{\partial \dot{\theta}} \\
&= -mr^2\sigma\cdot\frac{\partial}{\partial\theta}(\dot{\psi}\hat{\mathbf{e}}_\theta - \dot{\theta}\sin\psi\hat{\mathbf{e}}_\psi) + \frac{d}{dt}(-mr^2\sin\psi\sigma\cdot\hat{\mathbf{e}}_\psi) \\
&= -mr^2\sigma\cdot(\dot{\psi}(-\sin\psi\hat{\mathbf{e}}_r - \cos\psi\hat{\mathbf{e}}_\psi) - \dot{\theta}\sin\psi(\cos\psi\hat{\mathbf{e}}_\theta))
\end{aligned}$$

$$
\begin{aligned}
& -m(2r\dot{r}\sin\psi\sigma\cdot\hat{\mathbf{e}}_\psi + r^2\dot{\psi}\cos\psi\sigma\cdot\hat{\mathbf{e}}_\psi + r^2\sin\psi\sigma\cdot(-\dot{\psi}\hat{\mathbf{e}}_r + \dot{\theta}\cos\psi\hat{\mathbf{e}}_\theta)) \\
= \; & 2mr^2\dot{\psi}\sin\psi\sigma\cdot\hat{\mathbf{e}}_r - 2mr\dot{r}\sin\psi\sigma\cdot\hat{\mathbf{e}}_\psi \\
= \; & 2mr\sin\psi\sigma\cdot(r\dot{\psi}\hat{\mathbf{e}}_r - \dot{r}\hat{\mathbf{e}}_\psi) \\
= \; & -2mr\sin\psi\sigma\cdot(\hat{\mathbf{e}}_\theta\times\mathbf{v}) \\
= \; & 2r\sin\psi\hat{\mathbf{e}}_\theta\cdot(\sigma\times\mathbf{p})
\end{aligned}
$$

Now, we will prove that the force vector $\mathbf{F} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}}$ and the generalized forces $Q_r, Q_\psi, Q_\theta$ as obtained above, are related by the expression (1.49), using the expression we derived earlier for the force vector:

$$
\begin{aligned}
Q_i &= \mathbf{F}\cdot\frac{\partial\mathbf{e}_r}{\partial q_i} \\
&= \left(-\frac{dV(r)}{dr}\hat{\mathbf{r}} + 2(\sigma\times\mathbf{p})\right)\cdot\frac{\partial}{\partial q_i}(r\hat{\mathbf{e}}_r)
\end{aligned}
$$

$$
\begin{aligned}
Q_r &= \left(-\frac{dV(r)}{dr}\hat{\mathbf{e}_r} + 2(\sigma\times\mathbf{p})\right)\cdot\frac{\partial}{\partial r}(r\hat{\mathbf{e}}_r) \\
&= \left(-\frac{dV(r)}{dr}\hat{\mathbf{r}} + 2(\sigma\times\mathbf{p})\right)\cdot\hat{\mathbf{e}}_r \\
&= -\frac{dV(r)}{dr} + 2\hat{\mathbf{e}}_r\cdot(\sigma\times\mathbf{p})
\end{aligned}
$$

$$
\begin{aligned}
Q_\psi &= \left(-\frac{dV(r)}{dr}\hat{\mathbf{e}_r} + 2(\sigma\times\mathbf{p})\right)\cdot\frac{\partial}{\partial\psi}(r\hat{\mathbf{e}}_r) \\
&= \left(-\frac{dV(r)}{dr}\hat{\mathbf{r}} + 2(\sigma\times\mathbf{p})\right)\cdot(r\hat{\mathbf{e}}_\psi) \\
&= 2r\hat{\mathbf{e}}_\psi\cdot(\sigma\times\mathbf{p})
\end{aligned}
$$

$$
\begin{aligned}
Q_\theta &= \left(-\frac{dV(r)}{dr}\hat{\mathbf{e}_r} + 2(\sigma\times\mathbf{p})\right)\cdot\frac{\partial}{\partial\theta}(r\hat{\mathbf{e}}_r) \\
&= \left(-\frac{dV(r)}{dr}\hat{\mathbf{r}} + 2(\sigma\times\mathbf{p})\right)\cdot(r\sin\psi\hat{\mathbf{e}}_\theta) \\
&= 2r\sin\psi\hat{\mathbf{e}}_\theta\cdot(\sigma\times\mathbf{p})
\end{aligned}
$$

Part(c) of this problem was not included in the homework set, but it maybe the most interesting one: the equations of motion of the system are, using spherical polar coordinates:

$$L = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\psi}^2 + r^2\sin^2\psi\dot{\theta}^2) - V(r) - mr^2\sigma\cdot(\dot{\psi}\hat{\mathbf{e}}_\theta - \sin\psi\dot{\theta}\hat{\mathbf{e}}_\psi)$$

$$\begin{aligned}
0 &= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} \\
&= \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_j} + \frac{\partial U}{\partial q_j} \\
Q_j &= \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}
\end{aligned}$$

We can use the expressions we obtained earlier for the generalized forces. We need to express the equations in terms of the cartesian components of the vector $\sigma$, since those are constants (the spherical components are not constants, since the spherical unit vectors themselves are not constant). Since the vector $\sigma$ is constant, we can choose the z-axis along the $\sigma$ vector without any loss of generality. Then

$$\begin{aligned}
\hat{\mathbf{e}}_j\cdot(\sigma\times\mathbf{p}) &= m\sigma\cdot(\mathbf{v}\times\hat{\mathbf{e}}_j) \\
&= m\sigma\cdot(\dot{r}\hat{\mathbf{e}}_r + r\dot{\psi}\hat{\mathbf{e}}_\psi + r\sin\psi\dot{\theta}\hat{\mathbf{e}}_\theta)\times\hat{\mathbf{e}}_j \\
\hat{\mathbf{e}}_r\cdot(\sigma\times\mathbf{p}) &= m\sigma\cdot(-r\dot{\psi}\hat{\mathbf{e}}_\theta + r\sin\psi\dot{\theta}\hat{\mathbf{e}}_\psi) \\
&= -m\sigma r\dot{\theta}\sin^2\psi \\
\hat{\mathbf{e}}_\psi\cdot(\sigma\times\mathbf{p}) &= m\sigma\cdot(\dot{r}\hat{\mathbf{e}}_\theta - r\sin\psi\dot{\theta}\hat{\mathbf{e}}_r) \\
&= -m\sigma r\dot{\theta}\sin\psi\cos\psi \\
\hat{\mathbf{e}}_\theta\cdot(\sigma\times\mathbf{p}) &= m\sigma\cdot(-\dot{r}\hat{\mathbf{e}}_\psi + r\dot{\psi}\hat{\mathbf{e}}_r) \\
&= m\sigma(-\dot{r}\sin\psi + r\dot{\psi}\cos\psi)
\end{aligned}$$

$$\begin{aligned}
Q_r &= \frac{d}{dt}\frac{\partial T}{\partial \dot{r}} - \frac{\partial T}{\partial r} \\
-\frac{dV(r)}{dr} + 2\hat{\mathbf{e}}_r\cdot(\sigma\times\mathbf{p}) &= m\ddot{r} - mr(\dot{\psi}^2 + \sin^2\psi\dot{\theta}^2) \\
-\frac{dV(r)}{dr} - 2m\sigma r\dot{\theta}\sin^2\psi &= m\ddot{r} - mr(\dot{\psi}^2 + \sin^2\psi\dot{\theta}^2) \\
mr\dot{\psi}^2 + mr\sin^2\psi\dot{\theta}(\dot{\theta} - 2\sigma) &= m\ddot{r} + \frac{dV(r)}{dr}
\end{aligned}$$

$$Q_\psi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\psi}} - \frac{\partial T}{\partial \psi}$$

$$
\begin{aligned}
2r\hat{\mathbf{e}}_\psi \cdot (\sigma \times \mathbf{p}) &= (2mr\dot{r}\dot{\psi} + mr^2\ddot{\psi}) - mr^2 \sin\psi\cos\psi\dot{\theta}^2 \\
-2m\sigma r^2\dot{\theta}\sin\psi\cos\psi &= 2mr\dot{r}\dot{\psi} + mr^2\ddot{\psi} - mr^2\sin\psi\cos\psi\dot{\theta}^2 \\
r\dot{\theta}\sin\psi\cos\psi(\dot{\theta} - 2\sigma) &= 2\dot{r}\psi + r\ddot{\psi}
\end{aligned}
$$

$$
\begin{aligned}
Q_\theta &= \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta} \\
2r\hat{\mathbf{e}}_\theta \cdot (\sigma \times \mathbf{p}) &= 2mr\dot{r}\sin^2\psi\dot{\theta} + 2mr^2\sin\psi\cos\psi\dot{\theta} + mr^2\sin^2\psi\ddot{\theta} \\
m\sigma(-\dot{r}\sin\psi + r\dot{\psi}\cos\psi) &= mr\sin\psi(2\dot{r}\dot{\theta}\sin\psi + 2r\dot{\theta}\cos\psi + r\ddot{\theta}\sin\psi)
\end{aligned}
$$

3. **Problem 1-19: Spherical pendulum**

A spherical pendulum is a mass point $m$ suspended by a rigid weightless rod of length $l$. We set up a coordinate system with the origin at the attachment point, and the z-axis pointing down, along the gravitational force. In spherical polar coordinates, the position vector of the mass is

$$\mathbf{r} = l\hat{\mathbf{e}}_r,$$

its velocity is

$$\mathbf{v} = l\dot{\psi}\hat{\mathbf{e}}_\psi + l\sin\psi\dot{\theta}\hat{\mathbf{e}}_\theta,$$

its kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2(\dot{\psi}^2 + \sin^2\psi\dot{\theta}^2),$$

and the potential energy is

$$V = -m\mathbf{g}\cdot\mathbf{r} = -mgl\cos\psi.$$

The Lagrangian is

$$L = T - V = \frac{1}{2}ml^2(\dot{\psi}^2 + \sin^2\psi\dot{\theta}^2) + mgl\cos\psi$$

There are two generalized coordinates, $\psi$ and $\theta$. Lagrange's equations of motion are

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} &= ml^2\frac{d}{dt}(\dot{\theta}\sin^2\psi) = 0 \\
\frac{d}{dt}\frac{\partial L}{\partial\dot{\psi}} - \frac{\partial L}{\partial\psi} &= ml^2\ddot{\psi} - ml^2\dot{\theta}\sin\psi + mgl\sin\psi = 0
\end{aligned}
$$

The first equation is another way of saying the z-component of the angular momentum is constant.

4. **Problem 21: A table with a hole**

A mass $m_1$ on a table is connected to another mass $m_2$ by a string of length $l$ passing through a hole, with $m_2$ moving a vertical direction.

We set up a coordinate system with the origin at the center of the hole, the x-y plane on the table and the z-axis pointing down. We use cylindrical coordinates $r, \theta, z$. The constraints are easily solved, with mass $m_1$ moving in the plane $z = 0$ and mass $m_2$ moving in the z-axis:

$$\begin{aligned} \mathbf{r}_1 &= r\cos\theta\hat{\mathbf{i}} + r\cos\theta\hat{\mathbf{j}} \\ \mathbf{r}_2 &= z\hat{\mathbf{k}} \end{aligned}$$

The length of the string is $L = r + z$, so the constraint is $L = r + z$, or $\dot{r} = -\dot{z}$. We can choose $r, \theta$ as generalized coordinates for the system.

The kinetic energy is

$$T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}m_2\dot{z}^2 = \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\theta}^2$$

The gravitational potential energy, using $\mathbf{g} = g\hat{\mathbf{k}}$, is

$$V = -m_1\mathbf{g}\cdot\mathbf{r}_1 - m_2\mathbf{g}\cdot\mathbf{r}_2 = -m_2 g z = -m_2 g(L - r)$$

The Lagrangian is

$$L = T - V = \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\theta}^2 - m_2 g r$$

where we have omitted the constant term $m_2 g L$, since we know it doesn't change the equations of motion.

Lagrange's equation for $\theta$ is:

$$\begin{aligned} 0 &= \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} \\ 0 &= \frac{d}{dt}(m_1 r^2\dot{\theta}) \end{aligned}$$

Since $\theta$ is a cyclical coordinate, we see that there is a conserved quantity, $l = m_1 r^2\dot{\theta}$. This is the angular momentum of the mass $m_1$ on the table, which only has a vertical component:

$$\mathbf{r}_1 = r(\cos\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}})$$

$$\mathbf{v}_1 = \dot{r}(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}) + r\dot{\theta}(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}})$$
$$\mathbf{L}_1 = \mathbf{r}_1 \times m_1\mathbf{v}_1 = m_1 r^2\dot{\theta}\hat{\mathbf{k}} = l\hat{\mathbf{k}}$$

Lagrange's equation for the coordinate $r$ is:

$$\begin{aligned}
0 &= \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} \\
0 &= (m_1 + m_2)\ddot{r} - m_1 r\dot{\theta}^2 + m_2 g \qquad (1) \\
0 &= (m_1 + m_2)\ddot{r} - m_1 r\left(\frac{l}{m_1 r^2}\right)^2 + m_2 g \\
0 &- = (m_1 + m_2)\ddot{r} - \frac{l^2}{m_1 r^3} + m_2 g \\
0 &= = (m_1 + m_2)\ddot{r}\dot{r} - \frac{l^2}{m_1 r^3}\dot{r} + m_2 g\dot{r} \qquad (2) \\
0 &= \frac{d}{dt}\left(\frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{l^2}{2m_1 r^2} + m_2 g r\right) \\
0 &= \frac{d}{dt}\left(\frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\theta}^2 + m_2 g r\right) \\
0 &= \frac{d}{dt}(T + V) \\
0 &= \frac{dE}{dt}
\end{aligned}$$

By transforming the equation into a total derivative in numbered line (2), we recognize in this equation the conservation of mechanical energy. In numbered line (1), we also recognize the term $m_2 g$ representing the magnitude of the total external force in the system, equal to the rate of change in kinetic energy, $dT/dt = (m_1+m_2)\ddot{r} - m_1 r\dot{\theta}^2$. Notice that in polar coordinates, the rate of change in kinetic energy does not just have second time derivatives $(m_1 + m_2)\ddot{r}$, but also an "angular momentum" term $-m_1 r\dot{\theta}^2$.

If we wanted to use Newton's laws $\mathbf{F} = m\mathbf{a}$, we need to introduce the tension force, and then combine equations to eliminate it:

$$\begin{aligned}
m_2\mathbf{g} - T\hat{\mathbf{k}} &= m_2\ddot{\mathbf{r}}_2 \\
-T\hat{\mathbf{e}}_r &= m_1\ddot{\mathbf{r}}_1
\end{aligned}$$

We need the components of $m_1$'s acceleration in polar coordinates:

$$\mathbf{r}_1 = \; = \; r\cos\theta\ i + r\cos\theta\ j = r\hat{\mathbf{e}}_r$$

$$
\begin{aligned}
\dot{\mathbf{r}}_1 &= \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta \\
\ddot{\mathbf{r}}_1 &= \ddot{r}\mathbf{e}_r + 2\dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta - r\dot{\theta}^2\mathbf{e}_r
\end{aligned}
$$

Newton's equations are then:

$$
\begin{aligned}
m_2 g - T &= m_2\ddot{z} = -m_2\ddot{r} \\
-T &= m_1(\ddot{r} - r\dot{\theta}^2) \\
0 &= 2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r\dot{\theta}}\frac{d}{dt}(mr^2\dot{\theta}^2)
\end{aligned}
$$

We can combine the first two equations and obtain Lagrange' equation for $r$; the third equation is Lagrange's equation for $\theta$. Newton's laws give us a way to calculate the tension if we know $r(t)$: $T = m_2(g + \ddot{r})$; we don't get the tension from Lagrange's equations unless we use Lagrange's multipliers.

5. **Free fall with frictional forces**

A particle falls vertically under the influence of gravity, with fricitonal forces derivable from a dissipation function $\mathcal{F} = \frac{1}{2}kv^2$.

The Lagrangian is (choosing the z-axis pointing down)

$$
L = T - V = \frac{1}{2}m\dot{z}^2 + mgz
$$

Lagrange's equation is

$$
0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} + \frac{\partial \mathcal{F}}{\partial \dot{z}} = m\ddot{z} - mg + k\dot{z}
$$

We can write this as a differential equation for $v = \dot{z}$:

$$
m\dot{v} + kv - mg = 0
$$

We take another derivative and get an equation for $a = \dot{v}$:

$$
\ddot{v} + k\dot{v} = m\dot{a} + ka = 0
$$

which has an exponential solution

$$
a(t) = a_0 e^{-\lambda t},
$$

with $\lambda = k/m$. We integrate once to get the velocity:

$$
v(t) = v_0 + \frac{a_0}{\lambda}(1 - e^{-\lambda t})
$$

As $t \to \infty$, the velocity approaches a constant, $v \to v_0 + a_0/\lambda$, but this final maximum velocity seems to depend on arbitrary constants of integration: we know this is not so! If we integrate $v(t)$, we would obtain an expression for $z(t)$ with *three* constants of integration $z_0, v_0, a_0$, but we know we should only have two, since we started with a second order differential equation for $x$. However, the equation for $v$ tells us that $v_0, a_0$ are not independent:

$$\begin{aligned} 0 &= m\dot{v} + kv - mg \\ mg &= ma_0 e^{-\lambda t} + k\left(v_0 + \frac{a_0}{\lambda}(1 - e^{-\lambda t})\right) \\ mg &= kv_0 + ma_0 \\ a_0 &= g - \lambda v_0 \end{aligned}$$

and then the velocity is

$$v(t) = v_0 + \frac{g - \lambda v_0}{\lambda}(1 - e^{-\lambda t}) = v_0 e^{-\lambda t} + \frac{mg}{k}(1 - e^{-\lambda t})$$

and the final (maximum) terminal velocity is $mg/k$. At that point, gravity force $mg$ is in equilibrium with the friction force $-d\mathcal{F}/d\dot{z} = -k\dot{z}$.

6. **A pendulum hanging from a spring**

A spring of rest length $L_a$ (no tension) is connected to a support at one end and has a mass $M$ attached to the other. Assuming the motion is confined to a vertical plane, the position of the mass $M$ is, using polar coordinates $r, \theta$ with the x-axis pointing down:

$$\mathbf{r} = r\hat{\mathbf{e}}_r = r(\cos\theta, \sin\theta)$$

Notice that the length of the spring is not constrained like in a rigid pendulum, we do have two independent generalized coordinates in this case. The kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$$

The potential energy is

$$V = -m\mathbf{g} \cdot \mathbf{r} + \frac{1}{2}k(r - L_a)^2 = -mgr\cos\theta + \frac{1}{2}k(r - L_a)^2$$

and the Lagrangian is

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta - \frac{1}{2}k(r - L_a)^2$$

In equilibrium, the spring will be stretched to provide a force compensating for gravity on the mass: $L_0 = L_a + \Delta L = L_a + mg/k$. We can choose a generalized coordinate

$\xi$, instead of $r$, representing the length the spring has stretched from its equilibrium length $L_0$: $\xi = r - L_0$. The Lagrangian is then

$$L = \frac{1}{2}m(\dot{\xi}^2 + (\xi + L_0)^2\dot{\theta}^2) + mg(\xi + L_0)\cos\theta - \frac{1}{2}k(\xi + mg/k)^2$$

Lagrange's equation for $\xi$ is :

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\xi}} &= \frac{\partial L}{\partial \xi} \\
m\ddot{\xi} &= m(\xi + L_0)\dot{\theta}^2 + mg\cos\theta - k(\xi + mg/k) \\
m\ddot{\xi} &= m(\xi + L_0)\dot{\theta}^2 - mg(1 - \cos\theta) - k\xi
\end{aligned}$$

and Lagrange's equation for $\theta$ is:

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= \frac{\partial L}{\partial \theta} \\
m(\xi + L_0)^2\ddot{\theta} + 2\dot{\xi}\dot{\theta}(\xi + L_0) &= -mgL_0\sin\theta
\end{aligned}$$

If we assume small departures from equilibrium: $\xi \ll L_0$ and $\theta \ll 1$, we can keep leading oder terms in the equations, and obtain:

$$\begin{aligned}
m\ddot{\xi} &= -k\xi \\
mL_0^2\ddot{\theta} &= -mgL_0\theta
\end{aligned}$$

which have oscillatory solutions with frequencies $\omega_k^2 = k/m$ for the spring stretching and $\omega_g^2 = g/L_0$ for the angular displacement.