

Phys 7221 Homework #1

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1. Derivation 1-1: Kinetic energy and power

$$\begin{aligned}T &= \frac{1}{2}mv^2 = \frac{1}{2}\frac{p^2}{m} = \frac{\mathbf{p} \cdot \mathbf{p}}{2m} \\mT &= \frac{1}{2}\mathbf{p} \cdot \mathbf{p} \\ \frac{d(mT)}{dt} &= \frac{d\mathbf{p}}{dt} \cdot \mathbf{p} = \mathbf{F} \cdot \mathbf{p}\end{aligned}$$

If mass is constant,

$$\begin{aligned}\frac{d(mT)}{dt} &= m\frac{dT}{dt} = \mathbf{F} \cdot \mathbf{p} \\ \frac{dT}{dt} &= \mathbf{F} \cdot \frac{\mathbf{p}}{m} = \mathbf{F} \cdot \mathbf{v}\end{aligned}$$

Notice that $\mathbf{F} \cdot \mathbf{v} = \mathbf{F} \cdot (d\mathbf{s})/dt = d(\mathbf{F} \cdot \mathbf{s})/dt = dW/dt$ is the power (work per unit time).

2. Derivation 1-4: Non-holonomic constraints

If a constraint is written in the differential form

$$\sum_{i=1}^n g_i(x_1, \dots, x_n) dx_i = 0$$

can be integrated, it means it is also the exact differential of a function $f(x_1, \dots, x_n)$, which is constant and represents an algebraic relationship between the coordinates:

$$\sum_{i=1}^n g_i(x_1, \dots, x_n) dx_i = df = 0.$$

If this function exists, the constraint equation tells us about partial derivatives of the function f :

$$\frac{\partial f}{\partial x_i} = g_i$$

and since

$$\frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_i} = \frac{\partial}{\partial x_i} \frac{\partial f}{\partial x_j},$$

the functions g_i must satisfy the condition

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}$$

For a rolling disk on a plane, one of the constraints is

$$dx - a \sin \theta d\phi = 0$$

so $g_x = 1$, $g_\phi = -a \sin \theta$, $g_\theta = 0$. We see that

$$\frac{\partial g_\phi}{\partial \theta} = -a \cos \theta \quad \text{but} \quad \frac{\partial g_\theta}{\partial \phi} = 0$$

and then, we see that the constraint cannot be equal to an exact differential, and cannot be integrated: it is non-holonomic, *unless* $\theta = \pi/2$. If $\theta = \pi/2$, the disk is rolling along the x-direction, and the constraint can be integrated:

$$dx - a d\phi = 0 \rightarrow x = a(\phi - \phi_0)$$

This is a common problem in introductory physics, for objects rolling down an inclined plane.

The other constraint for the disk rolling on a plane is

$$dy + a \cos \theta d\phi = 0$$

so $g_y = 1$, $g_\phi = a \cos \theta$, $g_\theta = 0$. We see again that

$$\frac{\partial g_\phi}{\partial \theta} = -a \sin \theta \quad \text{but} \quad \frac{\partial g_\theta}{\partial \phi} = 0$$

Again, the constraint is non-holonomic unless $\theta = 0$ (the disk is rolling along the y-direction).

Counting degrees of freedom

A rigid body such as a disk has 6 degrees of freedom, or coordinates needed for the unique description of its position and orientation: for example, three coordinates of any point, and three angles defining the body's orientation in space; or three coordinates for two different points.

A disk can be described by the coordinates of the center of the disk $\mathbf{r} = (x, y, z)$; two components defining a unit vector $\hat{\mathbf{n}}$ along the axis of the disk, and a rotation angle ϕ of the disk about its axis. If the disk is constrained to be rolling on a plane, it has four constraints:

- (i) moving on a plane: $z = \text{constant}$;
- (ii) being oriented perpendicular to the plane: $\hat{n}_z = 0$ (and thus $z=a/2$);
- (iii) staying perpendicular to the plane: $\hat{\mathbf{n}} \cdot \mathbf{v} = 0$
- (iv) and rolling without slipping: $|\mathbf{v}| = a\dot{\phi}$.

Condition (ii) which can be used to define a coordinate θ : $\hat{\mathbf{n}} = (\cos \theta, \sin \theta, 0)$: constraints (i) and (ii) have been used to reduce the number of coordinates from six to four: x, y, θ, ϕ .

Condition (iii) can be expressed as a condition on the velocity $\mathbf{v} = v(\sin \theta, -\cos \theta)$; and condition (iv) is then the constraints as in (1.39) in the textbook

$$\begin{aligned} \dot{x} - a \sin \theta \dot{\phi} &= 0 \\ \dot{y} + a \cos \theta \dot{\phi} &= 0 \end{aligned}$$

There are four coordinates x, y, θ, ϕ , and two constraints, so there are only two degrees of freedom, but we cannot find two independent generalized coordinates associated with them because the constraints are non-holonomic.

There are six Newton's equations of motion for a rigid body ($\mathbf{F} = \dot{\mathbf{P}}$ and $\mathbf{N} = \dot{\mathbf{L}}$). However, since the motion of the center of mass is on a plane, and the axis of rotation is horizontal, there are only four (not six) non-trivial equations. We also have two constraint equations, for a total of 6 differential equations to solve, and four coordinates, x, y, θ, ϕ . We then see that there are two forces of constraint, also unknown (the components of the friction force, a horizontal vector), so we have six equations for six unknowns.

If the disk is rolling in a straight line, there is another constraint:

- (v) the direction of the velocity is constant.

From the previous expression $\mathbf{v} = v(\sin \theta, -\cos \theta)$, this means that the angle θ is constant, $\theta = \theta_0$. The constraint equations are then holonomic and *can* be integrated:

$$x = a \sin \theta_0 (\phi - \phi_0) \quad \text{and} \quad y = -a \cos \theta_0 (\phi - \phi_0).$$

We have then used five constraints to reduce the six disk coordinates to a single generalized coordinate ϕ : the system has only one degree of freedom. Newton's equations now only have two non-trivial equations (for the single components of the linear and angular momentum): these are two equations for the coordinate ϕ and for the constraint force (friction, which now also has a single component).

3. **Problem 1-5: Two wheels of radius a are mounted on the ends of an axle of length b such that the wheels rotate independently. The whole combination rolls without slipping on a plane.**

Consider first a single wheel (a disk) as in the previous problem. We have seen we can describe the system using four coordinates x, y, θ, ϕ , constrained by two differential equations:

$$\begin{aligned}\dot{x} &= a\dot{\phi} \sin \theta \\ \dot{y} &= -a\dot{\phi} \cos \theta\end{aligned}$$

We can also write the constraint using dx, dy :

$$\begin{aligned}dx - a \sin \theta d\phi &= 0 \\ dy + a \cos \theta d\phi &= 0\end{aligned}$$

In a problem with two wheels, each wheel satisfies the same constraints than the single rolling disk. We use $\mathbf{r}_1, \mathbf{v}_1$ for the center of wheel 1 and $\mathbf{r}_2, \mathbf{v}_2$ for the second wheel. Since the wheels are connected by a common axle, the angles θ_1, θ_2 that define each wheel's axis are the same: $\theta_1 = \theta_2 = \theta$, the angle of the common axle. The rotation angles ϕ_1, ϕ_2 are different, since the wheels can rotate independently. Thus, we can write the constraints as:

$$\begin{aligned}dx_1 - a \sin \theta d\phi_1 &= 0 \\ dy_1 + a \cos \theta d\phi_1 &= 0 \\ \\ dx_2 - a \sin \theta d\phi_2 &= 0 \\ dy_2 + a \cos \theta d\phi_2 &= 0\end{aligned}$$

The center of the axle (which is the center of mass) has a position vector $\mathbf{r} = (\mathbf{r}_2 + \mathbf{r}_1)/2$, so $x = (x_1 + x_2)/2$ and $y = (y_1 + y_2)/2$. Thus, we can write constraints for dx, dy :

$$\begin{aligned}dx - a \sin \theta (d\phi_1 + d\phi_2)/2 &= 0 \\ dy + a \cos \theta (d\phi_1 + d\phi_2)/2 &= 0\end{aligned}$$

Multiplying each equation by trigonometric factors $\sin \theta, \cos \theta$ and adding or subtracting them, we can write the equations as

$$\begin{aligned}\cos \theta dx + \sin \theta dy &= 0 \\ \sin \theta dx - \cos \theta dy &= \frac{a}{2}(d\phi_1 + d\phi_2)\end{aligned}$$

So far, these have been the equations for the disks not rolling, but we also have the constraint that the center of the wheels are at a constant distance b . The constraint can be written as $\mathbf{r}_2 - \mathbf{r}_1 = b(\cos \theta \hat{i} + \sin \theta \hat{j})$, or $x_2 - x_1 = b \cos \theta$, $y_2 - y_1 = b \sin \theta$. Taking derivatives, the x- constraint is

$$\begin{aligned} \dot{x}_2 - \dot{x}_1 &= -b\dot{\theta} \sin \theta \\ a \sin \theta (\dot{\phi}_2 - \dot{\phi}_1) &= -b\dot{\theta} \sin \theta \\ \dot{\theta} &= -\frac{a}{b}(\dot{\phi}_2 - \dot{\phi}_1) \\ \theta &= C - \frac{a}{b}(\phi_2 - \phi_1) \end{aligned}$$

(If we follow on with the y-constraint, we get the same equation: the constraint is only on the magnitude of the distance, not on direction; the constraint on direction was used to define the angle θ for the axle's direction, perpendicular to the wheels). Notice that the constraint was holonomic to begin with ($|\mathbf{r}_2 - \mathbf{r}_1| = b$), we transformed into one on velocities in order to involve θ, ϕ , but we then were able to integrate those equations to get a holonomic constraint on θ, ϕ . This is to say, we can have constraints involving velocities that are holonomic, if they are integrable.

4. Problem 1-13: Rocket motion.

The equation of motion is $\mathbf{F} = d\mathbf{p}/dt$. Consider the rocket with fuel at time t moving at some vertical velocity v with respect to the Earth: the momentum is mv . An instant later $t + dt$, there are two parts to the system: the rocket, with smaller mass $m - dm$, moving at an increased velocity $v + dv$; and the expelled fuel with mass dm , moving at a velocity v_f with respect to the Earth. The total momentum is now $p + dp = (m - dm)(v + dv) + dm v_f = mv + mdv + dm(v_f - v) = p + mdv + v' dm$, so $dp = mdv + v' dm$. From $F = dp/dt = -mg$, we have $-mg = mdv/dt + v' dm/dt$, or

$$m \frac{dv}{dt} = -mg - v' \frac{dm}{dt}.$$

If $dm/dt = \dot{m}$ and v' are constant, then we can write

$$dv = -gdt - \frac{v'}{m} dm = -\frac{g}{\dot{m}} dm - \frac{v'}{m} dm,$$

and we can integrate the equation:

$$v(m) = -\frac{g}{\dot{m}} m - v' \ln(m) + C.$$

If the initial mass is m_0 , and the initial velocity is zero, then we can solve for the constant C : $0 = (g/\dot{m})m_0 + v' \ln(m_0) + C$, and

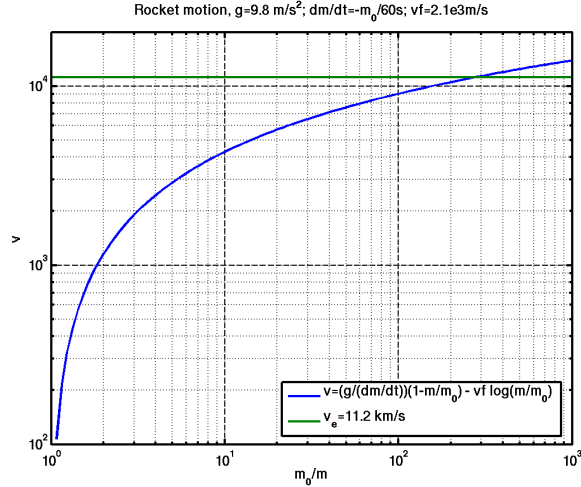
$$v(m) = -\frac{g}{\dot{m}}(m - m_0) - v' \ln \frac{m}{m_0}.$$

Notice that m decreases with time, so \dot{m} is a negative constant, and the first term is negative (it is of course the usual gravity term for free fall). The second term is positive because $m/m_0 < 1$, and it is the one that makes the velocity increase with time, at least in the beginning. If we want the velocity as a function of time, we use $m(t) = m_0 + \dot{m}t$:

$$v(t) = -gt - v' \ln \left(1 + \frac{\dot{m}t}{m_0} \right).$$

For late times, the second term is much larger than the first, and we have $v \approx -v' \ln m/m_0$, or $m/m_0 \approx e^{-v/v'}$.

The figure below shows the velocity as a function of the ratio of initial mass to mass, for $v' = 2.1 \text{ km/s}$, $\dot{m} = -m_0/60 \text{ s}$, and $g = 9.8 \text{ m/s}^2$. As it can be seen, it reaches Earth's escape velocity 11.2 km/s when the final mass is about $1/300$ of the initial mass. Since the final mass is at least the mass of the rocket (plus any fuel left), the fuel mass at launch was about 300 times the empty rocket mass.



5. **Problem 1-14: Two points of mass m are joined by a rigid weightless rod of length l , the center of which is constrained to move on a circle of radius a . Express the kinetic energy in terms of generalized coordinates.**

A system of two particles has 6 coordinates; there is one constraint due to the rod between them, and two constraints for the center of the rod moving in a circle, so we should be able to find three generalized coordinates.

Take the circle where the end of the rod is constrained to move on as a circle centered on the origin, in the horizontal plane. The position of the center of the rod- also the center of mass- is determined by a single angle Φ : $\mathbf{R} = a(\cos \Phi \hat{i} + \sin \Phi \hat{j})$.

We can choose as the other two generalized coordinates two angles θ and ϕ defining the rod's orientation (and thus the particles' positions). We choose angles in a spherical coordinate system with the origin in the center of mass (a non-inertial frame!). If particle 1 has angular coordinates $\theta = \theta_1, \phi = \phi_1$, then particle 2 has angular coordinates $\theta_2 = \pi - \theta$, and $\phi_2 = \phi + \pi$. Both particles are at a radial coordinate $l/2$ from the center of mass. The particles' positions with respect to the center of mass are:

$$\begin{aligned}\mathbf{r}'_1 &= (l/2)(\sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k}) \\ \mathbf{r}'_2 &= (l/2)(-\sin \theta \cos \phi \hat{i} - \sin \theta \sin \phi \hat{j} - \cos \theta \hat{k})\end{aligned}$$

The kinetic energy of each particle is $T_i = \frac{1}{2}mv_i^2$. The velocity of each particle is

$$\begin{aligned}\mathbf{v}_i &= \frac{d}{dt}(\mathbf{R} + \mathbf{r}'_i) = \mathbf{V} + \mathbf{v}'_i \\ &= a\dot{\Phi}(\sin \Phi \hat{i} + \cos \Phi \hat{j}) + \frac{l}{2}\dot{\theta}(\pm \cos \theta \cos \phi \hat{i} \pm \cos \theta \sin \phi \hat{j} \mp \sin \theta \hat{k}) \\ &\quad + \frac{l}{2}\dot{\phi} \sin \theta (\mp \sin \phi \hat{i} \pm \cos \phi \hat{j})\end{aligned}$$

The square of the speed of each particle is

$$\begin{aligned}v_i^2 &= V^2 + 2\mathbf{V} \cdot \mathbf{v}'_i + v_i'^2 \\ &= (a\dot{\Phi})^2 \pm 2al\dot{\Phi} \sin \Phi (\dot{\theta} \cos \theta \cos \phi - \dot{\phi} \sin \theta \sin \phi) \\ &\quad \pm 2al\dot{\Phi} \cos \Phi (\dot{\theta} \cos \theta \sin \phi + \dot{\phi} \sin \theta \cos \phi) + (l\dot{\theta}/2)^2 + (l\dot{\phi} \sin \theta/2)^2\end{aligned}$$

The total kinetic energy is then

$$T = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 = ma^2\dot{\Phi}^2 + \frac{1}{4}ml^2(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)$$

where we recognize the terms in formula 1.31 in the textbook:

$$T = \frac{1}{2}MV^2 + \sum \frac{1}{2}m_i v_i'^2$$

We also saw explicitly the identity used in the derivation of Formula 1.31, $\sum v_i' = 0$, since the velocity components of each particle have different signs, and cancel out in the sum.

The kinetic energy of the center of mass is

$$\frac{1}{2}MV^2 = \frac{1}{2}(2m)(a\dot{\Phi})^2 = ma^2\dot{\Phi}^2.$$

and the kinetic energy of motion about the center of mass is

$$\begin{aligned}\sum \frac{1}{2} m_i v_i'^2 &= 2 \times \frac{1}{2} m (l/2)^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \\ &= \frac{1}{4} m l^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)\end{aligned}$$

where we have used the formula for speed in spherical coordinates (see, for example, appendix F.3 in Marion and Thorne):

$$v_i'^2 = \dot{r}'^2 + r'^2 \dot{\theta}^2 + r'^2 \sin^2 \theta \dot{\phi}^2 = (l/2)^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)$$

(it's the same speed for both particles).