Physics 7221 Fall 2005 : Final Exam

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NAME: ....

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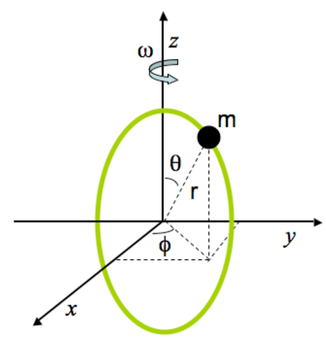
Please write as much as you can when answering questions *and* problems, explaining your steps in as many words as you can. When drawings are asked for, please annotate them if it makes them more clear.

You can use more pages if needed. Please write in the form in this page how many extra pages you returned for grading.

(10 pts) Question: a mass on a rotating hoop.

Consider a bead of mass m constrained to move along a vertical hoop of radius R. The hoop is rotating along a vertical axis through the center of the hoop, with constant angular velocity ω.



1. (1pt) **If the particle were unconstrained and free to move in a gravitational field, how many generalized coordinates would the system have?**

   A point particle in 3 dimensions has three degrees of freedom, so we would need three generalized coordinates, such as the cartesian coordinates x, y, z or spherical coordinates r, θ, ϕ.

2. (2pts) **How many constraints are there in the system? How many generalized coordinates are there in the constrained system?**

   The particle in the system is constrained to be at a distance from the origin, and to rotate about the z-axis with uniform velocity: those are two constraints. The mass has only one degree of freedom left, so it is described by just one generalized coordinate, such as the azimuth angle θ.

3. (3 pts) **What are the forces acting on the mass? Which of the forces acting on the mass can be derived from a potential, and which are constraint forces?**

   The forces on the mass are gravity, and the contact force exerted by the hoop on the mass. The gravitational force can be derived from a potential $V_g = -m\mathbf{g} \cdot \mathbf{r}$. The force of the hoop on the mass cannot be derived from a potential (except in special cases), and is a constraint force.

4. (4 pts) **How would you find the constraint forces using a Lagrangian formulation?**

   We would write the Lagrangian using all three coordinates r, θ, ϕ, and use two Lagrange multipliers λ_r, λ_ϕ for the constraints $r - R = 0$ and $\phi - \omega t - \phi_0 = 0$, respectively. There would be a total of five equations (three Lagrange equations, plus two constraint equations) for five variables. The Lagrange multipliers would be associated with the components of the hoop's force on the mass in the radial direction, and in the horizontal direction perpendicular to the position vector, respectively (i.e., along $\hat{e}_r$ and $\hat{e}_\phi$).

(25 pts) Problem : a mass on a rotating hoop.

Consider a bead of mass m constrained to move along a vertical hoop of radius R. The hoop is rotating along a vertical axis through the center of the hoop, with constant angular velocity ω. (You can refer to the drawing of the question in the previous page).

1. (5 pts) **Write down the Lagrangian of the system to describe the position of the mass on the hoop in terms of independent generalized coordinates.**

   The kinetic energy of the mass in spherical coordinates is

   $$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\sin^2\theta\dot{\phi}^2 + r^2\dot{\theta}^2\right) = \frac{1}{2}mR^2(\omega^2\sin^2\theta + \dot{\theta}^2)$$

   The potential energy is

   $$V = -m\mathbf{g}\cdot\mathbf{r} = mgz = mgR\cos\theta$$

   and the Lagrangian is then

   $$L = T - V = \frac{1}{2}mR^2(\omega^2\sin^2\theta + \dot{\theta}^2) - mgR\cos\theta$$

2. (8 pts) **Write down the Hamiltonian of the system.**

   The momentum canonically conjugate to θ is $p_\theta = \partial L/\partial\dot{\theta} = mR^2\dot{\theta}$, so $\dot{\theta} = p_t/mR^2$. . The Hamiltonian is

   $$H = p_t\dot{\theta} - L = \frac{p_t^2}{2m} - \frac{1}{2}mR^2\omega^2\sin^2\theta + mgR\cos\theta$$

3. (5 pts) **Is the Hamiltonian equal to the energy? Discuss conservation properties of the Hamiltonian and the energy.**

   The Lagrangian does not depend explicitly on time, so the Hamiltonian is a constant of motion. However, the energy of the particle is

   $$E = T + V = \frac{1}{2}mR^2\omega^2\sin^2\theta + \frac{1}{2}m\dot{\theta}^2 + mgR\cos\theta = H + mR^2\omega^2\sin^2\theta$$

   is not equal to the Hamiltonian, and is not a constant of motion, unless $\theta(t)$ is constant.

4. (7 pts) **Write the canonical equations of motion for the mass. Is there any equilibrium position?**

   The canonical equations of motion are

   $$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m}$$

   and

   $$\dot{p}_\theta = -\frac{\partial H}{\partial\theta} = mR^2\omega^2\sin\theta\cos\theta + mgR\sin\theta = -m\sin\theta(g + R\omega^2\cos\theta)$$

   The equilibrium conditions are $\dot{p}_\theta = 0, \dot{\theta} = 0$. From the canonical equation for $\dot{\theta}$, we see that $\dot{\theta} = 0$ if $p_\theta = 0$. From the canonical equation for $\dot{p}_\theta$, we see that there are equilibrium positions when $\sin\theta = 0$, or when $\cos\theta = -g/\omega^2$. The equilibrium position at $\theta = 0$ (top of the hoop) is unstable; the equilibrium position at $\theta = \pi$ (bottom of the hoop) is stable. The third equilibrium position exists only if $R\omega^2 > g$, and is in the lower half of the hoop.

(25 pts) Problem: A central force

Consider a mass m free to move in 3-dimension, subject to an isotropic spring potential of the form $V(r) = \frac{1}{2}kr^2$.

1. (8pts) **Plot the effective potential for the radial motion of the particle, when the particle has angular momentum of magnitude l.**

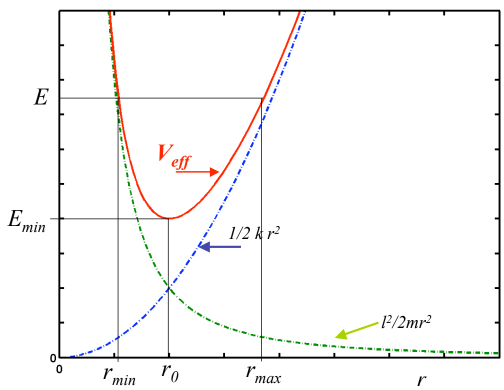


Figure 1: The effective potential is $V_{\text{eff}} = \frac{1}{2}kr^2 + \frac{1}{2}\frac{l^2}{mr^2}$

2. (8 pts) **Can the particle have an unbound orbit? Are the bound orbits closed orbits? How may turning points do the orbits have? Briefly explain your answers.**

   All orbits are bound, because the effective potential grows with distance: the particle cannot be at an infinite radial distance. If the angular momentum (a constant of motion) is not zero, the orbit has two turning points for a minimum and maximum radial distance.

   If the angular momentum is zero, orbits are still bound but don't have a minimum radial distance, they only have one turning point at a maximum radial distance. In this case, the "orbit" is a one-dimensional linear motion of the particle subject to a spring restoring force, between maximum displacement points on either side of the the origin.

   If $l \neq 0$, the orbit is bound and closed (this is one of the few potentials that have closed orbits), but not elliptical. The orbit will precess and come back to the initial point in a finite time.

3. (9 pts) **What is the energy the particle must have if its orbit is circular? How much of that energy is kinetic energy and how much potential energy?**

   The orbit will be circular when the energy is equal to the minimum of the effective potential. The effective potential has a minimum when its derivative vanishes, which is the equation for the radius of the circular orbit:

$$V'_{\text{eff}} = kr - \frac{l^2}{mr^3} = 0 \Rightarrow r_0^4 = \frac{l^2}{km}$$

   The energy of the orbit is the value of the effective potential evaluated at $r = r_0$. The kinetic energy is $\frac{1}{2}m\dot{r}_0^2 + l^2/2mr_0^2 = l^2/2m(l/\sqrt{km}) = l\sqrt{k/m}/2$. The potential energy is $V = \frac{1}{2}kr_0^2 = \frac{1}{2}kl/\sqrt{km} = l\sqrt{k/m}/2$. The total energy is thus half kinetic and half potential.

(15 pts) Question: Masses and springs

Four identical masses are at the corner of a square, attached by identical springs along the sides of the square, with equal spring constant k.

1. (2 pts) **Assuming the system can move in all three dimensions, how many normal modes there will be?**

   There are four point masses and no constraints, so the system has 12 degrees of freedom, and will have 12 normal modes.

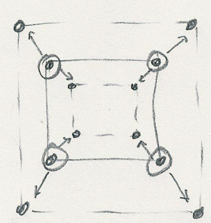
2. (5 pts) **How many normal modes will have a null eigenfrequency? Describe the motion of the system in each of those modes.**

   The null eigenfrequencies are associated with motion of the system that has no change in potential energy, i.e., when the springs do not stretch or compress.

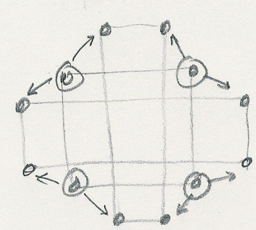
   There are three"rigid body" modes: three translation modes, one for each direction x, y, z; and three rotation modes, one about each of x, y, z axis.

   There are also three other modes where the springs do not compress or stretch, and thus have constant potential energy, but are not "rigid body" modes: a mode where the square transforms into a rhombus; and two modes in which one mass and its two adjacent square sides rotate about the diagonal line (there are two such independent modes, one for each of two masses on the same side).

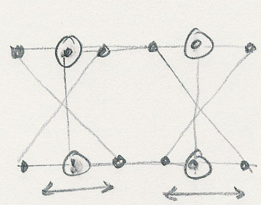
3. (8 pts) Sketch the motion of the system in at least three different normal modes with non-zero eigenfrequency.



All springs compress and stretch in phase.



Springs on opposite sides move in phase; springs on adjacent sides move in opposite phase

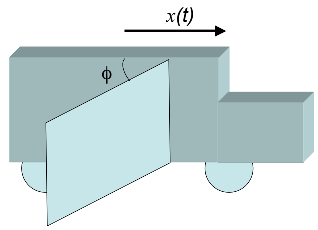


Three springs move out of phase with fourth spring.

5

(25 pts) Problem: A car driving with an open door

A car begins moving on a horizontal road, with a door accidentally left open with an initial angle ϕ_0 (where $\phi = 0$ indicates the door is closed). The motion of the car is described by a function $X(t)$. The door has mass M, width W, height H, and negligible thickness. Assume this is a primitive prototype of a car, where the hinges allow a full rotation of the door (!).



1. (7 pts) **Write the Lagrangian of the door, considering it as a rotating rigid body.**

   The system (the door) rotates about a vertical axis on the side, which is one of the principal axes of the body, although not through the center of mass. We will choose the other two principal axes so that the door is along the x' axis (i.e., with points in the door having $y' = 0$, in the $x'z'$ plane). The moment of inertia with respect to a vertical axis through the center of mass is $I_0 = MW^2/12$. The moment of inertia with respect to the axis of rotation is $I = I_0 + M(W/2)^2 = MW^2/3$.

   we choose an origin for the body system at the center of the door, on the axis of rotation. Points in the door will have a position vector with respect to an inertial system $\mathbf{r} = X\hat{\imath} + \mathbf{r}'$, where $\mathbf{r}'$ is the position vector with respect to the origin of the body frame. The velocity of mass elements in the door will be $\mathbf{v} = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}'$, where $\mathbf{V} = \dot{X}\hat{\imath}$ and $\boldsymbol{\omega} = \dot{\phi}\hat{k}$. The kinetic energy of the door will be

$$
\begin{aligned}
T &= \frac{1}{2}\int dm v^2 = \frac{1}{2}\int dm(V^2 + 2\boldsymbol{\omega}\cdot(\mathbf{V}\times\mathbf{r}') + |\boldsymbol{\omega}\times\mathbf{r}'|^2) \\
&= \frac{1}{2}M\dot{X}^2 + M\boldsymbol{\omega}\cdot(\mathbf{V}\times\mathbf{R}'_{cm}) + \frac{1}{2}I\omega^2 \\
&= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}MW\dot{X}\dot{\phi}\sin\phi + \frac{1}{2}I\dot{\phi}^2
\end{aligned}
$$

   There is no potential energy (the gravitational potential energy is constant), so the Lagrangian is simply

$$L = T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}MW\dot{X}\dot{\phi}\sin\phi + \frac{1}{2}I\dot{\phi}^2$$

2. (6 pts) **Find a differential equation for the angle of the door with the car, in terms of the known $X(t)$ of the car (assumed to be known).**

   Lagrange's equation for the coordinate ϕ is

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} - \frac{\partial L}{\partial\phi} = \frac{d}{dt}(\frac{1}{2}M\dot{X}\sin\phi + I\dot{\phi}) - \frac{1}{2}M\dot{X}\dot{\phi}\cos\phi = \frac{1}{2}M\ddot{X}\sin\phi + I\ddot{\phi} = 0$$

   or

$$\ddot{\phi} = -\frac{1}{2}\frac{M}{I}\ddot{X}\sin\phi = -\frac{3\ddot{X}}{2W}\sin\phi$$

3. (6 pts) **Describe (without an analytic solution) the door's motion when the car moves**

   (a) **with uniform velocity:** If $\ddot{X} = 0$, then $\ddot{\phi} = 0$. If the door starts open but at rest, it will remain open with the car moving (this also follows from Galileo's relativity principle).

   (b) **with uniform positive acceleration:** If $\ddot{X} > 0$ and $\sin\phi_0 > 0$, the door will have negative angular acceleration and will close itself (or keep rotating and move into the car if possible). So, you can close the door by accelerating forward if you forgot it open (not an advisable move, though).

   (c) **with uniform negative acceleration:** If $\ddot{X} < 0$ and $\sin\phi_0 > 0$, the door will have positive angular acceleration: the door will open wider, and keep increasing the angle until $\phi > \pi$ and the acceleration becomes negative. Of course, in a real car, the door will slam into the front of the car when $\phi = \pi$, if it can get that far.

4. (6 pts) Under what conditions can the door have small oscillations about an equilibrium position? What would the frequency of those oscillations be?

   For small ϕ and uniform positive acceleration $\ddot{X} = A$, the acceleration of the car acts like a restoring force, since $\ddot{\phi} \approx -(3A/2W)\sin\phi$. The solutions will be oscillatory (if the door can oscillate freely with positive and negative angle, into and out of the car), with frequency $\omega^2 = 3A/2W$.