Physics 7221 Fall 2006 : Final Exam

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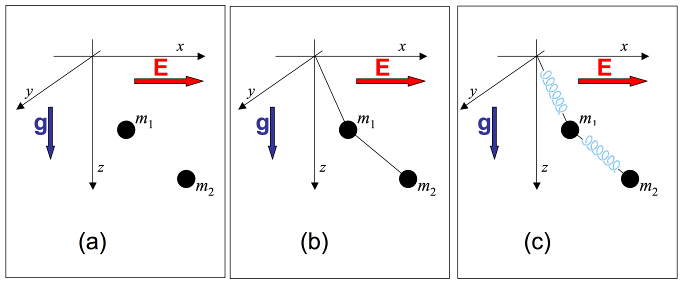
NAME: ....

(10 pts) Question 1: A two mass system.

Consider the following arrangements of two masses, shown in the figure: (a) two masses not connected to each other; (b) two masses connected by rigid rods (a double pendulum); (c) two masses connected by springs (a double pendulum with elastic strings). In all cases, there is a gravity force in the z direction, and a uniform electric field in the x direction. Both masses have mass m and positive electrical charge q.



1. (2 pts) **How many constraints does each system have?**

   (a) There are no constraints.

   (b) Two constraints: $|\mathbf{r}_1| = l, |\mathbf{r}_2 - \mathbf{r}_1| = l$.

   (c) There are no constraints.

2. (2 pts) **How many Hamilton equations are there for each system?**

   There are two Hamilton equations for each canonical coordinate (one for coordinate, one for momentum). The number of canonical coordiantes is equal tot he number of generalized coordinates, or number of degrees of freedom. The number of degrees of freedom is equal to 3 times the number N of particles (N=2 in all cases), minus the number of constraints.

   (a) 3N = 6 degrees of freedom (i.e., x, y, z for each particle); 12 Hamilton equations.

   (b) 3N-2= 4 degrees of freedom (i.e., two angles for each particle); 8 Hamilton equations.

   (c) 3N = 6 degrees of freedom (i.e., x, y, z for each particle); 12 Hamilton equations.

3. (6 pts) **Consider the following quantities: mechanical energy E, cartesian components of total linear momentum P_x, P_y, P_z; and cartesian components of total angular momentum L_x, L_y, L_z. Which of these quantities are conserved in each case?**

   Energy is conserved in all three systems (no dissipative forces, no time dependent constraints)

   In (a) there is a translational symmetry in the y-direction, so P_y is conserved, but not P_x, P_z. In (b) and (c), there are no translational symmetries, so no components of $\mathbf{P}$ are conserved.

   Because of the electric field, there aren't any rotational symmetries in all cases, so no component of $\mathbf{L}$ is conserved.

(15 pts) Question 2: Changing the Lagrangian by a total derivative.

Consider the most general Lagrangian of a one-dimensional harmonic oscillator:

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + \frac{dF(x,t)}{dt}$$

1. (5 pts) **Give a non-constant example of F such that the canonical momentum $p = \partial L/\partial \dot{x}$ is equal to the linear momentum $\mathbf{P}_x = m\dot{x}$.**

   Expanding the time derivative in the Lagrangian, we have

   $$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + \frac{dF(x,t)}{dt} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + \frac{\partial F}{\partial x}\dot{x} + \frac{\partial F}{\partial t}$$

   so the canonical momentum is

   $$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + \frac{\partial F}{\partial x}$$

   If $p = m\dot{x}$, we need $\partial F/\partial x =$. For example, $F = F_0 \sin \Omega t$ (or any function of t).

2. (5 pts) **Is there a non-constant choice for F such that the mechanical energy $E = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$ is not conserved? If yes, give an example; if not, explain why.**

   The mechanical energy is constant as a consequence of the equations of motion; the equations of motion are independent of the choice of $F(x,t)$. Thus, there is no choice of F that would make the energy not conserved.

3. (5 pts) **Give a non-constant example of F such that the Hamiltonian H is not equal to the energy, but it is conserved.**

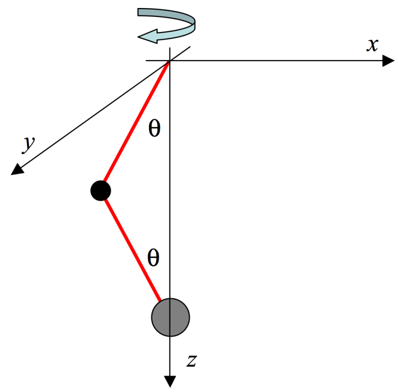
   The energy $E = T + V$ is conserved. The Hamiltonian is equal to

   $$H = p\dot{x} - L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 - \frac{\partial}{\partial t} = E - \frac{\partial F}{\partial t}$$

   If $\partial F/\partial t$ is constant, then $H \neq E$ but is still constant. For example, $F = Ct + A$. This is a non constant F, but it only adds a constant to the Lagrangian.

(25 pts) Problem 1. A constrained rotating system.

Consider a mass m attached with massless rods of length l to a fixed point at the origin, and to a bead of mass m on the z-axis, as shown in the figure. The whole system is made to rotate with constant angular velocity Ω about the vertical axis. The bead can move up or down on the axis. The gravitational force is assumed uniform, pointing down in the z direction. The angle the top rod makes with vertical axis is θ, which we choose as the single generalized coordinate needed for the system.



1. (12 pts) **Write Lagrange's equation for θ.**

   The position of the masses are

$$\mathbf{r}_1 = l(\sin\theta\cos\Omega t, \sin\theta\sin\Omega t, \cos\theta)$$

$$\mathbf{r}_2 = (0, 0, 2l\cos\theta)$$

   The velocities are

$$\mathbf{v}_1 = l\dot{\theta}(\cos\theta\cos\Omega t, \cos\theta\sin\Omega t, -\sin\theta) + l\Omega(-\sin\theta\sin\Omega t, \sin\theta\cos\Omega t, 0)$$

$$\mathbf{v}_2 = -2l\dot{\theta}(0, 0, \sin\theta)$$

   The kinetic energy is

$$T = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 = \frac{1}{2}ml^2(\dot{\theta}^2 + \Omega^2\sin^2\theta) + \frac{1}{2}m(4l^2\sin^2\theta)\dot{\theta}^2$$

   The gravitational potential energy is

$$V = -mgz_1 - mgz_2 = -3mgl\cos\theta$$

   The Lagrangian is then

$$L = \frac{1}{2}ml^2(\dot{\theta}^2 + \Omega^2\sin^2\theta) + 2ml^2\dot{\theta}^2\sin^2\theta + 3mgl\cos\theta$$

   Lagrange's equation is

$$\begin{aligned}
0 &= \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} \\
&= \frac{d}{dt}\left(ml^2\dot{\theta} + 4ml^2\dot{\theta}\sin^2\theta\right) - \left(ml^2\Omega^2\sin\theta\cos\theta + 4ml^2\dot{\theta}^2\sin\theta\cos\theta - 3mgl\sin\theta\right) \\
&= ml^2\ddot{\theta} + 4ml^2\ddot{\theta}\sin^2\theta + 8ml^2\dot{\theta}^2\sin\theta\cos\theta - \left(ml^2\Omega^2\sin\theta\cos\theta + 4ml^2\dot{\theta}^2\sin\theta\cos\theta - 3mgl\sin\theta\right) \\
&= ml^2\ddot{\theta}(1 + \sin^2\theta) + 4ml^2\dot{\theta}^2\sin\theta\cos\theta - ml^2\Omega^2\sin\theta\cos\theta + 3mgl\sin\theta
\end{aligned}$$

2. (3 pts) **How many equilibrium configurations are there? Describe the equilibrium positions in the limits for slow and fast rotation (small and large Ω).**

   In equilibrium, $\dot{\theta} = \ddot{\theta} = 0$, so Lagrange's equation will be satisfied if

   $$ml^2\ddot{\theta}(1+\sin^2\theta)+4ml^2\dot{\theta}^2\sin\theta\cos\theta-ml^2\Omega^2\sin\theta\cos\theta+3mgl\sin\theta = ml\sin\theta(l\Omega^2\cos\theta-3g) = 0$$

   This is an equation for the equilibrium angle θ with three solutions:

   $$\theta = 0, \pi, \cos^{-1}(\sqrt{3(g/l)/\Omega^2})$$

   The first solution $\theta = 0$ is a stable equilibrium with both masses on the positive z axis; the second solution with $\theta = \pi$ is an unstable equilibrium with one mass at the origin and another at the top (negative z axis); these solutions are independent of the rotation velocity Ω. These solutions are only possible for point masses, of course.

   The third solution is only possible if $\Omega^2 \geq 3g/l$. For slow rotation, then there are only two equilibrium positions; for fast rotation, there is a third equilibrium position that tends to $\theta = \pi/2$ (top mass on the horizontal axis) as $\Omega \to \infty$.

3. (10 pts) **Find the Hamiltonian for the system. Is it conserved? Is it equal to the energy?**

   The canonical momentum is

   $$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = ml^2(1+4\sin^2\theta)\dot{\theta}$$

   The Hamiltonian is

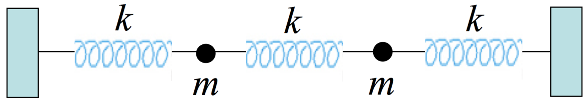
   $$\begin{aligned} H = p_\theta\dot{\theta} - L &= \frac{1}{2}ml^2(1+4\sin^2\theta)\dot{\theta}^2 - \frac{1}{2}ml^2\Omega^2\sin^2\theta - 3mgl\cos\theta \\ &= \frac{p_\theta^2}{2ml^2(1+4\sin^2\theta)} - \frac{1}{2}ml^2\Omega^2\sin^2\theta - 3mgl\cos\theta \end{aligned}$$

   The Hamiltonian is not equal to the energy, since the kinetic energy is not quadratic in $\dot{\theta}$.

   The Hamiltonian is conserved, though, because the Lagrangian does not depend explicitly on time.

(25 pts) Problem 2: Two masses and three springs

Consider two masses m and three springs, all with identical spring constants k and equilibrium length a. The masses can only move longitudinally.



1. (10 pts) **What are the normal frequencies ω_k and the normal modes $\mathbf{a}_k$ of the system? (If you can guess the *right* answer, you don't need to derive it - but you don't get partial credit if it's wrong...)**

   We set up the origin on the left wall, with the position of the two masses being $x_1 = a+\eta_1, x_2 = 2a + \eta_2$, where η_1, η_2 are deviations from equilibrium. The kinetic energy is

   $$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 = \frac{1}{2}m\dot{\eta}_1^2 + \frac{1}{2}m\dot{\eta}_2^2$$

   so the kinetic energy matrix is proportional to the identity matrix: $\mathbf{T} = m\mathbf{1}$

   The potential energy is

   $$V = \frac{1}{2}k(x_1 - a)^2 + \frac{1}{2}k(x_2 - x_1 - a)^2 + \frac{1}{2}k(3a - x_2 - a)^2 = \frac{1}{2}k\eta_1^2 + \frac{1}{2}k(\eta_2 - \eta_1)^2 + \frac{1}{2}k\eta_2^2$$

   so the potential energy matrix is

   $$\mathbf{V} = k\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

   The secular equation for eigenfrequencies is

   $$|V - \omega^2\mathbf{T}| = \begin{vmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{vmatrix} = (2k - m\omega^2)^2 - k^2$$

   with solutions $\omega_\pm^2 = 3k/m, k/m$

   The normal mode vectors $\mathbf{a}_\pm = (a_{\pm 1}, a_{\pm 2})$ will satisfy $(\mathbf{V} - \omega_\pm^2\mathbf{T}) \cdot \mathbf{a}_\pm = 0$. The matrices $(\mathbf{V} - \omega_\pm^2\mathbf{T})$ are

   $$\mathbf{V} - \omega_\pm^2\mathbf{T} = \begin{pmatrix} 2k - m\omega_\pm^2 & -k \\ -k & 2k - m\omega_\pm^2 \end{pmatrix} = \begin{pmatrix} \mp k & -k \\ -k & \mp k \end{pmatrix}$$

   so the (unnormalized) normal modes are $\mathbf{a}_\pm = a_\pm(1, \mp 1)$. Using the kinetic energy to normalize modes, we have

   $$\mathbf{a}_\pm = \frac{1}{\sqrt{2m}}(1, \mp 1)$$

   The first mode, with the higher frequency $\omega_+^2 = 3k/m$, has the two masses moving iout of phase, so the end springs are compressed when the middle spring is stretched. The second mode with lower frequency $\omega_-^2 = k/m$ has the two masses moving in phase, with the middle spring not stretching or compressing, and the two end springs compressing and stretching out of phase with each other.

2. (15 pts) **If the system starts at rest, with the first mass at a distance $a + \Delta$ from the left wall, and the second mass at a distance $2a$ from the wall, what are the displacements $x_1(t), x_2(t)$ as a function of time for each mass, and $x_{CM}(t)$ for the center of mass of the system?**

The general solution for the masses' motion is

$$x_1(t) = a + \eta_1(t) = a + C_+ \cos(\omega_+ t + \phi_+) + C_- \cos(\omega_- t + \phi_-)$$

$$x_2(t) = 2a + \eta_2(t) = 2a - C_+ \cos(\omega_+ t + \phi_+) + C_- \cos(\omega_- t + \phi_-)$$

The four constants $C_\pm, \phi_\pm$ are obtained from initial positions and velocities:

$$\begin{aligned} x_1(0) - a = \Delta &= C_+ \cos\phi_+ + C_- \cos\phi_- && (1)\\ \dot{x}_1(0) = 0 &= -C_+\omega_+ \sin\phi_- - C_-\omega_- \sin\phi_- && (2)\\ x_2(0) - 2a = 0 &= -C_+ \cos\phi_+ + C_- \cos\phi_- && (3)\\ \dot{x}_2(0) = 0 &= C_+\omega_+ \sin\phi_- - C_-\omega_- \sin\phi_- && (4) \end{aligned}$$

From (2) and (4) we see that $\sin\phi_+ = \sin\phi_- = 0$, and thus $\cos\phi_+ = \cos\phi_- = 1$. Using this in (3), we see that $C_+ = C_- = C$. Using this in 91), we see that $C = \Delta/2$. The masses' coordinates are then

$$x_1(t) = a + \frac{\Delta}{2}(\cos\omega_+ t + \cos\omega_- t)$$

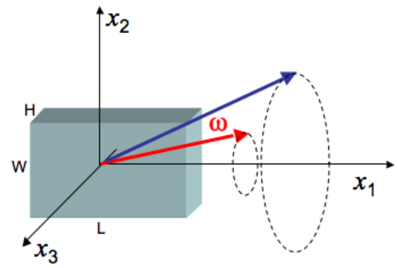
$$x_2(t) = 2a + \frac{\Delta}{2}(-\cos\omega_+ t + \cos\omega_- t)$$

The center of mass is

$$x_{com}(t) = \frac{1}{2}(x_1(t) + x_2(t)) = \frac{3}{2}a + \Delta\cos\omega_- t$$

(25 pts) Problem 3: A free rigid body.

Consider the motion of a rigid body with principal moments of inertia $I_1 < I_2 < I_3$, in absence of external forces and torques (i.e., a free rigid body). Assume the body is a rectangular figure of width W, height H and length L (i.e., a book), with H<W<L, as shown in the figure.



The angular velocity vector of the rigid body, in the body system, is $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$. The conserved energy of the top is E, and the conserved angular momentum vector is $\mathbf{L}$, which has magnitude L and, in the body system, has components $\vec{L} = (L_1, L_2, L_3) = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$.

1. (5 pts) **Prove that for a given energy E, the value for the angular momentum has minmum and maximum values $2EI_1 < L^2 < 2EI_3$. (Hint: write expressions for $2EI_1$, $2EI_3$ and L^2 in terms of $\omega_1, \omega_2, \omega_3$.)**

   The magnitude squared of the angular momentum is

   $$L^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$$

   The energy is just kinetic energy, which is

   $$E = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

   From this, we see that the differences $2EI_3 - L^2$ and $2EI_3 - L^2$ have a definite sign:

   $$2EI_3 - L^2 = I_3(I_1\omega_1^2 + I_2\omega_2^2) - (I_1^2\omega_1^2 + I_2^2\omega_2^2) = I_1(I_3 - I_1)\omega_1^2 + I_2(I_3 - I_2)\omega_2^2 \geq 0$$

   $$2EI_1 - L^2 = I_1(I_2\omega_2^2 + I_3\omega_3^2) - (I_2^2\omega_2^2 + I_3^2\omega_3^2) = -I_2(I_2 - I_1)\omega_2^2 - I_3(I_3 - I_1)\omega_3^2 \leq 0$$

   Thus, we see that $2EI_1 \leq L^2 \leq 2EI_3$. If the angular momentum has its minimum value $L = \sqrt{2EI_1}$, then $\omega_2 = \omega_3 = 0$, and the body rotates around its I_1 axis with angular velocity ω_1.

   If the angular momentum has its maximum value $L = \sqrt{2EI_3}$, then $\omega_1 = \omega_2 = 0$, and the body rotates around its I_3 axis with angular velocity ω_3.

2. (15 pts) **Assume the angular momentum is only slightly larger than its minimum value, and $\omega_2, \omega_3 \ll \omega_1$. Use Euler's equations to prove that to leading order, ω_1 is constant. Obtain solutions for $\omega_2(t), \omega_3(t)$ in this approximation.**

   If the angular momentum is only slightly larger than its minimum value, we can assume $\omega_2, \omega_3 \ll \omega_1$. Euler's equation for ω_1 is to leading order in $\omega_2/\omega_1, \omega_3/\omega_1$:

   $$0 = I1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) \approx I1\dot{\omega}_1$$

   so ω_1 is approximately constant.

Euler's equations for ω_2, ω_3 are

$$\begin{aligned}\dot{\omega}_2 &= \left(\omega_1 \frac{I_3 - I_1}{I_2}\right)\omega_3 \\ \dot{\omega}_3 &= -\left(\omega_1 \frac{I_2 - I_1}{I_3}\right)\omega_2\end{aligned}$$

which can be combined in a single second order differential equation:

$$\ddot{\omega}_2 = \left(\omega_1 \frac{I_3 - I_1}{I_2}\right)\dot{\omega}_3 = -\Omega^2\omega_2$$

with $\Omega^2 = \omega_1^2((I_3 - I_1)(I_2 - I_1)/(I_2 I_3))$. The solutions for ω_2, ω_3 are then

$$\omega_2(t) = \omega_0 \cos(\Omega t + \phi)$$

$$\omega_3(t) = \frac{I_2}{I_3 - I_1}\frac{\dot{\omega}_2}{\omega_1} = -\omega_0\sqrt{\frac{I_2(I_2 - I_1)}{I_3(I_3 - I_1)}}\sin(\Omega t + \phi)$$

The angular velocity vector $\vec{\omega}$ is seen to rotate in the body frame around the I_1 axis, with angular velocity Ω. To be consistent with our assumptions, we need $\omega_0 \ll \omega_1$, so the angular velocity vector makes a small (but not constant!) angle with the I_1 axis. Notice that the end point of the vector will describe an ellipse, not a circle like in the free symmetric top. The ratio ω_3/ω_2 is smaller than unity, so the ω_3 component is larger than the ω_2 component.

3. (5 pts) **Draw a snapshot of the angular velocity vector and the angular momentum vector in the figure. Are these vectors constant in inertial space? Are these vectors constant in the body frame?**

   The largest moment of inertia of the object in the figure is $I_3 = W^2 + L^2$, about an axis perpendicular to the "book's cover"; the smallest moment of inertia is $I_1 = H^2 + W^2$, about an axis parallel to the books binding edge. The moment of inertia about an axis parallel to the book's width is the intermediate one $I_2 \propto H^2 + L^2$.

   From the solutions we obtained, and being consistent with our assumptions, we need $\omega_0 \ll \omega_1$, so the angular velocity vector makes a small (but not constant!) angle with the I_1 axis. Notice that the end point of the vector will describe an ellipse, not a circle like in the free symmetric top.

   The angular momentum vector is $\mathbf{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$, so it will also have a constant component along x_3, and also precess around x_1 with angular velocity Ω, but at a different, larger angle, since the $L_1/L_{2,3} = (I_1/I_{2,3})\omega_1/\omega_{2,3}$ ratio is smaller than the $\omega_1/\omega_{2,3}$ ratio. The ellipse described by the end point of the angular momentum vector does not have the same axes ratio than the ellipse described by the angular momentum vector, it will even less round.

   Both the angular momentum and the angular velocity vector precess about the x_3 axis in the body frame.

   Since there are no torques, the angular momentum vector is conserved in inertial space, and it will be fixed. The angular velocity vector, however, will not be constant: we can use Euler's angles to rotate the vector we calculated in the body frame to inertial space (but we won't).