Energy function

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Consider a Lagrangian describing a system of N particles. The Lagrangian is a function of generalized coordinates q_j (where the index $j = 1...n \leq 3N$ is the number of degrees of freedom of the system), their time derivatives $\dot{q}_j$, and time:

$$L = L(q_j, \dot{q}_j, t)$$

and Lagrange's equation in the presence of a dissipation function $\mathcal{F}$ and of forces $\mathbf{F}'_i$ that cannot be derived from a potential (where the index $i = 1...N$ indicates the particle on which the force is acting):

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = -\frac{\partial \mathcal{F}}{\partial \dot{q}_j} + \sum_i \mathbf{F}'_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$$

The energy function is defined as

$$h = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L$$

We want to calculate the time derivative of the energy function. The time derivative of the Lagrangina function is

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \sum_j \dot{q}_j \frac{\partial L}{\partial q_j} + \sum_j \ddot{q}_j \frac{\partial L}{\partial \dot{q}_j}$$

The time derivative of the energy function is:

$$\begin{aligned} \frac{dh}{dt} &= \sum_j \ddot{q}_j \frac{\partial L}{\partial \dot{q}_j} + \sum_j \dot{q}_j \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{dL}{dt} \\ &= \sum_j \dot{q}_j \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial t} - \sum_j \dot{q}_j \frac{\partial L}{\partial q_j} \end{aligned}$$

We use Lagrange's equation in the first term to obtain:

$$\begin{aligned}
\frac{dh}{dt} &= \sum_j \dot{q}_j \left(\frac{\partial L}{\partial q_j} - \frac{\partial \mathcal{F}}{\partial \dot{q}_j} + \sum_i \mathbf{F}'_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right) - \frac{\partial L}{\partial t} - \sum_j \dot{q}_j \frac{\partial L}{\partial q_j} \\
&= -\frac{\partial L}{\partial t} - \sum_j \dot{q}_j \frac{\partial \mathcal{F}}{\partial \dot{q}_j} + \sum_j \dot{q}_j \sum_i \mathbf{F}'_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}
\end{aligned}$$

The dissipation function is a homogeneous function of second degree with respect to $\dot{q}_j$, so

$$\sum_j \dot{q}_j \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 2\mathcal{F}$$

We assume the constraints are holonomic, and the particles' positions $\mathbf{r}_i$ are functions of only coordinates q_j but not of their derivatives $\dot{q}_j$: $\mathbf{r}_i = \mathbf{r}_i(q_j, t)$. Thus,

$$\begin{aligned}
\sum_j \dot{q}_j \sum_i \mathbf{F}'_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} &= \sum_i \mathbf{F}'_i \cdot \left(\sum_j \dot{q}_j \frac{\partial \mathbf{r}_i}{\partial q_j} \right) \\
&= \sum_i \mathbf{F}'_i \cdot \left(\frac{d\mathbf{r}_i}{dt} - \frac{\partial \mathbf{r}_i}{\partial t} \right) \\
&= \sum_i \mathbf{F}'_i \cdot \mathbf{v}_i - \sum_i \mathbf{F}'_i \cdot \frac{\partial \mathbf{r}_i}{\partial t}
\end{aligned}$$

The complete expression for the time derivative of the energy function is then

$$\frac{dh}{dt} = -\frac{\partial L}{\partial t} - 2\mathcal{F} + \sum_i \mathbf{F}'_i \cdot \mathbf{v}_i - \sum_i \mathbf{F}'_i \cdot \frac{\partial \mathbf{r}_i}{\partial t}$$