Single and Double plane pendulum

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1 Introduction

We will write down equations of motion for a single and a double plane pendulum, following Newton's equations, and using Lagrange's equations.

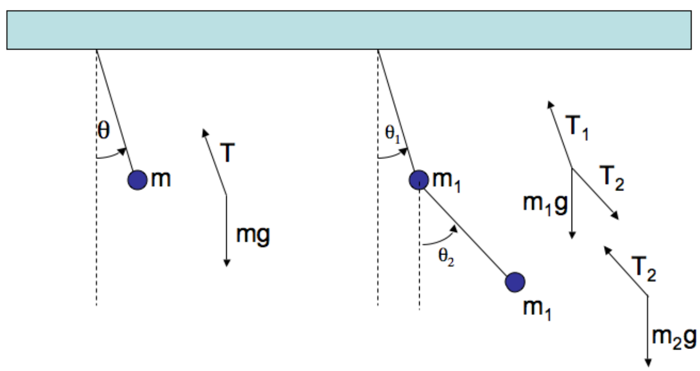

Figure 1: A simple plane pendulum (left) and a double pendulum (right). Also shown are free body diagrams for the forces on each mass.

2 Newton's equations

The double pendulum consists of two masses m_1 and m_2, connected by rigid weightless rods of length l_1 and l_2, subject to gravity forces, and constrained by the hinges in the rods to move in a plane. We choose a coordinate system with the origin at the top suspension point, the x-axis as a horizontal axis in the plane of motion, and the y-axis pointing down (so that gravity forces have positive components). The single plane pendulum, a simpler case, has a single particle hanging from a rigid rod.

2.1 Constraints

The simple pendulum system has a single particle with position vector $\mathbf{r} = (x, y, z)$. There are two constraints: it can oscillate in the (x,y) plane, and it is always at a fixed distance from the suspension point. Mathematically,

$$\begin{aligned} z &= 0 && (1) \\ |\mathbf{r}| &= l. && (2) \end{aligned}$$

The double pendulum system has two particles (N=2) with position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, each with components (x_i, y_i, z_i). There are four constraints: each particle moving in the x-y plane, and each rod having constant lengths. These constraints can be expressed as

$$\begin{aligned} z_1 &= 0 && (3) \\ z_2 &= 0 && (4) \\ |\mathbf{r}_1| &= l_1 && (5) \\ |\mathbf{r}_2 - \mathbf{r}_1| &= l_2 && (6) \end{aligned}$$

These constraints are holonomic: they are only algebraic relationships between the coordinates, not involving inequalities or derivatives.

In the single pendulum case, we only have one particle (N=1), so we have 3N=3 coordinates. Since we have two constraints (m=2), we are left with n=3N-m=3-2=1: only one generalized coordinate. This is the angular position of the pendulum θ, which we can use to write:

$$\mathbf{r} = l(\sin\theta, \cos\theta, 0). \tag{7}$$

In the double pendulum We know there should be only two generalized coordinates, since there are 3N=6 coordinates, and m=4 constraints, so n=3N-m=6-4=2. We can find expressions for $\mathbf{r}_1$, $\mathbf{r}_2$ in terms of two angles θ_1, θ_2:

$$\begin{aligned} \mathbf{r}_1 &= l_1(\sin\theta_1, \cos\theta_1, 0) && (8) \\ \mathbf{r}_2 &= \mathbf{r}_1 + l_2(\sin\theta_2, \cos\theta_2, 0) && (9) \end{aligned}$$

We can express velocity an acceleration vectors in terms of generalized coordinates. For the single pendulum,

$$\begin{aligned}
\mathbf{r} &= l(\sin\theta, \cos\theta) && (10)\\
\dot{\mathbf{r}} = \mathbf{v} &= l\dot{\theta}(\cos\theta, -\sin\theta) && (11)\\
\ddot{\mathbf{r}} = \mathbf{a} &= l\ddot{\theta}(\cos\theta, -\sin\theta) - l\dot{\theta}^2(\sin\theta, \cos\theta) && (12)\\
&= l\ddot{\theta}\hat{\mathbf{v}} - l\dot{\theta}^2\hat{\mathbf{r}} && (13)
\end{aligned}$$

The velocity vector $\mathbf{v}$ is perpendicular to the position vector $\mathbf{r}$, which is the expression of the constraint $|\mathbf{r}| = l =$ constant. We should recognize the tangential and centripetal acceleration terms, proportional to the velocity and to the inverted radial directions, respectively.

For the double pendulum, we derive the same expressions for the first particle

$$\begin{aligned}
\mathbf{r}_1 &= l_1(\sin\theta_1, \cos\theta_1) && (14)\\
\dot{\mathbf{r}}_1 = \mathbf{v}_1 &= l_1\dot{\theta}_1(\cos\theta_1, -\sin\theta_1) && (15)\\
\ddot{\mathbf{r}}_1 = \mathbf{a}_1 &= l_1\ddot{\theta}_1(\cos\theta_1, -\sin\theta_1) - l_1\dot{\theta}_1^2(\sin\theta_1, \cos\theta_1) && (16)\\
&= l_1\ddot{\theta}_1\hat{\mathbf{v}}_1 - l_1\dot{\theta}_1^2\hat{\mathbf{r}}_1 && (17)
\end{aligned}$$

and the second particle:

$$\begin{aligned}
\mathbf{r}_2 &= \mathbf{r}_1 + l_2(\sin\theta_2, \cos\theta_2) && (18)\\
\dot{\mathbf{r}}_2 = \mathbf{v}_2 &= \mathbf{v}_1 + l_2\dot{\theta}_2(\cos\theta_2, -\sin\theta_2) && (19)\\
\ddot{\mathbf{r}}_2 = \mathbf{a}_2 &= \mathbf{a}_1 + l_2\ddot{\theta}_2(\cos\theta_2, -\sin\theta_2) - l_2\dot{\theta}_2^2(\sin\theta_2, \cos\theta_2) && (20)
\end{aligned}$$

2.2 Forces

In the single pendulum case, the forces on the particle are gravity and tension. Gravity is along the y-direction, or the direciton of gravitational acceleration $\mathbf{g}$, and the tension is pointing towards the origin, along the direction of $-\mathbf{r}$:

$$\mathbf{F} = T\frac{-\mathbf{r}}{|\mathbf{r}|} + m\mathbf{g} = -\frac{T}{l}\mathbf{r} + m\mathbf{g} \tag{21}$$

In the double pendulum, the forces on m_1 are the tension in the two rods, and gravity. The tension in the upper rod is along the direction $-\mathbf{r}_1$, the tension force on m_1 due to the lower rod is along the direction $\mathbf{r_2} - \mathbf{r_1}$, so we can write the force $\mathbf{F}_1$ as

$$\mathbf{F}_1 = T_1\frac{-\mathbf{r}_1}{|\mathbf{r}_1|} + T_2\frac{\mathbf{r_2} - \mathbf{r_1}}{|\mathbf{r_2} - \mathbf{r_1}|} + m_1\mathbf{g} = -\frac{T_1}{l_1}\mathbf{r}_1 + \frac{T_2}{l_2}(\mathbf{r_2} - \mathbf{r_1}) + m_1\mathbf{g} \tag{22}$$

The forces on m_2 are the tension in the lower rod, and gravity. The tension on m_2 is along the direction of $-(\mathbf{r_2} - \mathbf{r_1})$:

$$\mathbf{F}_2 = T_2\frac{-(\mathbf{r_2} - \mathbf{r_1})}{|\mathbf{r_2} - \mathbf{r_1}|} + m_2\mathbf{g} = -\frac{T_2}{l_2}(\mathbf{r_2} - \mathbf{r_1}) + m_2\mathbf{g} \tag{23}$$

2.3 Equations of motion

2.3.1 Single Pendulum

In the single pendulum case, Newton's law is $\mathbf{F} = m\ddot{\mathbf{r}}$. Writing the two non-trivial components, we have

$$m\ddot{\mathbf{r}} = F = -\frac{T}{l}\mathbf{r} + m\mathbf{g} \tag{24}$$

$$ml\left(\ddot{\theta}(\cos\theta, -\sin\theta) - \dot{\theta}^2(\sin\theta, \cos\theta)\right) = -T(\sin\theta, \cos\theta) + mg(0, 1). \tag{25}$$

We thus have two equations:

$$ml\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) = -T\sin\theta \tag{26}$$

$$-ml\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) = -T\cos\theta + mg \tag{27}$$

Notice that although we only have one generalized coordinate (θ), we have two equations. That is because the equations also have the magnitude of the tension as an unknown, so we have two equations for two unknowns, θ and T.

The equations are not only coupled, but are also non-linear, involving trigonometric functions (ugh!). A common trick with expressions involving trig is to make use of trig identities like $\cos^2\theta + \sin^2\theta = 1$. For example, multiplying 43 by $\cos\theta$, and adding Eq. 45 multiplied by $-\sin\theta$, we obtain a simpler equation for $\ddot{\theta}$. Using these identities, we can write the equations as

$$l\ddot{\theta} = -g\sin\theta \tag{28}$$

$$ml\dot{\theta}^2 = T \tag{29}$$

We cannot solve Eq. 28 analytically, but we could either solve it numerically, or in the small angle approximation. There will be two constants of integration, because it is a second order differential equation: we can relate those constants to the initial position and velocity, or to conserved quantities such as the total energy (but not linear momentum, nor angular momentum: the forces and torques are not zero!). Whichever way, once we have a solution for $\theta(t)$, we can use it in Eq.29 to solve for the other unknown, the tension T. Eq.29 does not invovle derivatives of T, so there are no new constants of integration for the problem.

2.3.2 Double Pendulum

In the double pendulum, Newton's second law on each particle is $\mathbf{F}_i = m_i\ddot{\mathbf{r}}_i$:

$$m_1\ddot{\mathbf{r}}_1 = -\frac{T_1}{l_1}\mathbf{r}_1 + \frac{T_2}{l_2}(\mathbf{r_2} - \mathbf{r_1}) + m_1\mathbf{g} \tag{30}$$

$$m_2\ddot{\mathbf{r}}_2 = -\frac{T_2}{l_2}(\mathbf{r_2} - \mathbf{r_1}) + m_2\mathbf{g} \tag{31}$$

Are these six equations (each equation has three componsents) for two coordinates θ_1, θ_2? Again, not quite: the equations only have two non-zero components in the x,y plane, and we have four unknowns: θ_1, θ_2, T_1 and T_2, so we have four equations for four unknowns, just as expected.

We write the equation of motion for the two particles, split into their two components in the plane:

$$\begin{aligned} m_1 l_1 \left(\ddot{\theta}_1 \cos\theta_1 - \dot{\theta}_1^2 \sin\theta_1\right) &= -T_1 \sin\theta_1 + T_2 \sin\theta_2 && (32)\\ -m_1 l_1 \left(\ddot{\theta}_1 \sin\theta_1 + \dot{\theta}_1^2 \cos\theta_1\right) &= -T_1 \cos\theta_1 + T_2 \cos\theta_2 + m_1 g && (33) \end{aligned}$$

$$\begin{aligned} m_2 \left(l_1\ddot{\theta}_1 \cos\theta_1 - l_1\dot{\theta}_1^2 \sin\theta_1 + l_2\ddot{\theta}_2 \cos\theta_1 - l_2\dot{\theta}_2^2 \sin\theta_2\right) &= -T_2 \sin\theta_2 && (34)\\ -m_2 \left(l_1\ddot{\theta}_1 \sin\theta_1 + l_1\dot{\theta}_1^2 \cos\theta_1 + l_2\ddot{\theta}_2 \sin\theta_2 + l_2\dot{\theta}_2^2 \cos\theta_2\right) &= -T_2 \cos\theta_2 + m_2 g. && (35) \end{aligned}$$

We have then four differential equations, for four unknowns $(\theta_1, \theta_2, T_1, T_2)$.

Trigonometric identities such as $\cos^2\theta + \sin^2\theta = 1$, and $\sin\theta_2 \cos\theta_1 - \cos\theta_2 \sin\theta_1 = \sin(\theta_2 - \theta_1)$ can be used to write the equations of motion as:

$$\begin{aligned} l_1\ddot{\theta}_1 &= (T_2/m_1)\sin(\theta_2 - \theta_1) - g\sin\theta_1 && (36)\\ l_1\dot{\theta}_1^2 &= (T_1/m_1) - (T_2/m_1)\cos(\theta_2 - t_1) - g\cos\theta_1 && (37) \end{aligned}$$

$$\begin{aligned} l_1\ddot{\theta}_1 \cos(\theta_2 - \theta_1) + l_1\dot{\theta}_1^2 \sin(\theta_2 - \theta_1) + l_2\ddot{\theta}_2 &= -g\sin\theta_2 && (38)\\ -l_1\ddot{\theta}_1 \sin(\theta_2 - \theta_1) + l_1\dot{\theta}_1^2 \cos(\theta_2 - \theta_1) + l_2\dot{\theta}_2^2 &= (T_2/m_2) - g\cos\theta_2. && (39) \end{aligned}$$

We now can use Eqns. 36,37 to make substitutions in Eqns. 38, 39, and use more trig identities to simplify these equations further:

$$\begin{aligned} l_2\ddot{\theta}_2 &= -g\sin\theta_2 - ((T_2/m_1)\sin(\theta_2 - \theta_1) - g\sin\theta_1)\cos(\theta_2 - \theta_1)\\ &\quad - ((T_1/m_1) - (T_2/m_1)\cos(\theta_2 - t_1) - g\cos\theta_1)\sin(\theta_2 - \theta_1)\\ &= -(T_1/m_1)\sin(\theta_2 - \theta_1) && (40) \end{aligned}$$

$$\begin{aligned} l_2\dot{\theta}_2^2 &= (T_2/m_2) - g\cos\theta_2 + ((T_2/m_1)\sin(\theta_2 - \theta_1) - g\sin\theta_1)\sin(\theta_2 - \theta_1)\\ &\quad - ((T_1/m_1) - (T_2/m_1)\cos(\theta_2 - t_1) - g\cos\theta_1)\cos(\theta_2 - \theta_1)\\ &= (T_2/m_2) + (T_2/m_1) - (T_1/m_1)\cos(\theta_2 - \theta_1) && (41) \end{aligned}$$

The four equations of motion are then

$$\begin{aligned} l_1\ddot{\theta}_1 &= (T_2/m_1)\sin(\theta_2 - \theta_1) - g\sin\theta_1 && (42)\\ l_1\dot{\theta}_1^2 &= (T_1/m_1) - (T_2/m_1)\cos(\theta_2 - \theta_1) - g\cos\theta_1 && (43)\\ l_2\ddot{\theta}_2 &= -(T_1/m_1)\sin(\theta_2 - \theta_1) && (44)\\ l_2\dot{\theta}_2^2 &= (T_2/m_2) + (T_2/m_1) - (T_1/m_1)\cos(\theta_2 - \theta_1) && (45) \end{aligned}$$

Since the equations do not have derivatives of T_1, T_2, the best way to cast these equations for analytical or numerical solution is to obtain two differential equations for θ_1, θ_2 without T_1, T_2 terms, and use their solutions in expressions for T_1, T_2 in terms of θ_1, θ_2 and their derivatives.

We obtain such expressions for T_1, T_2 from Eqns. 44, 42:

$$\begin{aligned} T_1 &= -m_1 \frac{l_2 \ddot{\theta}_2}{\sin(\theta_2 - \theta_1)} && (46) \\ T_2 &= m_1 \frac{l_1 \ddot{\theta}_1 + g \sin\theta_1}{\sin(\theta_2 - \theta_1)} && (47) \end{aligned}$$

We use these expressions in Eqn. 43:

$$\begin{aligned} l_1 \dot{\theta}_1^2 &= (T_1/m_1) - (T_2/m_1)\cos(\theta_2 - \theta_1) - g\cos\theta_1 \\ &= -\frac{l_2 \ddot{\theta}_2}{\sin(\theta_2 - \theta_1)} - \frac{l_1 \ddot{\theta}_1 + g\sin\theta_1}{\sin(\theta_2 - \theta_1)} \cos(\theta_2 - \theta_1) - g\cos\theta_1 \\ -l_1 \dot{\theta}_1^2 \sin(\theta_2 - \theta_1) &= l_2 \ddot{\theta}_2 + l_1 \ddot{\theta}_1 \cos(\theta_2 - \theta_1) + g \sin\theta_2 && (48) \end{aligned}$$

3 Lagrange's equations

3.1 Simple Pendulum

We have one generalized coordinate, θ, so we want to write the Lagrangian in terms of $\theta, \dot{\theta}$ and then derive the equation of motion for θ.

The kinetic energy is $T = (1/2)mv^2 = (1/2)ml^2\dot{\theta}^2$ (using Eq. 11 for the velocity). The potential energy is the gravitational potential energy, $V = -mgy = -mgl\cos\theta$. Notice we can derive the gravitational force from the potential, $\mathbf{F}_g = -\nabla V = mg(0,1) = m\mathbf{g}$, but not the tension force on the pendulum: that is a constraint force.

The Lagrangian is

$$L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta \qquad (49)$$

The Lagrange equation is

$$\begin{aligned} 0 &= \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} && (50) \\ &= \frac{d}{dt}\left(ml^2\dot{\theta}\right) - (-mgl\sin\theta) && (51) \\ &= ml^2\ddot{\theta} + mgl\sin\theta && (52) \\ l\ddot{\theta} &= -g\sin\theta && (53) \end{aligned}$$

This is, of course, the same equation we derived from Newton's laws, Eq. 28. We do not have, however, an equation to tell us about the tension, similar to Eq. 29: we need to use Lagrange multipliers to obtain constraint forces.

3.2 Double Pendulum

We need to write the kinetic and potential energy in terms of the generalized coordiantes θ_1, θ_2. We already wrote velocity vectors in terms of the angular variables in Eqns. 15,19. Using those expressions, the kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 && (54) \\ &= \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1)\right) && (55) \\ &= \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1) && (56) \end{aligned}$$

The potential energy is the gravitational potential energy;

$$\begin{aligned} V &= -m_1 g y_1 - m_2 g y_2 && (57) \\ &= -m_1 g l_1\cos\theta_1 - m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2) && (58) \\ &= -(m_1+m_2)g l_1\cos\theta_1 - m_2 g l_2\cos\theta_2 && (59) \end{aligned}$$

The Lagrangian is then

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 \\ &\quad + m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1) + (m_1+m_2)g l_1\cos\theta_1 + m_2 g l_2\cos\theta_2 && (60) \end{aligned}$$

We begin calculating the terms needed for the Lagrange equation for θ_1:

$$\begin{aligned} \frac{\partial L}{\partial\dot{\theta}_1} &= (m_1+m_2)l_1^2\dot{\theta}_1 + m_2 l_1 l_2\dot{\theta}_2\cos(\theta_2-\theta_1) && (61) \\ \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}_1} &= (m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2\ddot{\theta}_2\cos(\theta_2-\theta_1) \\ &\quad - m_2 l_1 l_2\dot{\theta}_2^2\sin(\theta_2-\theta_1) + m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1) && (62) \\ \frac{\partial L}{\partial\theta_1} &= m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1) - (m_1+m_2)g l_1\sin\theta_1 && (63) \end{aligned}$$

Lagrange's equation for θ_1 is then

$$\begin{aligned} 0 &= \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}_1} - \frac{\partial L}{\partial\theta_1} \\ &= (m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2\ddot{\theta}_2\cos(\theta_2-\theta_1) \\ &\quad - m_2 l_1 l_2\dot{\theta}_2^2\sin(\theta_2-\theta_1) + (m_1+m_2)g l_1\sin\theta_1 && (64) \end{aligned}$$

Similarly, the Lagrange's equation for θ_2 is

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}_2} &= m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_2 - \theta_1) && (65)\\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} &= m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_2 - \theta_1) \\
&\quad + m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_2 - \theta_1) - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2 - \theta_1) && (66)\\
\frac{\partial L}{\partial \theta_2} &= -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2 - \theta_1) - m_2 g l_2 \sin\theta_2 && (67)
\end{aligned}$$

$$\begin{aligned}
0 &= \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} \\
&= m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_2 - \theta_1) \\
&\quad + m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_2 - \theta_1) + m_2 g l_2 \sin\theta_2 && (68)
\end{aligned}$$

We collect the two Lagrange equations of motion, which are, of course, the same ones we got from Newton's law:

$$\begin{aligned}
(m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_2 - \theta_1) &= m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2 - \theta_1) - (m_1 + m_2) g \sin\theta_1 && (69)\\
l_2 \ddot{\theta}_2 + l_1 \ddot{\theta}_1 \cos(\theta_2 - \theta_1) &= -l_1 \dot{\theta}_1^2 \sin(\theta_2 - \theta_1) - g \sin\theta_2 && (70)
\end{aligned}$$