Wavelets and quantum algebras

A. Ludu
Department of Physics and Astronomy, Louisiana State University,
Baton Rouge, Louisiana 70803-4001

M. Greiner
Institut für Theoretische Physik, Technische Universität, D-01062 Dresden, Germany

J. P. Draayer
Department of Physics and Astronomy, Louisiana State University,
Baton Rouge, Louisiana 70803-4001

(Received 23 October 1995; accepted for publication 12 November 1997)

Wavelets, known to be useful in nonlinear multiscale processes and in multiresolution analysis, are shown to have a $q$-deformed algebraic structure. The translation and dilation operators associate to any scaling equation a nonlinear, two parameter algebra. This structure can be mapped onto the quantum group $sl_q(2)$ in one limit, and approaches a Fourier series generating algebra, in another limit. A duality between any scaling function and its corresponding nonlinear algebra is obtained. Examples for the Haar and $B$-wavelets are worked out in detail. © 1998 American Institute of Physics. [S0022-2488(98)00703-8]

I. INTRODUCTION

Interest in wavelet theory and its applications in the multiresolution analysis\(^1\) has grown over the last decade, involving new and linking disparate fields of research from pure mathematics and physics to down-to-earth signal engineering. It is being widely applied in signal processing and data compression,\(^2\) pattern recognition,\(^3\) statistical physics and turbulence,\(^4\) jet dynamics,\(^5\) field theory,\(^6\) solid state physics,\(^7\) quantum mechanical applications,\(^8\) nonlinear dynamics,\(^9\) soliton theory,\(^10\) etc.

The rise in interest in wavelet theory is motivated by the fact that Fourier analysis is quite ineffective when dealing with nonlinear models and localize sharp features.\(^6\)\(^–\)\(^18\) A central property of wavelet theory is the ability to expand and analyze functions with respect to a self-similar set of localized basis functions (scaling functions and wavelets) and to processes them locally, without affecting the scale. The wavelet method is recursive and therefore ideal for computational applications. Moreover, the scaling functions and the corresponding wavelets are very well localized both in the time and frequency domains. Wavelets are the natural tool in the analysis of phenomena where different space/time scales occur. They provide mathematical representations that can handle both analytical and numerical difficulties due to singular phenomena.\(^6\)\(^,\)\(^7\) Like Fourier analysis, wavelet theory uses expansion functions with different characteristic scales. However, Fourier analysis has the advantage of being based on a simple and solid algebraic foundation.\(^20\) A challenge is to see if an algebraic framework for the foundation of the wavelet analysis can be constructed.

This paper is a first step in this direction. It is mainly concerned in obtaining a closed algebraic method for the construction of the scaling and wavelet functions. We present results for the Haar and $B$-wavelets. The purpose of this paper is to show that wavelets are not only multi-resolution bases of self-similar functions but they have a definite nonlinear symmetry associated with a $q$-deformation of the Fourier series generating algebra. In a following second paper we will show that wavelets also have variational properties; all multiscale equations follow from Hamilton’s equation of an infinite-dimensional Hamiltonian system.

We also show that the central object of wavelet theory, the two-scale equation, has an algebraic counterpart, arising from a quantum algebra.\(^21\) Applications of this result can be realized using supercomputers and efficient numerical schemes which have led to discrete modeling of complex continuous systems or of genuine discrete physical systems defined on lattices. The
discrete symmetries related to complete solvability\cite{22} or to discrete-continuous transitions,\cite{23} and their basic tool, the finite difference equations, are new and central items in any analysis of discrete systems. On uniform lattices the symmetry algebras of partial differential equations is left unchanged by the discretization.\cite{24} In the case of nonuniform lattices one needs to introduce generalized symmetries, like quantum algebras.\cite{25} This is an example when the $q$-deformation of some initial symmetries can play an important role. An outcome of the present paper is an extension of the traditional Fourier analysis towards wavelet expansions that can be accomplished by $q$-deforming the Fourier algebra.

The finite-difference and scaling operators involved in the dilation equation are algebraically closed with respect to certain nonlinear commutation relations. This symmetry is better emphasized when the operators are realized in terms of $q$-deformed derivatives.\cite{26} By expressing the translation and dilation operators as $q$-deformed derivatives, the scaling and difference operators can be mapped onto the generators of nonlinear algebra and the action of the $q$-deformed derivatives is extended to nondifferentiable functions. Moreover, such nonlinear algebras can be undeformed to the Fourier series generating algebra or can be mapped onto the $\mathfrak{sl}_q(2)$ algebra.

After devoting Sec. II to definitions and notations, we present in Sec. III the deformation of a Fourier algebra into a scaling function generating algebra, and identify a spectral problem with the Haar dilation equation. In Sec. IV we introduce the definition of the most general scaling function and wavelet algebra. The uniqueness of the algebraic formulation of the dilation equation is proved. A duality relation is obtained between any scaling function and certain nonlinear algebra, which associates to any given scaling function (wavelet) an algebra and, for a given algebra of a certain type, one finds the appropriate wavelet structure. In Sec. V we give examples of scaling function algebras for the Haar and the $B$-waves. We use some new limiting procedures in order to solve the $q$-difference and finite-difference equations. In Sec. VI we have used a variant and extension of the deforming functional technique,\cite{28,29,31} wherein we can obtain a mapping of the generators of nonlinear algebras to those of $\mathfrak{sl}_q(2)$. Finally, Sec. VII comprises concluding comments, further extensions, and remarks.

II. BASIC ELEMENTS

An algebraic structure can be provided for any scaling function system and wavelet basis. In order to realize this construction we need three building blocks: the algebraic structure of the Fourier system (starting point), the wavelet theory (final aim), and $q$-deformation (intermediate tool).

A. Fourier algebras

In order to demonstrate our approach, we give an example for Fourier series, i.e., algebras having Fourier series as basis for their representation spaces.\cite{20} In the following we shall denote $\delta f/\delta x^k = \delta f = \delta^k$. The trigonometric (Fourier) system $\{|k> = e^{ikx}\}_{k \in \mathbb{Z}}$ diagonalizes all translation invariant operators acting on $L^2([0,2\pi])$. We introduce three generators within a differential realization $J_0 = -i\partial, J_\pm = e^{\pm i\partial}(-i\partial)^p$, satisfying the commutation relations for $p = 0, 1$ only

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 1 - (1 - 2i\partial)^p.$$

(1)

For $p = 0$ Eqs. (1) describe an algebra, denoted $\mathcal{F}_0$, that is isomorphic to an analytical prolongation, $e(2,C)$, of the Euclidean algebra $e(2,\mathbb{R})$ of rigid motions in the plane. The algebra $e(2,\mathbb{R})$, generated by $P_x, P_y$, and $R = -R^\dagger$, with the commutators $[R, P_{x,y}] = \pm P_{y,x}$, $[P_x, P_y] = 0$, is mapped through the generator mapping

$$J_0 = iR, \quad J_\pm = P_x \pm iP_y,$$

onto the $\mathcal{F}_0$ algebra. The unitary irreducible representations (unirreps) of $\mathcal{F}_0$ are based on the eigenvectors of the self-adjoint operator $J_0, J_0|n> = n|n>$. For any two distinct eigenstates, $|n>$ and $|n'>$, by using the first commutator in Eq. (1), we obtain $n' = n + 1$ and hence the spectrum of $J_0$ is unbounded, discrete and consists in equidistant eigenvalues. Hence the space of representations is generated by the Fourier system and the other generators $J_\pm|n> = |n \pm 1>$ act like ladder operators on the $|n>$ states, increasing/decreasing the scale by unity. For $p = 1$ Eqs. (1) describe another Lie algebra, $\mathcal{F}_1$, isomorphic with the symplectic algebra $sp(2,\mathbb{R}) = su(1,1) = so(2,1)$. This differ-
ential realization has the same representation space as \( \mathcal{F}_0 \) but it is not irreducible. The action of the generators of \( \mathcal{F}_1 \) is similar to that of the generators of \( \mathcal{F}_0 \) for \( n \neq 0 \). Unlike the \( \mathcal{F}_0 \) case, however, we have \( J_{\pm} \) and the subspaces \( \{ e^{i n t} \}_{n \in \mathbb{N}} \) and \( \{ e^{-i n t} \}_{n \in \mathbb{N}} \) are invariant subspaces of \( \mathcal{F}_1 \). The algebras \( \mathcal{F}_{0,1} \) suggest the possibility of other constructions in terms of the operator \( \partial \) and the complex exponential functions of different scales.

B. Scaling functions and wavelets

Wavelets are able to reconstruct a signal through regular sampling, i.e., by analyzing the signal at different scales (which increase/decrease exponentially) with the step size between each scale being the same. Different from the ordinary Fourier transform, which reproduces a function as a superposition of complex exponentials, or from the windowed Fourier transform, which introduces a scale into the analysis of signals, multiresolution analysis processes the signal locally, using the appropriate local scale. Basically, wavelets are constructed with a pair of operators: the dilation (scaling) and finite difference (combinations of translations) operators, acting on \( L^2(\mathbb{R}) \) and defined by \( T^a f(x) = f(x + \alpha) \), \( \partial^bf(x) = 2^bf(2^b x) \), with \( \alpha, b \) arbitrary real numbers. In the following we will use the operator \( D^\beta = 2^{-\beta} \partial^\beta \) for simplicity in equations (\( \partial^\beta \) is the dilation operator consistent with wavelet theory). They are invertible, unitary, and fulfill \( T^a D^\beta = D^\beta T^a = T^{a+\beta} \), \( D^\alpha D^\beta = D^{\alpha+\beta} \). The formal Taylor series of these operators are \( T^\alpha = e^{\alpha \partial} \) and \( D^\beta = 2^\beta e^{\beta \ln 2 \partial} \). On the space of compact supported or rapidly decreasing functions, any holomorphic function \( f(T) \) is locally polynomial. Indeed, if \( f \) is holomorphic, and its action is taken on the compact supported function subspace of \( \Phi \in L^2(\mathbb{R}) \), then the action of \( f(T) = \sum_{k \in \mathbb{Z}} C_k T^k \) reduces to that of a Laurent polynomial by keeping only a finite number of terms in the sum.

The wavelet system is given by a set of scaled and translated copies of a pair of functions: the scaling function \( \Phi \) and the mother wavelet \( \Psi \). The basic fact about wavelets is that both these fundamental functions are finite linear combinations of \( \Phi \), reflecting the self-similar character of the wavelet system. The defining equation for the scaling function (the dilation equation) is a linear finite-difference equation, including a scale change (\( q \)-difference\(^7\))

\[
\Phi(x) = D \sum_{k=0}^{n} C_k T^{-k} \Phi(x) = D g(T) \Phi(x),
\]

where the RHS sum is a polynomial in \( T g(T) \). Equation (2) is a fixed-point equation and consequently it has only one unique solution.\(^{1-7,11,13}\) The scaling function \( \Phi(x) \), as a solution of Eq. (2), is required to have two properties:\(^1-5\)

1. \( \int \Phi(x) dx = 1 \), the average value property;
2. \( \langle \Phi_n, \Phi \rangle = \int \Phi(x + n) \Phi(x) dx = \delta_{n,0} \), for any \( n \in \mathbb{Z} \), the orthogonality condition.

These conditions introduce restrictions on the coefficients \( C_k \) in Eq. (2),\(^1-4\)

\[
\sum_{k=0}^{n} C_k = 2, \quad \sum_{k=0}^{n} C_k C_{k+2l} = \delta_{l0}, \quad l \in \mathbb{Z}.
\]

The condition Eq. (3) yield a pattern of \( L^2(\mathbb{R}) \) as a chain of subspaces \( V_j \subset V_{j+1} \), each one being generated by all the translations of \( D^j \Phi \), \( j \in \mathbb{Z} \). By repeated application of \( D \) and \( T \) on \( \Phi \), \( D^j T^a \Phi(x) = (2^j x + a) \Phi(x) \), one obtains a nonorthogonal basis in \( L^2(\mathbb{R}) \). The action of the operator \( - T^{-\lambda} D g(-T^{-1}) \) on \( \Phi \), \( \lambda \) being a unique odd integer, provides the mother wavelet function,

\[
\Psi(x) = - T^{-\lambda} D g(-T^{-1}) \Phi(x) = - T^{-\lambda} \sum_{k=0}^{\infty} (-1)^k C_{-k+1} \Phi(2x-k).
\]

The wavelet \( \Psi(x) \) has the property that \( \{ \Psi_{j,n} \}_{j,n \in \mathbb{Z}} \) is an orthonormal basis of \( L^2(\mathbb{R}) \) Refs. 1–3 and \( L^2(\mathbb{R}) \) is a direct sum of the orthogonal subspaces \( W_j \) (the orthogonal complements of \( V_j \)), each of them generated by all possible translations of \( D^j \Psi = \Psi_{j,0} \) with integral \( m L^2(\mathbb{R}) \)
functions and wavelets. The simplest example is provided by the Haar wavelet, defined by the scaling function $\Phi_{\text{Haar}}(x) = 1$ if $|x - 1/2| \leq 1/2$ and 0 otherwise. In this case we have $D(1 + T^{-1})\Phi_{\text{Haar}} = \Phi_{\text{Haar}}$ and $g_{\text{Haar}}(T) = 1 + T^{-1}$ ($C_0 = C_{-1} = 1$ and the rest 0 in Eq. (2)). The corresponding Haar wavelet is $\Psi_{\text{Haar}} = D(1 - T^{-1})\Phi_{\text{Haar}}$, i.e., $\lambda = 1$ in Eq. (4).

There are similarities and differences between the Fourier and the wavelet approaches. From an algebraic point of view, in both approaches, there are eigenfunction equations, which keep the scale constant, and the ladder operators, which change the scale. The dilation equation, Eq. (2), has its analog in the Fourier formalism, though it is a fixed-point equation and has only one solution. The mother wavelet, Eq. (4), has no analog in Fourier analysis. Each Fourier eigenfunction carries one scale; a wavelet function $\Psi(x)$ involves two scales, e.g., $\Phi_{0,k}$ and $\Phi_{2,k}$. Because of their localization, wavelets possess a degree of freedom beyond that of a Fourier system; namely, the width of the support of $\Phi$. In the present approach this degree of freedom is associated with the deformation parameter, $q$.

C. Quantum deformations

A possibility for constructing scaling functions/wavelets within an algebraic approach is to deform the Fourier algebraic structure into a nonlinear system. In general, the scaling functions and wavelets are not differentiable. This suggests the use of finite-difference operators instead of derivatives. Finite-difference operators are closed with respect to commutation but only within nonlinear algebraic constructions. Consequently, it is natural to use $q$-deformed derivatives to find a foundation for wavelet theory in $q$ deformed algebras.

Quantum algebras refer to some specific deformations of Lie algebras, to which they reduce when the deformation parameter $q$ is set equal to unity (for a recent monograph see Ref. 21, and references therein). The simplest example of a $q$-algebra is $sl_q(2)$ whose Jordan–Schwinger realization is given in terms of $q$-bosonic operators. The $q$-deformed algebras have been applied in various branches of physics like spin-chain models, noncommutative spaces, rotational spectra of deformed nuclei, Hamiltonian quantization, and dynamical symmetry breaking. The basic element is the $q$-deformation of a certain object $x$, which can be a number, an operator, or a function,

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \rightarrow x,$$

where $q = e^{i\epsilon}$. The Taylor expansion of $T$ and $D$ are related to the $q$-deformation of the derivative operator, according to the definition of the coordinate description of the $q$-deformed oscillator introduced in Ref. 26,

$$[\partial]_s f = \frac{T^s - T^{-s}}{2 \sinh(s)} f(x).$$

Analogously, we introduce the operator

$$[x\partial]_{s \ln 2} f(x) = \frac{f(2^s x) - f(2^{-s} x)}{2 \sinh(s \ln 2)} = \frac{D^s - D^{-s}}{2 \sinh(s \ln 2)} f(x).$$

When $s \rightarrow 0$, $[\partial] \rightarrow \partial$ and $[x\partial] \rightarrow x \partial$, Eqs. (6) and (7) can be inverted and the occurrence of the $q$-deformed derivative and the translation/dilation operators becomes immediate

$$T^{\pm s} = \frac{1}{2} \left( \pm \frac{[\partial]}{\eta(s)} + \frac{[\partial]^2_{s \ln 2}}{\eta'(s \ln 2)} \right),$$

$$D^{\pm s} = \frac{1}{2} \left( \pm \frac{[x\partial]}{\eta(s \ln 2)} + \frac{[x\partial]^2_{s \ln 2}}{\eta'(s \ln 2)} \right).$$
where \( s \in \mathbb{N} \) (or more general \( \in \mathbb{R} \)) and \( \eta(s) = 1/(e^s - e^{-s}) \). The \( q \)-deformed algebra \( \mathfrak{sl}_q(2) \), generated by \( \{J_0, J_\pm\} \), is defined by the \( q \)-deformed version of the commutation relation

\[
[J_+, J_-] = [2J_0]_s,
\]

while the other two commutator relations remain undeformed, as for \( \mathfrak{sl}(2) \). In addition to this traditional version of \( \mathfrak{sl}_q(2) \), several generalized forms have been introduced, in two different ways. First, by deforming only the commutator \( [J_+, J_-] \) by using some arbitrary function \( F(J_0, q) \), \( [J_+, J_-] = F(J_0, q) \) independently proposed in Refs. 28, 29, and 30. The second way is by deformations including all three commutation relations, using two functions \( G(J_0, q) \) and \( F(J_0, q) = [J_+, J_-] \), introduced in Ref. 31. Unlike the former, for which the spectrum of \( J_0 \) is linear, the latter is characterized by an exponential spectrum for \( J_0 \). Since in wavelet theory the domains of analysis are divided exponentially, rather than linearly with bands of equal widths, such algebras are used in the following analyses.

### III. HAAR SCALING FUNCTION ALGEBRA, \( \mathscr{S}_{s, \alpha} \)

The aim in this section is to obtain operators depending on \( D \) and \( T \), closed under some nonlinear algebra. This structure must provide an algebraic form for Eqs. (2), (4) and must be mapped into one Fourier algebra, \( \mathscr{F}_{0,1} \). Consider an operator depending on \( T \) and two real parameters \( s, \alpha \),

\[
W_0(s, \alpha) = \frac{T^s e^{-s \alpha} \cos s \pi}{2 \xi(\alpha) \sinh \frac{s}{2}},
\]

where \( \xi(\alpha) = [1/\sinh(1)] \sin(\alpha \pi/2) - 2i \cos(\alpha \pi/2) \). The \( (s, \alpha) \) parameters allow \( W_0 \) to approach \( \partial \) or combinations of \( T \),

\[
W_{0}(0, \alpha) = \frac{\alpha}{\xi(\alpha)} \partial, \quad (11)
\]

\[
W_{0}(1, \alpha) = \frac{T^\alpha + T^{-\alpha}}{2 \sinh(1) \xi(\alpha)}, \quad (12)
\]

\[
W_{0}(2, \alpha) = \frac{T^{2\alpha} - T^{-2\alpha}}{2 \sinh(2) \xi(\alpha)}.
\]

In the limit \( s \to 0 \), \( W_0 \) reduces to the normal derivative with respect to \( x \), Eq. (11). In the limit \( s = \frac{1}{2} \), we obtain a power of the translation operator \( T \), Eq. (12). Equation (13) defines a \( q \)-deformation of unity, i.e., \( W_{0}(1,0) = [i/4 \sinh(1)] \). \( W_0(s, \alpha) \) can be a Hermitian or anti-Hermitian operator, \( W_{0}(1, \alpha) = \pm W_{0}(1, \alpha)^\dagger \), depending on \( \alpha \). The last limit, Eq. (14), represents a finite difference operator and is proportional to the \( q \)-derivative with respect to the \( \alpha \)-deformation \( [2\partial]_\alpha \), similar to Eqs. (6)–(9). For \( \alpha = 2k \) in Eq. (12), \( \alpha = k \) in Eq. (13) and \( \alpha = k/2 \) in Eq. (14), the corresponding \( W_0(s, \alpha) \) are Laurent polynomials in \( T \), providing a direct connection to the wavelet operators. For some values of \( \alpha \), Eqs. (12)–(14) give invertible operators with respect to \( T, T = T(W_0) \).

In order to introduce the dilation operator \( D \), we define

\[
W_\pm(s) = \frac{1}{2} e^{\mp is(s-1)/2} D^{s} e^{\mp is(s-1)/2} (1 + T^{\pm s}).
\]

In order to fulfill the Hermiticity condition \((W_+) = W_-^\dagger \) on \( L^2(\mathbb{R}) \) these generators can be redefined in the form

\[
W_\pm \rightarrow \overline{W}_\pm = 2^{-1 - s/2} e^{\mp is(s-1)/2} D^{s} e^{\mp is(s-1)/2} (1 + e^{-i(s(s-1)/2)} [1 + e^{-i(s(s-1)/2)}]) D^{s} e^{\pm is(1/2)(1 \pm 1)}.
\]
By using the commutators relations between $D$ and $T$ and by performing an integration by parts, one has a direct check of the relation $(\vec{W}_+)^\dagger = \vec{W}_-$.

$$\langle f_1, \vec{W}_+ f_2 \rangle = \int_R f_1^*(x) \vec{W}_+ f_2(x) dx = \langle \vec{W}_- f_1, f_2 \rangle.$$  

From Eqs. (10) and (15) one obtains the commutators,

$$[W_0(s,\alpha), W_+(s)] = G(s,T)W_+(s),$$  

$$[W_0(s,\alpha), W_-(s)] = -W_- (s) G(s,T),$$  

$$[W_+(s), W_-(s)] = F(s,T),$$  

with

$$G(s,T) = W_0(s,\alpha) - \frac{T^{2s}\alpha e^{is\alpha(1+2t)(s-1)/2} - T^{-2s}\alpha e^{-is\alpha(1+2t)(s-1)/2}}{2 \sinh(s) \xi(\alpha)} \cos s \pi$$  

and

$$F(s,T) = \frac{1}{4} (1 + e^{is(1+2t)(s-1)/2}) (1 + T^{-s}) - (1 + e^{is(1+2t)(s-1)/2}) (1 + T^s).$$  

Equations (10), (15)–(20) describe a nonlinear associative algebra, generated by $T^\pm, D^\pm$, denoted $\mathcal{A}_{s,\alpha}$. The eigenvalue problem for $W_\pm$ provides the algebraic form for the scaling equation. When $W_0(s,\alpha)$ is invertible with respect to $T$, the algebra has $W_0, W_\pm$ as generators which depend on two parameters $(s,\alpha)$ and is homeomorphic with a special $q$-deformed algebra, namely, the two-color quasitriangular Hopf algebra $\mathcal{A}_q$. The spectrum of $W_0$ is not equidistant because $F(s,T) \neq \text{const} \times W_0$. The Casimir operator of the algebra $\mathcal{A}_{s,\alpha}$ is given by

$$C = W_+ W_+ + H(W_0) = W_+ W_- + H(W_0) - F(W_0),$$

where $H$ is a real function, holomorphic in a neighborhood of 0 and which must satisfy the functional equation

$$H(\xi) - H(\xi - G(\xi)) = F(\xi)$$

with $\xi$ generic. The irreps of $\mathcal{A}_{s,\alpha}$ are labeled by the eigenvalues $\alpha$ of $W_0$ and those of the Casimir operator. The commutator in Eq. (16) can be written in the form

$$W_+ W_0 = (W_0 - G) W_+.$$  

If $|\alpha| \neq |\alpha'|$ are eigenvectors of $W_0$, with the corresponding eigenvalues $\alpha, \alpha'$, we have from Eq. (23) a nonlinear recursion relation for all the eigenvalues

$$\alpha' = \alpha - G(\alpha).$$

The algebra $\mathcal{A}_{s,\alpha}$ can be mapped into $e(2,C)$, the Lie algebra of a Fourier series, in the limit $s \to 0$, $\alpha \to 2$: $W_\pm = e^{is\alpha} J_\pm$, $W_0 = -i \partial = J_0$, where we take these limits in the order $\mathcal{A}_{s,\alpha} \to \mathcal{A}_{0,2} = e(2,C)$ is an algebraic morphism and one can consider wavelets as $q$-deformed generalizations of the Fourier series, in the above sense.

As a first example, the Haar scaling function can be associated with a particular case of $\mathcal{A}_{s,\alpha}$, i.e., the algebra $\mathcal{A}_{1,1}$. In the algebra $\mathcal{A}_{1,1}$, the eigenproblem for $W_-$ provides the dilation equation for the Haar scaling function $\Phi(x)$ defined by Eq. (2) with $g(T) = 1 + T^{-1}$. We have in this case $G = W_0 - 2W_0^2 + 1$, and $W_\pm = \frac{1}{2} D^{\pm 1}(1 + T^{\pm 1})$ and the commutators,
\[ [W_0, W_+] = (W_0 - 2W_0^2 + 1)W_+, \]
\[ [W_0, W_-] = -W_-(W_0 - 2W_0^2 + 1), \]
\[ [W_+, W_-] = \frac{1}{4} \left( T^2(W_0) - T^{-1/2}(W_0) - T^{-1/2}(W_0) + T^{-1}(W_0) \right), \]  
where \( T(W_0) \) is the solution of the equation \( 2W_0 = T + T^{-1} \), a pseudodifferential operator. The Casimir operator of \( \mathcal{A}_{1,1} \) is a constant. Indeed, Eq. (22) for the general Casimir operator becomes
\[ H(W_0) - H(2W_0^2 - 1) = F(W_0), \]  
and has, in terms of the \( T = T(W_0) \) operator, the form
\[ \chi(T) - \chi(T^2) = \frac{1}{4} \left( T^2 - T^{1/2} - T^{-1/2} + T^{-1} \right), \]  
where \( \chi(T) = H[(T + T^{-1})/2] \). The unique analytical solution for Eq. (27) reads
\[ \chi(T) = \frac{1}{4} \left( \text{const} - T - T^{1/2} - T^{-1/2} \right), \]  
which gives for the Casimir operator a constant.

The spectrum of \( W_0 \) consists of periodic functions satisfying the equation \( \Phi(x + 1) + \Phi(x - 1) = 2\Phi(x) \). This takes one back to Fourier analysis. The way to the Haar scaling function is to
\[ \text{const} \]  
for any \( \tilde{a} \). The corresponding eigenfunctions have the form \( |a_n\rangle \sim e^{\pm 2^n \tilde{a} x} \) which results in an exponential spectrum for \( a_n \) similar to the sequence of scales in wavelet theory. This is a self-similar spectrum with respect to \( \tilde{a} \), like the sequence of scales in wavelet theory. A part of this spectrum is shown in Fig. 1. The action of \( W_\pm \) is
\[ W_\pm |a_n\rangle \sim \frac{1 + e^{\pm 2^n \tilde{a}}}{2} |a_{n \pm 1}\rangle \].  

Another basis for a representation of \( \mathcal{A}_{1,1} \) is given by \( |k\rangle = e^{i\pi x/k}, k \in \mathbb{Z} \). We have the action
\[ W_0 |k\rangle = \cos \frac{\pi}{k} |k\rangle, \quad W_\pm |k\rangle = \frac{1}{2} \left( 1 + e^{\pm i\pi/k} \right) |2\pm 1k\rangle, \]  
and on this basis \( (W_+)^3 = W_- \). There are two invariant spaces, \( |2^k\rangle \) and \( |2^{-k}\rangle, k \in \mathbb{N} \), with \( W_\pm |0\rangle = 1 \).

The same procedure can be followed for any set of the parameters \( s, \alpha \). For example, if we choose the algebra \( \mathcal{A}_{2,1/2} \) with \( W_0 \) defined in Eq. (12), we obtain the spectrum of \( W_0 \) described by the recursion relation
\[ a' = a \left( a + \sqrt{a^2 + 1} + \frac{1}{a + \sqrt{a^2 + 1}} \right), \]  
which also gives a nonlinear, unbounded, discrete representation of \( \mathcal{A}_{2,1/2} \).

The last thing to prove is the uniqueness of the two-scale equation in the \( \mathcal{A}_{1,1} \) scaling function generating algebra. The two-scale equation, Eq. (2), in its algebraic form \( 2W_\pm \Phi = \Phi \) is not unique.
in \( \mathcal{A}_{1,1} \) if there exists an operator, similar with that occurring in the dilation equation, which commutes with \( W_2 \). This operator should contain higher powers in \( T \) and consequently is an element of \( U(\mathcal{A}_{1,1}) \).

**Proposition 1:** In \( U(\mathcal{A}_{1,1}) \) there exists a unique operator \( X = \delta x(T^{-1}) \) such that \( [W_2, X] = 0 \). The function \( x(\xi) \) is an integer, it is not a polynomial and it is unbounded, for generic \( \xi \).

**Proof:** We take for \( X \) a Laurent series \( x(T) = \sum_{k \in \mathbb{Z}} C_k T^{-k} \), then the condition \( [W_2, X] = 0 \) results in a recursion relation for the coefficients \( C_k \),

\[
C_{2k+1} = C_k - C_{2k-1}, \quad C_{2k} = C_k - C_{2k-2},
\]

for any \( k \in \mathbb{Z} \). Equations (31) have one trivial solution which reproduces \( W_2 x(T^{-1}) = C_{-1} (1 + T^{-1}) \) and only one different solution, with \( C_{-1} = \pm C_{-1} \) or 0, uniquely defined (\( C_0 = C_{-3} = C_{-4} = C_5 = C_{-6} = C_7 = C_{-8} = C_{-11} = \cdots = 0 \), \( C_{-2} = C_{-5} = C_6 = C_{-9} = C_{-10} = \cdots = C_{-1} \) and \( C_1 = C_4 = C_{-7} = C_8 = C_9 = C_{-12} = \cdots = C_{-1} \), etc.). The sequence of nonzero coefficients is infinite, and \( \{C_k\} \) is not a Cauchy sequence. (q.e.d.) We note that the \( \mathcal{A}_{1,1} \) algebra is also a Hopf algebra,\(^{21,30,31} \) defined by Eqs. (25) and by the coproduct, counit and antipode in the form,

\[
\Delta T^\pm = T^\pm \otimes T^\pm, \quad \Delta D^\pm = D^\pm \otimes D^\pm, \quad \epsilon(T^\pm) = \epsilon(D^\pm) = 1, \\
S(T^\pm) = T^\mp, \quad S(D^\pm) = D^\mp.
\]

All its generators are primitive elements.

**IV. GENERAL SCALING AND WAVELET ALGEBRA**

This algebraic approach for the Haar scaling function can be generalized to yield a nonlinear algebra for any scaling function, and conversely, to find the dilation equation and scaling function for a certain type of algebra. The procedure starts with an algebra generated by dilation and translation and constructs, within this algebra, the two-scale equation. The generators of the \( \mathcal{A}_{s,a} \) algebra can be generalized as

\[
W_0 \rightarrow j_0(T), \quad W_\pm \rightarrow j_\pm(D, T) = e^{\mp ix(x - 1)/2}D^\mp e^{\mp ix(x - 1)/2}j(T^{\pm 1}),
\]
with \( j_0(T) \) and \( j(T) \) being arbitrary functions of \( T \) and \( s \), holomorphic in a neighborhood of \( T = 1 \), with their dependence on \( s \) being such that in the limit \( s \to 0, j_0(T) \to -i \partial, j(T) \to 1 \). In this case the commutation relations, Eqs. (16)–(20), become

\[
[j_0, j_+] = \tilde{G} j_+, \quad [j_0, j_-] = -j_- \tilde{G}, \quad [j_+, j_-] = \tilde{F},
\]

(33)

with \( \tilde{G}, \tilde{F} \) depending on \( T \) through \( j_0, j_\pm \), respectively,

\[
\tilde{G} = \tilde{G}(s, T) = j_0(s, T) - j_0(s, T^{2^s} e^{i(s+2^s)(s-1)/2}).
\]

(34)

\[
\tilde{F} = \tilde{F}(s, T) = j(s, e^{i(s-1/2)(s + 2^s)} T^{2^s}) j(s, T^{-1}) - j(s, T^{-2^s} e^{i(s-1/2)(s + 2^s)}) j(s, T).
\]

(35)

Equations (33)–(35) define a nonlinear algebra denoted \( \mathcal{A}_{j_0,j} \) as a generalization of \( \mathcal{A}_{s,a} \). If the function \( j(T) \) is invertible with respect to \( T \), \( \mathcal{A}_{j_0,j} \) can be expressed in terms of the generators \( j_\pm \) only and then \( \tilde{G}(T) = \mathcal{S}(j_0), \tilde{F}(T) = \mathcal{S}(j_0) \). The algebraic morphism \( j_0(T) \to W_0 \) and \( j(T) \to 1 + T^s \) provides the mapping \( \mathcal{A}_{j_0,j} \to \mathcal{A}_{s,a} \). Since the first two commutators of \( \mathcal{A}_{j_0,j} \), Eq. (33), do not depend on the function \( j(T) \), and the third commutator in Eq. (33) does not depend on the function \( j_0(T) \), the closure condition for \( \mathcal{A}_{j_0,j} \) is independent of the functions \( j(T) \) and \( j_0(T) \) and hence they can be chosen in a convenient way to provide any dilation equation in the form of the eigenproblem for \( j_- \). The algebraic closure conditions for \( \mathcal{A}_{j_0,j} \), together with the definitions of \( j, j_0 \) require

\[
j_0(T^2) - j_0(T) = \mathcal{S}(j_0(T^2)),
\]

(36)

\[
j(T^2) j(T^{-1}) - j(T^{-1}) j(T) = \mathcal{S}(j_0(T)).
\]

(37)

Both Eqs. (36) and (37) are nonlinear, and in general, difficult to solve analytically. The unique solution for the dilation equation are \( j_\pm \) since the Casimir operator of this algebra depends on \( T \) only. Choosing \( g(T) = j(T) \) yields the dilation equation in the form \( 2 j_- \Phi = \Phi \). In the limit of \( \mathcal{A}_{s,a} \) this provides again the Haar two-scale equation. The corresponding scaling function belongs to the basis of the representation of \( j_- \) with eigenvalue \( 1/2 \). The remaining arbitrary function \( j_0 \) can now be selected to obtain the wavelet generating operator \( j_- j_0 \Phi = \Psi \), in agreement with Eq. (4). One can formally construct a basis of the representation for \( j_- \) with the functions \( |n\rangle = (in)^n \Phi \). The procedure is the following: for a given algebra \( \mathcal{A}_{j_0,j} \) (therefore given functions \( \mathcal{S}, \mathcal{F} \)) solve Eqs. (36) and (37) with respect to the functions \( j_0, j_\pm \) and obtain the corresponding dilation equation in the form \( 2 j_- \Phi = \Phi \). A certain combination of generators produces the corresponding wavelet equation for \( \Psi \). Conversely, given a scaling/wavelet function, and consequently its dilation equation and \( g(T) \), one can choose \( j_- = D g(T) \) and then solves the equation \( D g(-T^{-1}) = j_- j_0 \) with respect to \( j_0 \). With \( j_0, j_\pm \) known one can solve Eqs. (36) and (37) with respect to \( \mathcal{S}, \mathcal{F} \) to construct the algebra.

This algebraic approach is in some sense universal, what is modified is the specific realization of the nonlinear algebra in terms of \( D \) and \( T \). Constraints are imposed by the two limiting approaches (two-scale equation and Fourier limit). Loosely speaking, the closure of the algebra provides the fixed-point dilation equation and the nonlinearity of the algebra provides the exponential scaling.

We also note that \( V_0 = \{ \sum_k C_k T^k \Phi \} \), i.e., the space generated by all the integer translations of \( \Phi \), and any other \( V_j = D^j V_0 \), is not invariant to the action of \( j_- \). For example, the condition \( j_- D \Phi = \sum_k C_k T^k D \Phi \) implies the existence of an operator containing \( D \), whose commutator with \( j_- \) is a function of \( T \) only. From the definition of the algebra this is impossible, and consequently this proves the above conjecture. The \( V_j \) spaces form, by recursion, a basis in \( L^2(R) \).

In order to prove the uniqueness of the two-scale equation in the general case, we again use Proposition 1, for \( W_j = D j(T^{-1}) = D \sum_{k \in Z} j_k T^k \). We have to solve the commutation equation \( [D j(T^{-1}), X] = 0 \) for \( X(T) = \sum_{k \in Z} X_k T^k, X = D x(T) \). This implies that the arbitrary functions \( x(\xi) \) and \( j(\xi) \) must fulfill the functional condition \( \xi \) a generic variable)

\[
x(\xi^2) j(\xi) = x(\xi) j(\xi^2).
\]

(38)
Equation (38) does not carry any restriction with respect to the values of the functions in 0 and 1. Since \( D_f(T) \) represents a scaling operator we have, according to the dilation equation, \( j(1) = 1 \), \( j(-1) = 0 \) which implies \( x(-1) = 0 \). If \( \Phi \) is the corresponding scaling function for \( j_+ \) then \( X\Phi \) is also an eigenfunction of \( j_+ \), \( j_-(X\Phi) = X\Phi \). In order for \( DX(T) \) to satisfy the average value property (1) in Sec. II B, we impose the additional condition \( x(1) = 1 \). The last condition requires \( DX(T) \), the orthogonality condition (2) of Sec. II B, be equivalent to the condition \( (39) \)

\[
|x(T)|^2 + |x(-T)|^2 = 1.
\]

By introducing Eq. (38) into Eq. (39) we obtain \( X = j_- \) which provides the trivial identity solution, and hence the uniqueness.

V. CONSTRUCTION OF THE SCALING AND WAVELET ALGEBRA

The link between the scaling function and wavelet, and the corresponding scaling algebra is supported by the solutions of Eqs. (36) and (37). In the following we present a method for solving these equations, from the wavelets towards the algebra. The dilation equation Eq. (2) has the form of a fixed-point equation. Therefore we must try to find its eigenvectors as limits of some functional sequences. The limits of these sequences should be compact supported or rapidly decreasing functions for two reasons; to obtain scaling functions with good localization, and to provide correct behavior of the action of the operators \( f(T) \). Since the scaling function will be expressed as a limit of a sequence, one has to look for operators which commute with the limit (are absorbed in the limit). We choose a test function \( \Delta_0(x) \) and a sequence of operators \( f_n \) from the universal covering \( U(\mathcal{A}_{J_0,j}) \) of the scaling function algebra introduced in the preceding section, such that the limit \( \Phi(x) = \lim_{n \to \infty} f_n \Delta_0(x) \) exists in the weak topology and it provides the scaling function.

In the following we show how to construct the scaling algebra \( \mathcal{A}_{J_0,j} \) from the dilation equation, written in the form \( j_-(T,D_x)\Phi = \Phi \). We have the following proposition in the framework of the algebra \( \mathcal{A}_{J_0,j} \), for \( s = 1 \).

**Proposition 2**: Let \( f_0(j_0) \) be a functional sequence in \( U(\mathcal{A}_{J_0,j}) \), \( s = 1 \), \( \Delta_0(x) \) a (test) function and \( \Phi(x) \) the scaling function. If the limit \( \Phi(x) = \lim_{n \to \infty} f_n(j_0) \Delta_0(x) \) exists, and \( \lim_{n \to \infty} f_n(j_0) \Delta_0(x) = \Delta_0(x) \), then \( j_+ \Phi(x) = \Phi(x) \).

**Proof**: From the RHS of the first condition in the hypothesis and from Eq. (33) we have

\[
f_n(j_0)j_0 = j_-f_n(j_0-\mathcal{F}(s,j_0))j_0^{-1} = j_-\frac{f_n(j_0-\mathcal{F}(s,j_0))}{f_{n-1}(j_3)}j_0^{-1}.
\]

It follows

\[
\phi_\lim_{n \to \infty} j_-f_n(j_0)\Delta_0(x) = \mathcal{F}(s,j_0) = \Phi(x),
\]

q.e.d. It follows from Proposition 2 that for a given dilation equation \( j_+ \Phi(x) = \Phi(x) \), (both \( j_- \) and \( \Phi \) given) we can find out a sequence \( f_n(j_0) \), the operator \( \mathcal{F}(s,j_0) \) and a test function \( \Delta_0(x) \), such that these objects satisfy the hypothesis of Proposition 2. Then it follows that one can construct the scaling algebra, since with \( j_0 \) and \( \mathcal{F}(s,j_0) \) found, \( \mathcal{F}(s,j_0) \) results from the last commutator in Eq. (33). In general, one starts with \( j_0 \) as an arbitrary function of \( T \) which can be deformed into \( -i\delta \) when mapping \( \mathcal{A}_{J_0,j} \) to \( \mathcal{A}_{J_0,j} \). As \( j_- \) is provided by the two-scale equation, \( j_+ \Phi = \Phi \), other solutions are forbidden. If \( \lim_{n \to \infty} f_n(j_0) = f_\Phi(j_0) \neq \text{const.} \), this limit should commute with \( j_- \) which is forbidden by Proposition 1. This procedure closes the construction of the algebra \( \mathcal{A}_{J_0,j} \).

In the following we give an algorithm for finding \( j_n(j_0) \) and \( j_0(T) \) and illustrate it with two examples. Since any mother scaling function is defined by the dilation equation \( (Dh(T)\Phi = \Phi \) or \( j_+ \Phi = \Phi \) it is natural to search for solutions by using a recursion algorithm. For instance, we can find a test function \( \Delta_0(x) \) and a triple of operators \( A,B,C \) such that the limit \( \Phi(x) = B \lim_{n \to \infty} A^n \Delta_0(x) \) exists and in addition \( CB = BA^k \), for a finite positive integer \( k \). Then, we have the property \( C\Phi(x) = \Phi(x) \). Indeed, \( C\Phi(x) = CB \lim_{n \to \infty} A^n \Delta_0 = B \lim_{n \to \infty} A^{n+k} \Delta_0 = B \lim_{n \to \infty} A^n \Delta_0 = \Phi(x) \). From the dilation equation we know that \( C \) should have the form \( C \).
$= Dc(T)$ with $c(T)$ a function of $T$. One of the simplest choices is to use $A = D$ and $B = b(T)$. Then we have $CB = Dc(T)b(T) = c(T^{1/2})b(T^{1/2})D = b(T)A$; that is, $k = 1$ and a restriction for the arbitrary functions $c(T)$ and $b(T)$ arises,

$$c(T^{1/2})b(T^{1/2}) = b(T).$$

(40)

This equation is useful in both directions (algebraic-dilation equation), since one can start with a given scaling function (given $c(T)$) and find the operator $b(T)$ involved in the algebra and conversely.

Now consider the algorithm for the Haar scaling function, $c(T) = 1 + T^{-1}$. We look for solutions of Eq. (40) as Laurent series for $b(T) = \sum_{k \geq 2} b_k T^k$. In this case Eq. (40) has a unique solution, $b(T) = \text{const} \cdot (1 - T^{-1})$. This gives again a unique solution for $F = (1 - T^{-1}) \lim_{n \to \infty} D^n \Delta_0$. We stress that $Db(T)$ is exactly the operator which gives the Haar wavelet.

Further, we can express $\Phi_{\text{Haar}}$ in the form $\Phi(x) = H(x) - H(x - 1)$, where $H(x)$ is the Heaviside distribution. For any sequence of $C^2$ functions $\delta_n(x) \to \delta(x)$, we have $\Delta_n(x) = \int \delta_n(x) dx \to H(x)$ and we can write

$$\Phi(x) = \lim_{n \to \infty} (\Delta_n(x) - \Delta_n(x - 1)).$$

(41)

For example, we can use the sequences $\delta_n = (1/\pi)[2^{-n}I(x^2 + 2^{-2n})]$ or $\delta_n = \left[2^{-n-1}/\cosh^2(2^n x)\right]$ and $\Delta_n = \left[\arctan(2^n x)/\pi\right]$ or $\Delta_n = \frac{1}{2} \tanh(2^n x)$, respectively. The latter example is a soliton-like shape, having good localization. These sequences converge to $\delta(x)$, respectively. In this way we have selected subsequences that step in powers of 2. Equation (41) can be written as

$$\Phi(x) = \lim_{n \to \infty} (1 - T^{-1}) D^n \Delta_0(x).$$

(42)

We can express this definition in terms of the algebra $\mathcal{A}_{1,1}$. By using Eq. (25) and the properties of the operators $D$ and $T$, we can write Eq. (42) in the form

$$\Phi(x) = \lim_{n \to \infty} (1 - T^{-2^{-n}}) (2 W_-)^n \Delta_0(x).$$

(43)

Indeed, from the commutation relation between $D$ and $T$ and the definition of $W_-$ we have

$$(2 W_-)^n = (D(1 + T^{-1}))^n = D^n (1 + T^{-1} + \cdots + T^{-2^n+1})
= (1 + T^{-2^{-n}} + (T^{-2^{-n}})^2 + \cdots + (T^{-2^{-n}})^{2^n-1}) D^n.$$

(44)

Hence, we can write $(1 - T^{-1}) D^n = (1 - T^{-2^{-n}}) (2 W_-)^n$. Moreover, we can write, by using the inverted form for $W_-(T)$, the dilation equation in a pure algebraic form,

$$\Phi(x) = \lim_{n \to \infty} (1 - T^{-2^{-n}} (W_0)) (2 W_-)^n \Delta_0(x).$$

(45)

From this last equation it follows that $2 W_- \Phi = \Phi$, i.e., the Haar dilation equation. If the function $\Delta_0(x)$ is chosen from a class of suitable functions for a wavelet analysis then its scaling function, $\Phi(x)$, Eq. (45), gives a rapidly convergent wavelet expansion. In order to exemplify we show in Fig. 2 scaling functions obtained by this procedure for different values of the parameter $s$ within $\mathcal{A}_{s,1}$. For $s = 0$ and 1 this functions yields the Haar scaling function.

We give another example for the $\Phi_2$ B-scaling function and the $\Psi_2$ B-wavelet. We define

$$\Phi_2(x) = \begin{cases} 0 & x \leq 0, \ x \geq 1 \\ 4x & 0 \leq x \leq 1/2 \\ -4x + 4 & 1/2 \leq x \leq 1 \end{cases},$$

(46)

or
\[ \Phi_2(x) = \lim_{n \to \infty} \frac{1}{2^{n+1}} (1 - 2T^{-1/2} + T^{-1}) D^n \Delta_{0,\text{tri}}(x), \]  

with

\[ \Delta_{0,\text{tri}}(x) = \frac{2x}{\pi} \arctan(2x) - \frac{\ln(1 + 4x^2)}{2\pi} = \int_0^{2\pi} \Delta_0(x) dx. \]

Following the above algorithm we obtain the corresponding dilation equation,

\[ j_{-1}.\Phi_2(x) = \frac{D}{2} (1 + T^{-1/2})^2 \Phi_2(x). \]

By using the commutation relation \( \partial D = 2D \partial \) and the dilation equation Eq. (49) we can obtain a self-contained equation for the wavelet,

\[ \Psi_{\text{tri}}(x) = D(1 + 2T^{-1/2} + T^2)^2 \Psi_{\text{tri}}. \]

The corresponding algebras for Haar and \( \Phi_2 \) B-scaling functions are different. For the B-wavelet we have, in \( \mathcal{H}_{1,1} \),

\[ G(W_0) = W_0 - W_0^2 - \frac{1}{2}. \]

This new algebra, fulfills different commutation relations and consequently has a different spectrum for \( W_0 \), not the Haar scaling algebra. The algorithm introduced above can also be used to generate other scaling functions and their corresponding algebras. An interesting line is to obtain

FIG. 2. The continuous deformation of the scaling function \( \Phi \) in \( \mathcal{H}_{s,a} \) as function of the parameter \( s \). For \( s = 0,1 \) this is the Haar scaling function.
polygon-like wavelets or, by following the same procedure which guided us to obtain the B-scaling function from the Haar scaling function, to obtain smoother scaling functions.

VI. THE sl_q(2) LIMIT OF $\mathcal{A}_{\partial_0,j}$

The quantum group $sl_q(2)$ or its extensions 21–26,28–31 are associative algebras over $C$ generated by three operators, $J_0 = (J_0)^\dagger$, $J_+$, and $J_-$, satisfying the commutation relations,

$$[J_0, J_+] = J_+,$$  
$$[J_0, J_-] = -J_-,$$  
$$[J_+, J_-] = [J_0]_s,$$  

(52)

where $[J_0]$ is a deformation of $J_0$, that is, a real, parameter-dependent function of $J_0$, holomorphic in the neighborhood of zero, approaching $2J_0$ in the limit $s \rightarrow 0$. We have to find a deforming functional that transforms the $\mathcal{A}_{\partial_0,j}$ generators into operators satisfying the commutation relations Eq. (52). For this purpose the first equation in Eqs. (33) can be written as

$$(j_0 - \varphi(s,J_0))j_+ = j_0,$$  

(53)

and hence, for every entire function $p(\xi)$, we can write

$$p(j_0 - \varphi(s,j_0))j_+ = j_0p(j_0).$$  

(54)

Let us consider the functional equation:

$$\Gamma(\xi - \varphi(s,\xi)) = \Gamma(\xi) - 1$$  

(55)

for a given function $\Gamma(\xi)$, with $\xi$ generic. If this equation has a solution $\Gamma(\xi)$ that is an entire function, then Eq. (55) can be written, for $p(\xi) = \Gamma(\xi)$, in the form $[\Gamma(j_0), j_+] = j_+$, and correspondingly $[\Gamma(j_0), j_-] = -j_-$. These equations give the correspondence between the wavelet algebras and the $q$-deformed algebras like $sl_q(2)$ or other extensions of them. They map the structure of $\mathcal{A}_{\partial_0,j}$, Eq. (33), onto the structure described by the first two equations in Eq. (52). For the third commutator of each of these algebras we use the function $\Gamma(\xi)$, which allows Eqs. (33) to be reduced to the third equation in Eqs. (52) through the mapping.

$$J_0 = \Gamma(j_0), J_+ = j_+.$$  

(56)

If $\Gamma$ is invertible, the third equation in Eqs. (33) in $\mathcal{A}_{\partial_0,j}$ can be reduced to the third equation of Eqs. (52) of an algebra defined by the function $[J_0]$ with the identification $\varphi(\Gamma)^{-1} = \Gamma$ where $[\Gamma^{-1}] \Gamma$ means the composition of the two functions, i.e., $[\Gamma^{-1}] \Gamma)(\xi) = [\Gamma^{-1} \Gamma(\xi)] = [\Gamma(\xi)]$. In the case of $sl_q(2)$, i.e., $[J_0] = [2J_0]_q$, the function $\varphi(s,j_0)$ becomes

$$\varphi(s,j_0) = \frac{\phi(j_0)^2 - \phi^* (j_0)^2}{q - q^{-1}},$$  

(57)

where $q = e^s$, $\phi(\xi) = q^{-\Gamma(\xi)}$ has to satisfy the equation $\phi(\xi - \varphi(s,\xi)) = q \phi(\xi)$. Consequently, Eqs. (53)–(55) provide the reduction of $sl_q(2)$ into $\mathcal{A}_{\partial_0,j}$, through $\Gamma$. For details of the above technique of mapping and reduction of nonlinear algebras one can see examples in Ref. 31. In the case of the algebra $\mathcal{A}_{1,1}$ Eqs. (53)–(55) give the condition

$$p(W_0 - 1) = W_0 - G(1,W_0),$$  

(58)

which has the solution

$$J_0 = p(W_0)|_{\mathcal{A}_{1,1}} = \cosh(2^{-W_0} b),$$  

(59)

for any arbitrary constant $b$. By inverting Eq. (59) we obtain

$$W_0 = \mp \ln(2) \ln \left( \frac{\partial}{b} \right).$$  

(60)
Conversely, starting from \( sl_q(2) \) and its deformation \([\cdot]_q \) and mapping it into \( \mathcal{A}_{1,1} \) we obtain for the operator \( F(s,T) = F(s,T(W_0)) \),

\[
F = f(\Gamma(W_0)) = f(p^{-1}(W_0)) = [2p^{-1}(W_0)].
\]  

A diagram containing these algebra morphisms is presented in Fig. 3.

**VII. FURTHER EXTENSIONS, COMMENTS, AND CONCLUSIONS**

In this paper we present a method for connecting classes of nonlinear \((q\text{-deformed})\) algebras, with scaling functions and wavelets. This connection is reciprocal. The algebra \( \mathcal{A}_{1,1} \) allows the choice of different dilation equations, i.e., for different functions \( j(1,T) \) one can obtain different scaling functions. The function \( j_0(1,T) \) should be chosen in order to give the correct wavelet function through the action of \( j_0 \). The connection with quantum groups is more transparent if we choose for \( j_0 = [\partial]_{-1/2} = W_0(2,1/2) \), Eq. (14), and for \( \Phi \), the Haar scaling function, \( j_0 = D(1 + T^{-1}) \). The action of the \( q\)-derivative \([\partial]_{1/2} \) on \( \Phi \) gives exactly the Haar wavelet. In this example the operator \( j_0 \) is anti-Hermitian and we can identify its spectrum (imaginary eigenvalues) from the commutation relations. We also have the relation \( \Psi = (j_0 - \tilde{G}(1,j_0)\Phi, \) so that one can express the eigenvectors \( |a\rangle \) of \( J_0 \) in the wavelet basis of the algebra. If \( |a\rangle = \sum_{j,n} C_{j,n} \Phi_{j,n} \), we obtain a recursion relation for the coefficients of \( |a\rangle \) and for the eigenfunctions of \( j_0 = \sum_k C_{j,p} C_{j-k} = aC_{j,p} \) for any \( j \) and \( p \) integer, where \( C_{j,k} \) are the Taylor coefficients of the function \( -\xi^{-\lambda}g(-\xi) \). We note that by using the \( q\)-derivative instead of a function of \( T \) we obtain exactly the same mother wavelet (Haar wavelet).

We want to note a natural extension of the above developments, coming from a different realization of \( sp(2,\mathbb{R}) \) in terms of the full algebra of symmetry of the real line: \( X_0 = x\partial \) (dilations), \( X_- = \partial \) (translations), and \( X_+ = x^2\partial \) (expansions). This is the maximal finitely generated real Lie algebra with generators in the form \( x^a\partial \). By exponentiation, the generators \( X_0, X_- \) provide \( D \) and \( T \) and the last generator has the action \( E^a = e^{itx^a} f(x) = f(x/1-ax) \). For \( a > 0 \) the action corresponds to a contraction of the function and for \(-1 < a < 0 \) it is a dilation. But, for \( a < -1 \) and \( |a| \geq 1 \) we obtain a splitting of the function in two Heaviside distributions, \( E^a f(x) = H(x) + H(-x + 1/|a|) \). A possible closed nonlinear algebra containing \( D, T \), and \( E \) could be an opportunity for wavelets with multiscale properties.\(^{18,19}\)

In conclusion, we found that the operators of dilation and translation \( (D,T) \) can be combined in such a way as to generate nonlinear algebras, depending on certain parameters \( (s,\alpha) \). We have investigated these algebras from the point of view of quantum groups, discussing their unitary, Casimir operators, and reductions of these algebras to other \( q\)-deformed algebras, like \( sl_q(2) \) or to the Fourier series generating algebra. It has been shown that such algebras provide an appropriate framework for the foundation of the wavelets analysis and for the obtaining the corresponding scaling functions. We have worked out two examples, the Haar and the B-scaling functions. The

\[
\begin{align*}
\mathcal{A}_{0,2} & \rightarrow ~ \mathcal{A}_{s,\alpha} \rightarrow ~ \mathcal{A}_{1,1}(\Phi_H) \\
\downarrow \rho & \rightarrow \rho \\
\text{sp}(2,\mathbb{R}) & \simeq \mathcal{F}_0 \\
\text{e}(2) & \simeq \mathcal{F}_1
\end{align*}
\]  

**FIG. 3.** A diagrammatic representation of the algebraic maps (arrows) and morphisms (\( \simeq \)) of \( \mathcal{A}_{1,1} \), \( \mathcal{A}_{s,\alpha} \), \( \text{sp}(2,\mathbb{R}) \), \( \text{e}(2) \), and \( \mathcal{F}_0 \).
algorithm provides a general algebraic method for finding specific scaling function. Direct applications of such an approach can be found in the theory of finite-difference equations and $q$-difference equations.\textsuperscript{21–25,27}

This study represents only a first step towards understanding the relation between the theory of nonlinear algebras (having exponential spectra) and wavelets with their finite-difference equations. We conjecture that such an algebraic approach (nonlinear algebras) to scale invariant structures can lead to interesting mathematical constructions like tools for identifying isolated coherent structures. The transition between such self-organized structures can be carried out through the modification of the deformation parameter $q$, with the intermediate domain representing “noise” (nonclosed algebraic structures).


* * *