$U(5)–O(6)$ Phase Transition in the SD-Pair Shell Model *

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$U(5)$-$O(6)$ transitional behaviour in the SD-pair shell model is studied. The results show that the $U(5)$-$O(6)$ transitional patterns of the interacting boson model can be reproduced in the SD-pair shell model approximately.

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Quantum phase transitions have been attracting a great deal of attention in many areas of physics. In atomic nuclei, such quantum phase transitions can be related to different geometrical shapes of the system. Theoretical tools for analysing nuclear shape (phase) transitions were provided in the early 1980s\(^[1-3]\) following earlier work\(^[4]\) of Gilmore. It is now widely accepted that the three limiting cases of the interacting boson model (IBM) correspond to three different geometric shapes of nuclei, referred to as spherical (\(U(5)\)), axially deformed (\(SU(3)\)), and \(\gamma\)-soft (\(O(6)\)), respectively, which is usually described in terms of the Casten triangle.\(^[5]\)

It is well known that building blocks in the IBM are \(s\)- and \(d\)-bosons mapped from \(S\) and \(D\) nucleon pairs.\(^[6]\) Since the \(SD\)-pair shell model (SDPSM)\(^[7-9]\) is also built up from \(SD\) pairs, it is expected that the SDPSM can reproduce similar transitional patterns to those of the IBM. Our previous study show that the \(U(5)–SU(3)\) transitional patterns can indeed be reproduced within the SDPSM.\(^[10]\) The \(U(5)-O(6)\) is another important transition region and have been studied extensively\(^[11-14]\) in the IBM as well as in some microscopic theories.\(^[16]\) Since the SDPSM can reproduce the \(U(5)-SU(3)\) transitional pattern, it is interesting to see if the \(U(5)-O(6)\) phase transition can be reproduced in the SDPSM, and this is the purpose of this Letter.

From Ref.\(^[17]\) it is known that the \(O(6)\)-limiting spectrum can be reproduced in 50–82 shell with a Hamiltonian composed of pairing+quadrupole-quadrupole interaction, and the \(U(5)\)-limit can be reproduced with a pure pairing Hamiltonian in \(gds\) shell. Ref.\(^[18]\) show that the vibrational character do not depend on the parity of the populated levels. Therefore, to study the \(U(5)-O(6)\) transition, 50–82 shell and a schematic Hamiltonian are adopted, which is a combination of the pairing interaction corresponding to the vibrational case and pairing+quadrupole–quadrupole interaction corresponding to the \(\gamma\)-soft case with

\[
H = -\alpha G S^4 S - (1 - \alpha)(G S^4 S + \kappa Q^{(2)}) , \quad Q^{(2)}
\]

\[
S^4 = \sum_a \frac{\hat{t}^a}{2} \left(C^a_0 \times C^a_0\right),
\]

\[
Q^{(2)} = \sqrt{16\pi/5} \sum_i r_i^2 Y^2(\theta_i, \phi_i),
\]

where \(G\) and \(\kappa\) are the pairing and quadrupole-quadrupole interaction strength, respectively, and \(0 \leq \alpha \leq 1\) is the control parameter. \(Q^{(2)}\) is the quadrupole operator, and its second quantized form is given by

\[
Q^2_{\mu} = \sum_{cd} q_{\mu}(cd) P^2_{\mu}(cd),
\]

\[
q_{(cd)} = (-)^{c-1} \frac{\hat{c}^d}{\sqrt{20\pi}} C^1_0 C^2_1 d^{1/2} \Delta_{cd} \langle NL_c| r^2 |NL_d\rangle,
\]

\[
P^4_{\mu}(cd) = (C^1_c \times C^d_d)^t_{\mu},
\]

where \(N\) is the principal quantum number of the harmonic oscillator wave function, such that the energy is \((N + 3/2)\hbar\omega_0\). The matrix elements for \(r^2\) are given as

\[
\langle NL_c| r^2 |NL_d\rangle = \begin{cases} \frac{(N + 3/2) r_0^2}{l_c = l_d}, & l_c = l_d, \\ \varphi(N + l_d + 2 \pm 1) \frac{(N - l_d + 1 \mp 1)}{1/2} r_0^2, & l_c = l_d \pm 2, \end{cases}
\]

where the phase factor \(\varphi\) can be taken either as \(-1\) or \(+1\), \(r_0^2 = \frac{\hbar^2}{m\omega_0^2} = 1.012A^{1/3}f_m^2\), and \(A = 100\) is used in this work.

The \(E2\) transition operator is simply

\[
T(E2) = e_{\text{eff}}Q^{(2)},
\]

where \(e_{\text{eff}}\) is the effective charge, and for simplicity

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where the intermediate quantum numbers
\[ S^I = \sum_a y(aa0)(C^I_a \times C^I_b)^0. \]  

(7)

In this study, the \( S^I \)-pair structure coefficient, as an approximation, is fixed to be \( y(aa0) = \sqrt{\frac{N}{\Omega_a - N}} \),

where \( \Omega_a \) is defined as \( \Omega_a = a + 1/2 \) and \( N \) is the number of pairs for like-nucleons. The \( D \)-pair is obtained by using commutator

\[ D^\dagger = \frac{1}{2}[Q^{(2)},S^\dagger] = \sum_{ab} y(ab2)(C^1_a \times C^1_b)^2. \]  

(8)

The overlap between two states with \( N \) fermion pairs is a key quantity. The analytical expression for the overlap between two \( N \)-pair states is

\[ \langle s_1s_2 \cdots s_N; J'_1 \cdots J'_{N-1}J|\tau_1 \tau_2 \cdots \tau_N; J_1 \cdots J_N \rangle \]

\[ = (\hat{J}^N_{N-1}/\hat{J}^N_N)(-)^{J_N+\sigma_N-J_{N-1}} \]

\[ \sum_{k=N}^{1} \sum_{k=1}^{L+1} H_N(s_N) \cdots H_{k+1}(s_N) \]

\[ \times \left[ \psi_k \delta_{k-k_1-k_1-1} \right. \langle s_1 \cdots s_{N-1}; J'_1 \cdots J'_{N-1}| r_1 \cdots r_1' \cdots \rangle \]

\[ \left. \times \sum_i r_{k+1} \cdots r_{k+1} \cdots r_{N}; J_1 \cdots J_{N-1} L_k \cdots L_{N-1} \right] \]

\[ + \sum_{i=k-1}^{1} \sum_{r_{k} \cdots r_{k-2}} \langle s_1 \cdots s_{N-1}; J'_1 \cdots J'_{N-1}| r_1 \cdots r_1' \cdots \rangle \]

\[ \times \sum_{i=k-1}^{1} \sum_{r_{k} \cdots r_{k-2}} \langle s_1 \cdots s_{N-1}; J_1 \cdots J_{N-1} L_k \cdots L_{N-1} \rangle \],

(9)

where \( H_k(s) = (-)^{J_k+J_{k-1}+L_k} U(r_k L_{k-1}; L_{k-1}J_k) \), and \( U(r_k L_{k-1}; L_{k-1}J_k) \) is unitary Racah coefficient, \( \psi_k = 2(-)^{J_k-J_{k-1}+\gamma_k} \left( \hat{J}_k/\hat{J}_{k-1} \right) \sum_{ab} \bar{y}(a_k b_k r_k) y(a_k b_k s_k) \), while \( r' \) represents a new collective pair \( A'^\dagger \) with

\[ A'^\dagger = \sum_{a_k a_k} y'(a_k a_k r_k') A^I'(a_k a_k)^1, \]  

(10)

and its structure coefficients are given by a rather complicated expression,

\[ y'(a_k a_k r_k') = z(a_k a_k r_k') - (-)^{\alpha_k + \alpha_k + r_k'} \cdot z(a_k a_k r_k'), \]

\[ z(a_k a_k r_k') = 4\sqrt{r_k s} \sum_{b_k b_k} \hat{G}_k(st) Q_k(t) \cdots Q_{k+1}(t) \]

\[ \cdot \hat{M}_k(t r_k') \sum_{b_k b_k} y(b_k b_k s_k)(a_k b_k r_k) y(a_k b_k r_k) \]

\[ \left\{ \begin{array}{c} r_k \quad s \quad t \quad r_k' \quad b_k \quad a_k \quad a_k \quad b_k, \end{array} \right\}, \]

\[ \hat{G}_k(st) = -U(r_k s L_{k-1}; t J_{k}), \]

\[ Q_k(t) = (-)^{J_k+J_{k-1}+L_k+L_k} U(r_k L_k; J_k t L_{k-1} J_{k-1}), \]

\[ \hat{M}_k(t r_k') = U(r_k t J_{k-1} L_k; r_k' J_k'), \]

(11)

where the intermediate quantum numbers

\( L_1 \cdots L_{k-2} L_{k-1}, \) and \( L_{k'} \) (\( i' = i \cdots k - 2, \ k - 1 \)) is the angular momentum of the first \( i' \) pairs in the bra vector on the right-hand-side of Eq. (9). The details of the model can be found in Refs. [7-9].

As in Ref. [10], to explore the \( U(5)-O(6) \) transitional patterns, an identical nucleon system with \( N = 4 \) is considered. The parameters \( G \) and \( \kappa \) are set to be 0.1 MeV and 0.001 MeV/\( r_0^4 \), respectively.

Some low-lying energy levels as a function of \( \alpha \) are shown in Fig. 1. It is seen that there is indeed a minimum in the excited levels corresponding to the coexistence of spherical and \( \gamma \)-soft shapes in the critical region. Also, the minimum energy points are not exactly the same for excited levels. One can also notice that the energy levels change slowly with \( \alpha \), a feature of the second phase transition. To see the change of the energy levels with \( \alpha \) clearly, \( 2^+_2 \) state as a function of \( \alpha \) is given in Fig. 2, from which one can clearly see that there are indeed a minimum point, around \( \alpha \sim 0.45 \).

\[ \text{Fig. 1. Some low-lying energy levels across the transitional region, where } \alpha = 0 \text{ corresponds to the } \gamma\text{-soft shape and } \alpha = 1 \text{ to the spherical shape with vibrational spectrum.} \]

\[ \text{Fig. 2. Characteristics of } 2^+_2 \text{ levels across the transitional region, where } \alpha = 0 \text{ corresponds to the } \gamma\text{-soft shape and } \alpha = 1 \text{ to the spherical shape with vibrational spectrum.} \]

To check the phase transition further, energy ratios

\[ R_{42} = E_{4^1}/E_{2^1}, \]

\[ R_{02} = E_{0^1}/E_{2^1}, \]

\[ R_{03} = E_{0^1}/E_{2^1}, \]

\[ R_{03} = E_{0^1}/E_{2^1}, \]
Fig. 3. \( R = E_{4_1}^+/E_{4_1}^-, R = E_{0_2}^+/E_{0_2}^-, R = E_{0_3}^+/E_{0_3}^- \) and \( R = E_{0_3}^+/E_{0_2}^- \) across the transitional region.

Fig. 4. Some overlaps of excited states. The full lines are the overlaps \(|\langle J^+; \alpha = 0 | J^+; \alpha \rangle|\) and the dash lines shows the overlaps \(|\langle J^+_1; \alpha = 1 | J^+_1; \alpha \rangle|\).

Fig. 5. \( B(E2) \) values as a function of \( \alpha \).

and \( R_{00} = E_{0_2}^-/E_{0_2}^+ \) as a function of \( \alpha \) are shown in Fig. 3. It is seen that the similar behaviour to those of the IBM in Ref. [14] can be reproduced in the SDPSM. From Fig. 3 one can also see that \( R_{42} \) for \( \alpha = 0 \) (\( O(6) \) limit) and \( \alpha = 1 \) (\( U(5) \) limit) are smaller than 2.5 and 2, the values of \( R_{42} \) in the IBM. However, if a larger space is used as in Ref. [17], the results will be close to those of the IBM.

To see the critical points for different states clearly, the overlap \(|\langle J^+_1; \alpha = 0 | J^+_1; \alpha \rangle|\) with \( \alpha_0 = 0 \) or 1 are shown in Fig. 4 for a few typical excited states. These results show that the critical points for the \( 2^+_1 \) and \( 2^+_2 \) states are around \( \alpha \sim 0.6 \), while it is \( \alpha \sim 0.55 \) and 0.5 for
$0^+_2$ and $0^+_2$ states, respectively.

It is well known that $B(E2)$ values can also be used to test the collectivity of the low-lying states. To further explore that nature of the vibration to $\gamma$-soft transition, the $B(E2)$ values for $4^+_1 \rightarrow 2^+_1$, $2^+_2 \rightarrow 2^+_1$, and $0^+_2 \rightarrow 2^+_1$ are presented in Fig. 5. One can see that the three $E2$ transitions are almost same when $\alpha \sim \gamma$. A typical feature of the $SO(6)$ limiting case in the IBM. With increasing $\alpha$, $B(E2; 0^+_1 \rightarrow 2^+_2)$ decreases, while the other two change slowly. When $\alpha \sim 1.0$, one finds that the $B(E2)$ values are dominated by $4^+_1 \rightarrow 2^+_1$ and $2^+_2 \rightarrow 2^+_1$, and it is forbidden for $0^+_2 \rightarrow 2^+_2$, which is a typical feature of the $U(5)$ limiting case in the IBM.

The relative $B(E2)$ ratios for $B(E2; 4^+_1 \rightarrow 2^+_1)/B(E2; 2^+_1 \rightarrow 0^+_1)$ and $B(E2; 0^+_2 \rightarrow 2^+_1)/B(E2; 2^+_2 \rightarrow 0^+_2)$ are presented in Fig. 6. It is seen that the similar results as those of the IBM can be reproduced approximately.[14]

In summary, vibration-$\gamma$-soft transitional patterns have been studied within the framework of the SDPSM. Though some drastic changes of some quantities within the critical region are smoothed in the SDPSM, the $U(5) - SO(6)$ transitional signatures of the IBM can indeed be reproduced. Once again we see that since the SDPSM is formulated in fermion pair basis, the results seem to confirm in yet another way that the IBM indeed has a sound shell-model foundation.

References

  Gilmore R 1979 J. Math. Phys. 20 89