U(3) \rightarrow R(3) INTEGRITY-BASED SPECTROSCOPY

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Abstract: The U(3) \rightarrow R(3) algebra, widely used in nuclear spectroscopy studies, is reviewed. The most general form of a R(3)-preserving interaction that is rotationally invariant and of group origin to the group generation is presented. Here the full purpose and beauty of the integrability idea concept is realized. In contrast to the above it is shown that the structure of the \( \text{SU}(3) \rightarrow R(3) \) symmetry basis can be deduced from a systematic counting of defining space matrix vectors of a single, 3-body vector operator, \( k = 0, 1, 2, \ldots \). The tensorial character of the so-called "missing labels" operator and, more importantly, of the operators responsible for the splitting and movement of rotational bands is obtained by relating integral basis multiplets through direct force in the U(3) generation to derive operators of a standard \( \text{SU}(3) \rightarrow R(3) \) 3-body spectroscopy. The results are used to show how A4-band splitting is as well as a [111 11] factor in the energy can be realized within a single representation of \( \text{SU}(3) \) by two-body interactions of the dia and higher order shell. Parameter sets of model interactions associated with both normal and inverse A4-band structures are given, as well as the results of a "free fit" theory for the ground and gamma bands of \( ^{26}\text{Mg} \).

1. Introduction

The three-dimensional isotropic harmonic-oscillator group structure, \( U(3) \rightarrow R(3) \), is a fundamental part of several physical models. Elliott was the first to recognize its significance for reproducing rotational features of light deformed d-shell nuclei, most notably the Ne and Mg isotopes \(^1\). Since that pioneering work it has also been shown to be an important symmetry structure for reproducing rotational features of heavy deformed rare-earth and actinide nuclei \(^2\). In the interacting boson model it occurs as one of three possible group chains and is a fundamental part of the so-called rotational limit of that model \(^3\). The symplectic collective model, which is a microscopic formulation of the Bohr-Mottelson theory of collective motion, extends the \( U(3) \rightarrow R(3) \) part of the Elliott

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model in a natural way to accommodate multishell quadrupole collective phenomena.\textsuperscript{43}

In each case the $U(3) \to R(3)$ structure occurs as part of a group chain which starts with some largest group $G$. For example, $G$ is $U(1)$ for the interacting boson model and $Sp(3, R)$ for the symplectic collective model. The full structure is $G \to U(3) \to R(3)$. By definition $G$ is a \textit{conserved} symmetry if the hamiltonian $H$ is expressible solely in terms of its generators. Then $H$ has no matrix elements connecting different irreducible representations (irreps) of $G$ and every $H$ eigenstate belongs to a single $G$ irrep. And $R(3)$ is an \textit{exact} symmetry if $H$ is a multiple of the identity in each of its irreps. All such operators are $R(3)$ scalars. The intermediate group $U(3)$ is normally neither exact nor conserved, it is an \textit{imposed symmetry}. $H$ both spreads states within an irrep of $U(3)$ and connects states of different $U(3)$ irreps.

Whenever one encounters a group structure that has in it an imposed symmetry, a natural question to ask is, how good is the symmetry? That question presupposes a fixed $H$ and the answers depend upon the measures chosen. Statistical spectroscopy provides a set of tools for quantifying "goodness of symmetry" measures.\textsuperscript{1} The measures are all representation dependent. However, if one relaxes the fixed-$H$ constraint, a less sharply focused question, but one of even greater potential significance can be posed: namely, how much symmetry breaking is required? The latter presupposes only in a fixed set of data, the explicit form of the model interaction is left an open question. In particular, one can determine differences that must be attributed to the larger group structure by picking for the model $H$ a parameterized interaction that is built solely of generators of the imposed symmetry, which is then for that $H$ a conserved symmetry, and by varying the parameters of that $H$ to obtain a "best fit" theory.

It might be argued that the latter procedure is troublesome because the model $H$ appears to no longer be tied to a physical interaction. Indeed, it is possible to arrange for an $H$ such that the eigenstates within a single irrep assume any desired ordering. But there are other considerations. For example, a theory which uses a parameterized interaction would have little merit if there are huge case-by-case variations in the strengths of those parameters. Also, if terms in the interaction associated with high powers of the generators prove to be essential the theory should be suspect. This can be appreciated by recalling that in any "normal mode" decomposition, and an expansion of $H$ in terms of generators can be viewed in that way, the low-order terms are usually associated with the dominant features of a system whereas high-order terms look after details.

Ideally, of course, an effective interaction theory should be used to produce an interaction appropriate to the restricted (single irrep) model space. Unfortunately, a suitable symmetry adapted effective interaction theory is not presently available. Nonetheless, the outcome of such an exercise, because it is a perturbation theory, would place, at a minimum, a constraint on the effective particle rank of the
model-space interaction. If the full space $H$ is a \{0+1+2\}-body interaction, a first-order theory would being in terms of particle rank $k \leq 2$ while a second-order theory would draw in terms quadratic in $H$ for which $k \leq 4$, etc. And, because there is a group structure and an associated underlying operator algebra, a particle-size cutoff constraint places a restriction on the degree to group generators that a model $H$ can have. So the degree of a multilinear representation of an interaction in terms of generation of an intermediate impose symmetry is constrained, either directly or indirectly, by the fact that a model-space interaction derives from a real two-body force.

The underlying purpose of this article is to revisit spectroscopy of the $\text{U}(3) \to \text{R}(3)$ group structure in light of the above. We are therefore interested in discovering the most general form of a $\text{U}(3)$ preserving interaction that is rotationally invariant and of given particle rank. Here the full purpose and beauty of the integrity basis concept will be realized. Indeed, we begin by showing how one can anticipate the structure of the $\text{U}(3) \to \text{R}(3)$ integrity basis set by counting reduced matrix elements of $p$-shell, $\text{\Sigma}$-body scalar operators, $k = 0, 1, 2, \ldots$. We then go on to discover the tensorial character of the so-called “missing label” operators and, more importantly, of the operators responsible for the splitting and inversion of rotational bands. This is done by relating integrity basis multilinearities through degree four in the $\text{U}(3)$-generators to $p$-shell density matrixes. The latter is then used to show how $K$-band splitting and inversion phenomena as well as an $(\ell \ell' + 1)$ factor in the energy can be realized within a single representation of $\text{SU}(3)$ by two-body interactions of the $\text{ds}$ and higher shells. And finally, to emphasize the importance and function of the complete set of integrity basis operators, we give parameter sets of model interactions associated with what might be considered typical spectra, including both normal and inverted $K$-band structures, as well as the results of a “best fit” theory for the ground and gamma bands of $^{24}$Mg.

2. Information/counting and the integrity basis operators

An interesting and instructive perspective on spectroscopy can be gained by reducing the information content of a theory to its minimum and simplest form. For example, in the nuclear $\text{ds}$ shell one knows that the most general $(1+1+2)$-body scalar interaction is determined by $1 \times 3 \times 63 = 189$ independent matrix elements. So, even though the $(\ell', \ell) = (0', 0)$ subspace of $\text{U}(3)$ has dimensionality 325 and therefore appears to have a total of $325 \times 326/2 = 52,975$ independent matrix elements, if the $\text{U}(3)$ is a $(0+1+2)$-body interaction they all derive from 69 pieces of information. The $\text{ds}$-shell geometry determines how that information is dispersed among the many-body matrix elements. This type of reduction emphasizes the importance of the so-called defining space matrix.
elements, that minimum set of information about \( H \) which for the chosen geometry uniquely determines the properties of a system. Only by imposing further constraints on the system can the required amount of input information be reduced. For example, picking a quadrupole-quadrupole plus point form for \( H \) places 61 constraints on the set of 63 d-d shell 2-body matrix elements.

Now consider the U(3) \( \to \) R(3) structure and, in particular, its shell-model realization using LS-coupled basis states. The number of defining space matrix elements associated with a general central interaction (R(3) scalar) of particle rank \( k \) is given in column 3 of Table 1. The numbers can be verified by simply counting reduced

<table>
<thead>
<tr>
<th>Particle rank ( k )</th>
<th>Representations ( \lambda_1, \lambda_2 )</th>
<th>Number of matrix elements</th>
<th>Integrity basis scalars</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(00)</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>(10)</td>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>(20), (2,0)</td>
<td>5</td>
<td>( \nu_{12}^4, \nu_{12}^4 )</td>
</tr>
<tr>
<td>3</td>
<td>(30), (3,0)</td>
<td>5</td>
<td>( \nu_{13}^4, \nu_{13}^4 )</td>
</tr>
<tr>
<td>4</td>
<td>(40), (4,0)</td>
<td>9</td>
<td>( \nu_{14}^4, \nu_{14}^4 )</td>
</tr>
<tr>
<td>5</td>
<td>(50), (5,0)</td>
<td>13</td>
<td>( \nu_{15}^4, \nu_{15}^4 )</td>
</tr>
<tr>
<td>6</td>
<td>(60), (6,0)</td>
<td>22</td>
<td>( \nu_{16}^4, \nu_{16}^4 )</td>
</tr>
<tr>
<td>7</td>
<td>(70), (7,0)</td>
<td>30</td>
<td>( \nu_{17}^4, \nu_{17}^4 )</td>
</tr>
</tbody>
</table>

At \( k = 0 \), due to the inessential \( L = 2 \) multiplicity in the (22) representation, no antisymmetric defining space matrix element can be reduced. This corresponds to the \( \nu_{12}^4 \) integrity basis element from the (12) integral-basis vectors, accounts for all additional antisymmetric defining space matrix elements that occur for \( k > 0 \).
matrix elements, \((l|k)H(l|k)\), where \(l|k\) ranges over all \(k\)-particle representations of SU(3). Rotational invariance requires that \(E_L = L\) and \((k|L) = (L|k)\) because \(U(3)\) is taken to be the largest group and as a consequence is a covariant symmetry. Associated with each \(L\) value are \(\frac{1}{2}(L+1)\) distinct matrix elements where \(L\) is the multiplicity of \(L\) in \(U(3)\); \(x\) matrix elements that are diagonal in \(x\) and \(\frac{1}{2}(L-1)\) off-diagonal ones. (Although we consider only real symmetric matrices here, the arguments extend to the additional \(\frac{1}{2}(L-1)\) off-diagonal antisymmetric forms as well.) The various \((l|k)L\) values for each \(k\) are given in column 2 of Table 1. Thus, for example, there are \(22\) \(k = \pm 4\) matrix elements and only \(21\) different \((l|k)L\) values. That is, because \((k|±4)L = (±4|k)L\) occurs with multiplicity 2 and so there are three rather than two matrix elements of that type; two that are diagonal in \(x\) and one off-diagonal one. For \(k = 7\) there is again just one off-diagonal combination but for \(k \geq 8\) the number grows rapidly. Note that every \(k\)-state includes those \((k|±4)\) combinations of the \((L|3)\) set, i.e., there is a repeat pattern of \(k\) modulo 3. This occurs because of the \((U(3)\) to \(SU(3)\) restriction. In particular, \(k = \lambda + 2\mu + 3\nu\) where \(\lambda\) labels the number of \((U(3)\) invariant structure in the \(k\)-particle \((\lambda, \mu, \nu)\) state. In terms of Young shape labels, \(\nu\) is the number of columns containing three boxes.

Associated with each of the \(k\)-particle defining space matrix elements referred to above is a unique scalar density operator or propagator. These propagators have vanishing matrix elements in all \(k' < k\) particle spaces and dictate how the \(k\)-particle piece of information each represents gets dispensed in the \(k > k\) particle spaces. These operators form a complete set; any \(k\)-body scalar operator has a unique expansion in terms of the scalar \(k\)-particle density operators. In particular, the \(H\) that is of maximum particle rank \(k\) can be expanded in terms of the \(k' \leq k\) particle density operators. It follows that if one can deduce simple generator formulas for density operators one has the sought after prescription for writing down the most general symmetry preserving interaction that is of a given maximum particle rank.

Noether has proven that whenever there is a group-subgroup structure, say \(G(\equiv U(3)) \rightarrow K(\equiv SU(3))\), all \(K\)-scalar operators in the enveloping \(G\) algebra can be expressed as multinomial functions of some basic set of \(K\) scalars. That minimum set of scalar operators is called the \(G \rightarrow K\) integrity basis*). Saltz et al. have shown that for \(SU(3) \rightarrow SU(2)\) the integrity basis contains five operators that give rise to real symmetric matrix forms, two of degree two in the generators, two of degree three, and one of degree four*). They show that these can be chosen to be \(U_1\), the second-order Casimir invariant of \(R(3)\); \(C_2\) and \(C_3\), the second- and third-order Casimir invariants of \(U(3)\), and two non-Casimir-invariant \(R(3)\) scalars, generally labelled \(X_3\) and \(X_4\) which are of degree three and four, respectively, in the generators. (Actually, there is in addition a sixth-order noninvariant scalar. (Note: We reserve the terminology "invariant" for Casimir invariants and "noninvariant scalar" for non-Casimir-invariant operators which transform as subgroup scalars.) Although this sixth-order noninvariant scalar cannot be written
as a simple polynomial function of the other five, it can be expressed in terms of the commutator of $X_j$ and $X_k$. It therefore gives rise to antisymmetric matrix forms. Its square, which is necessarily symmetric, can however be written as a polynomial function of the five other $R(1)$ scalars. Those five operators are therefore the integrity basis set for real symmetric forms. Of the three types of

scalars that can occur, invariants of $G$, invariants of $K$, and noninvariants $K$ scalars, only the last two can spread states within a $G$ irrep and only the last one can couple and mix multiple occurrences of a given $K$ irrep in $G$. It is the latter property that makes the noninvariant scalars candidates for resolving the so-called mixing-label problem. Indeed, it is this property that has motivated much of the work that has been done to date in this area."

Below we come to an appreciation of the full power and beauty of the integrity-basis concept.

As the density operators are scalars, they qualify under the above and can therefore be expanded as multinomial functions of the integrity-basis operators. A question that remains concerns the degree of such a series. This is an important question for the answer determines, ultimately, the number of parameters in a general symmetry-preserving $H$. But that number is known: it is just the total number of defining space matrix elements! Working backwards from the number of defining space matrix elements, see for example the cataloging done in table 1, it would appear that one should be able to deduce an answer to the "degree" question. But much more can be learned by directly establishing the (almost obvious) fact that the particle rank of a symmetry-preserving interaction fixes the degree of its representation in terms of the generators of the conserved symmetry. To see this, the $U(3) \rightarrow R(3)$ case will serve as our example but the arguments can obviously be extended. We proceed by inductive reasoning. Clearly a $0$-body operator is a multiple of the identity in every $U(3)$ irrep. In the $p$-shell the action of a $1$-body operator can be represented by the product of a creation-annihilation operator pair which maps directly onto a linear $U(3)$ generator form. Now a $(k+1)$-body operator can always be rewritten as a product of a $k$-body operator and a $1$-body operator with a residual $k$-body part. It then follows that if every $k$-body operator has a $(k+1)$th-degree multinomial representation in terms of $U(3)$ generators, every $(k+1)$-body operator has a $(k+1)$th-degree multinomial representation. The proof is then secured by noting that a multinomial of degree $k$ must have a $k$-body part. So a $k$-body scalar has a representation in terms of integrity-basis operators with the degree of that representation, as measured in terms of generators of the conserved symmetry, fixed by $k$.

To see how this can now be used to deduce the structure of an integrity-basis set, consider a representation of the scalar density operators, which remember are in one-to-one correspondence with the defining space matrix elements, in terms of the integrity-basis operators. If there are $z$ integrity-basis operators, $\{x_i; \ i = 1, 2, \ldots, z\}$, and the $\mathfrak{h}$th one is constructed of $k$ generators so it is of maximum particle rank $\mathfrak{k}$, the number of terms in a multinomial series of degree $k$ is just
equal to the number of sets of \( \alpha \) nonnegative integers \( \alpha_i \) that satisfy the constraint equation

\[
0 \leq x_1 + x_2 + \cdots + x_k \leq k.
\]

Each set \( \{\alpha_i\} \) corresponds to a specific polynomial term, \( \prod_{i=0}^{\alpha_i} (x_i!) \). Now consider the results given in columns 4 of Table 1 for the \( U(3) \rightarrow R(3) \) case. For \( \alpha = 0 \) there is but one matrix element and as shown, one operator, the identity. For \( k = 1 \) there is again one matrix element with \( \alpha \), the number operator, its propagator. For \( k = 2 \) there are three matrix elements. Now \( n^\prime \), or more appropriately \( f(L) \) qualifies as one of the operators but there must be two others. They can be chosen to be \( \hat{C}_1 \) and \( \hat{C}_2 \), the square of the angular momentum which is the invariant of \( R(2) \) and the second-order invariant of \( SL(3) \). For \( k = 3 \) the counting demands a total of five operators. As shown, these can be combinations of those already introduced but two additional ones are needed. The third-order invariant of \( SL(3) \) qualifies but in addition there must be one more. There are no other invariants available so it must be a noninvariant scalar which in the table is simply labeled \( X_3 \). Note that it arises independent of \( \{\alpha_i\} \rightarrow L \)-multiplicity. For \( k = 4 \) there is again a need for one additional noninvariant scalar, which is labeled \( X_4 \) in the table. Beyond \( k = 4 \) the counting of defining space matrix elements reveals that there is no need for additional operators. Here we acknowledge only semisimple forms of \( L \)-degeneracy of \( x \) in \( \{\alpha_i\} \rightarrow \{\alpha + 1\} \) rather than \( \{\alpha_i\} \) pieces of information. Note, however, that the counting can be extended to include antisymmetric forms as well. Indeed, the first occurrence of an \( L \)-multiplicity at \( k = 6 \) if \( L = 2 \) occurs twice in \( \{\alpha_1\} \rightarrow \{22\} \) requires the existence of an antisymmetric invariant of sixth degree. This is in agreement with the work of Jadc et al.\(^4\) who identified the commutator of \( X_2 \) and \( X_3 \) as an independent (antisymmetric) sixth-degree invariant scalar. For example, for \( k = 50 \) one finds that the number of sets of nonnegative \( \alpha \) satisfying the constraint relation

\[
0 \leq x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 50
\]

is \( 33993 \), which is just the number of defining space matrix elements, excluding, of course, the antisymmetric off-diagonal ones. This number can either be determined by direct counting or somewhat more simply by recognizing that the number of distinguishable arrangements of \( n \) objects taken \( k \) at a time is \( \binom{n+k-1}{k} \), so that a general result for the number of sets for any \( k \) is given by

\[
\sum_{0 \leq \alpha_i \leq k} \binom{\sum \alpha_i + \alpha_i - 1}{\alpha_i}.
\]

where \( T = (1,2, \ldots, n) \) with \( l_i = \alpha, \ f = (m) \) with \( m = \alpha_i \) equal to the number of integrality basis operators of degree \( \alpha \), and \( \delta = (\gamma) \) with each \( \gamma_i \) a nonnegative running integer index. For the \( U(3) \rightarrow R(3) \) case the vectors are four-dimensional
and \( m = (1, 2, 2, 1) \) because the integrity-basis consists of six scalars, \( \{ a, C_1, L, C_2, X_3, X_4 \} \), or for \( SU(3) \to R(3) \) a set of five, \( \{ X_1, L, C_5, X_3, X_4 \} \). Note that the identity is not included as it is just \( m \) to the zeroth power.

As noted above, of the six \( U(3) \to R(3) \) integrity basis scalars only \( L^2 = X_3, X_4 \) and \( L^2 \) split the \( xL \) degeneracy of a given \( \lambda \mu \) representation. The \( R(3) \) invariant \( L^2 \) produces a rotational \( L(L+1) \) sequence but does not distinguish between multiple occurrences of a given \( L \)-value. The non-invariant scalars \( X_3 \) and \( X_4 \) depend, in general, on both \( \lambda \) and \( L \). It is well known that either, or any linear combination of the two, suffices to resolve the \( SU(3) \to R(3) \) multiplicity problem \( ^{23} \). But we now see that that alone is far from complete picture. The full significance of the \( X_3 \) operators is only realized in terms of the propagation picture. They are required even when there is no \( L \)-multiplicity. Their real function is to propagate \( k = 3, 4, \ldots \) particle information into spaces of higher particle numbers. The fact that they have off-diagonal matrix elements \( \lambda \lambda \) is a natural consequence of that requirement. Nature is not redundant in providing extra means for resolving the \( L \)-multiplicity, the multiplicity problem is simply not the whole story. The full integrity basis is required for a complete spectroscopy. Examples of this are given above.

3. Matrix elements of the integrity-basis operators

Since the most general symmetry-preserving interaction can be expanded as a multinomial function of integrity-basis operators, it follows that knowing general expressions for matrix elements of those operators is most important. For example, below we will see that for the \( U(3) \to R(3) \) case there are nineteen terms in a fourth-degree integrity-basis multinomial series. But there are only six different integrity-basis operators. Additionally, of those six basic scalars four can be chosen to be group invariants which then leaves only two noninvariant ones, the noninvariant scalars \( X_3 \) and \( X_4 \). Therefore, as the eigenvalues of the \( U(3) \) and \( R(3) \) Casimir invariants are known, every \( U(3) \to R(3) \) eigenvalue problem can, in principle, be reduced to evaluating matrix elements of two noninvariant scalar operators. So of the nineteen terms in the fourth-degree theory referred to above, only \( X_3, X_4 \) and \( X_4 \) have noninvariant matrix elements and obviously \( X_3, X_4 \), is no more complex than \( X_3 \) itself. Indeed, the first slight complication that occurs is for \( X_3^2 \) in a sixth-degree theory. But products of noninvariant scalars present no real problem either for the \( X_3 \) operators preserve both \( \lambda \mu \) and \( L \) so the required intermediate sums are simple.

Before giving results for matrix elements of the \( U(3) \to R(3) \) noninvariant scalar operators, note that the grouping of integrity-basis operators into three classes, invariants of \( G \) which are just constants within a given \( G \) irrep, invariants of \( K \)
which serve to differentiate between different K irreps in a G rep, and noninvariant K scalars which can be used to distinguish between multiple occurrences of K irreps, is always possible. In addition, if the G → K structure supports an intermediate group H, G → H → K, some of the noninvariant K scalars can be replaced by the invariants of H. A complete replacement of the noninvariant K scalars by H invariants is, in general, impossible. This is so even when H is a trivial representation of the G multiplicity. The latter can be appreciated as follows: If the same K irrep occurs in two different H irreps, even though both of those occur but once in any given G irrep, there will be off-diagonal defining space matrix elements in the intermediate group labels and associated with those matrix elements are noninvariant K scalars which cannot be accounted for by H invariants. The U(1) → U(2) → U(1) chain offers an almost trivial example of this. The U(1) generators C12 and C13 are noninvariant U(1) scalars that lie outside the set of U(2) invariants.

Another possibility is that there is more than one choice for the intermediate group H. The invariants of the various H symmetries can then be substituted for some, perhaps all, of the noninvariant scalars. When this happens one has a model that encompasses special symmetry limits. For example, the U(6) → U(3) structure of the interacting boson model admits three intermediate symmetries, SU(5) associated with rotations, SU(5) with vibrations, and O(6) with so-called quasi-degenerate configurations [1]. The term "dynamical symmetries" has been applied to these intermediate group structures [1]. In terms of the language introduced in sect. 1, they are imposed symmetries for an interaction which is in general, different irreps of any particular H. Of course, relative to one choice of an intermediate symmetry group for the labelling of state vectors, the invariants of the other intermediate symmetries appear to be noninvariant scalars; the invariant properties of an operator can only be realized in the concomitant state labelling scheme. Now even if the number of intermediate group invariants is insufficient to replace all the noninvariant scalars, they may be sufficient to replace all those of degree less than or equal to that which is required for a nontrivial representation of an interaction. When this happens, the interaction has a particularly simple and beautiful form in that it can be represented solely in terms of group invariants, including those of the intermediate symmetries. A particular symmetry limit is realized when the coefficients associated with invariants of all but one of the intermediate groups vanish. This is how the three symmetry limits of the interacting boson model are achieved.

Let \( |\psi_{L,M}^{(a)}\rangle \) denote a \( p_{1/2} \)-shell U(1) → R(3) state vector. Here \( n \) labels the number of particles, \( (L) \) the irrep of SU(3), \( L \) the irrep of R(3) and \( M \) its projection, and \( a \) is a running index used to distinguish multiple occurrences of an \( L \) in \( \psi \). Our choice for \( a \) is that made by Vergados [35] and is therefore closely tied to the Elliott K-projection label [34]. Matrix elements of the integrity basis U(3)
invariant operators are independent of the \( n \)LM labels,

\[
\langle m | \epsilon | \mu \rangle \langle \mu \phi | \epsilon | n \rangle | LM \rangle = \left(\begin{array}{c}
\frac{m}{2} \\
\frac{\kappa}{2} + \frac{\mu}{2} + \frac{\mu}{2} - 1
\end{array}\right) \left(\begin{array}{c}
L^2 + 5\mu^2 + 2\mu^2 + 9(\kappa + \mu + 1)(\kappa - \mu)^2
\end{array}\right).
\]

Note that \( n \) counts the number of oscillator quanta and is equal to \( m \), the number of particles, because here we assume there is just one quantum per particle, i.e., a p-shell geometry. The \( R(3) \) invariant depends only on \( L \).

\[
\langle m | \epsilon | \mu \rangle \langle \mu \phi | \epsilon | n \rangle | LM \rangle = L(L + 1).
\]

The noninvariant scalar operators \( X_3 \) and \( X_4 \) can be defined in a variety of ways. As they are \( R(3) \) scalars, it is most convenient to define them in terms of contracted spherical tensors. The nine generators of \( U(3) \) can be written as spherical tensors of ranks 0, 1, and 2. The rank zero tensor is just \( n \), the number operator. The first-rank tensors are the generators of \( R(3) \); namely, the angular momentum operators, \( L_3 \). And the second-rank tensors form the components of an oscillator quadrupole operator, \( Q \). The \( L \) and \( Q \) operators are normalized so that \( Q^2 = n + 1 \) and \( Q^2 = n + 1 \). Candidates for \( X_3 \) include \( (L \times Q) \cdot L \), \( (Q \times Q) \cdot L \) and \( (Q \times Q) \cdot Q \). Note that at least one \( Q \) is required for \( (L \times L) \cdot L \) lies wholly within the \( R(3) \) subspace and cannot couple different realizations of the same \( R(3) \) irrep.

Below, for convenience, we use

\[
X_3 = (L \times Q) \cdot L = \sqrt{3} \left( L \times Q \times L \right)_3.
\]

\[
X_4 = (L \times Q) \cdot (Q \times Q) = \sqrt{2(n + 1)} \left( L \times Q \times Q \times L \right)_3
\]

with \( X_3 = X_4(L = 1) \) for then the noninvariant scalars involve a minimum number of \( Q \) operators and for \( X_4 \) the lowest-rank spherical tensor contraction. For future reference,

\[
(L \times Q) \cdot L = (L^2 + 1) \sqrt{3} (Q \times Q) \cdot Q - 4 \sqrt{3} C_3.
\]

The matrix elements of the \( X_3 \) can be determined in terms of Racah couplings given the \( SU(3) \rightarrow R(3) \) reduced matrix elements of \( L \) and \( Q \).

\[
\langle \mu | L(1) \mu | \mu \rangle \langle \mu \phi | L(1) \mu | n \rangle = \delta_{\mu \epsilon} \delta_{L} \delta_{L+1} \left( L \times Q \times Q \times L \right)_{3} + \left( L(1) + 1 \right)!
\]

\[
= [2L(1) + 1]!.
\]
\[ \langle \psi_1 | \mathcal{L}^{\mu} | \psi_2 \rangle = (-1)^\delta \times 2^{L_2} \langle \psi_1 | \mathcal{L}_1^{(L;1)} \mathcal{L}_2 \cdots | \psi_2 \rangle, \] (8)

The phase factor \( \phi \) is 1 if \( \rho \neq 0 \) and 0 if \( \rho = 0 \) and the SU(3) \( \to \) R(3) Wigner coefficients on the right-hand side can be evaluated using codes in the SU(3) package of Akyama and Dressler. Note that the reduced matrix elements as defined are unconventional; that is, the full matrix element is just an ordinary R(3) Wigner coefficient times the reduced matrix element with no extra \( \sqrt{2L+1} \) factors included. So finally,

\[ \langle \lambda \mu \nu | \mathcal{L}^{\mu} | \lambda \mu \nu \rangle = \langle \lambda \nu \lambda | \mathcal{L}^{\nu} | \lambda \nu \lambda \rangle = \langle \lambda \mu \nu | \mathcal{L}^{\nu} | \lambda \mu \nu \rangle = \langle \lambda \nu \lambda | \mathcal{L}^{\nu} | \lambda \nu \lambda \rangle = \sum_{\gamma \delta} (-1)^{L_1 + L_2 + L_3} \sqrt{2L + 1} \langle \nu | \mathcal{L}_1^{(L_1;1)} \mathcal{L}_2^{(L_2;1)} \mathcal{L}_3^{(L_3;1)} | \gamma \rangle \times \langle \mu | \mathcal{L}_1^{(L_1;1)} \mathcal{L}_2^{(L_2;1)} \mathcal{L}_3^{(L_3;1)} | \delta \rangle. \] (9)

\[ \langle \lambda \mu \nu | \mathcal{L}^{\mu} | \lambda \mu \nu \rangle = \langle \lambda \nu \lambda | \mathcal{L}^{\nu} | \lambda \nu \lambda \rangle = \langle \lambda \mu \nu | \mathcal{L}^{\nu} | \lambda \mu \nu \rangle = \langle \lambda \nu \lambda | \mathcal{L}^{\nu} | \lambda \nu \lambda \rangle = \sum_{\gamma \delta} (-1)^{L_1 + L_2 + L_3} \sqrt{2L + 1} \langle \nu | \mathcal{L}_1^{(L_1;1)} \mathcal{L}_2^{(L_2;1)} \mathcal{L}_3^{(L_3;1)} | \gamma \rangle \times \langle \mu | \mathcal{L}_1^{(L_1;1)} \mathcal{L}_2^{(L_2;1)} \mathcal{L}_3^{(L_3;1)} | \delta \rangle. \] (10)

4. Density operators and their matrix elements

This section is devoted to a review of the essential ingredients of a standard many-body U(3) \( \to \) R(3) spectroscopy. This is done in anticipation of results needed in the next section where a comparison is made with the equivalent, but technically very different, integrability-based theory. In particular, the results given below will be used to establish the SU(3) tensorial character of the integrability-based operators. The latter is essential to gaining an appreciation of how R-band splitting and inversion phenomena as well as an \( \{L+1\}^2 \) factor in the energy can be realized within a single representation of SU(3) by 2-body ds and higher shell interactions. For this it is necessary to carefully define the U(3) \( \to \) R(3) density operators and give results for evaluating their matrix elements.

If \( \psi_\lambda (k) \) denotes a model-space \( \lambda \)-particle state vector and \( \bar{\psi}_\lambda (k) \) its conjugate so that

\[ \int dx \bar{\psi}_\lambda (k) \psi_\lambda (k) = \delta_{\lambda \lambda} \delta_{\lambda \bar{\lambda}}, \] (11)

then the model-space density matrix representation of a \( \lambda \)-body operator is given by

\[ \hat{\rho}(k) = \sum_{\lambda \bar{\lambda}} \langle \lambda \bar{\lambda} | \mathcal{L}^{(\lambda;1)} \mathcal{L}^{(\bar{\lambda};2)} | \lambda \bar{\lambda} \rangle \bar{\psi}_\lambda (k) \psi_\lambda (k). \] (12)
In this expansion the product \( \phi_j(k) \psi_j^*(k) \) is a \( k \)-particle density operator which can be conveniently denoted \( \rho^{(k)}(k) \). A density operator has vanishing matrix elements in all \( k \leq 1 \) particle spaces except for one, its own, where its value is unity.

\[
\langle \phi_j(k) \psi_j^*(k) \rangle = \delta_{ij} \delta_{kj} \delta_{ll}
\]  

(13)

Let

\[
\phi_j(k) = \phi_j^{(1)}(k),
\]

(14a)

\[
\psi_j^*(k) = \psi_j^{(1)}(k)
\]

(14b)

where both \( \phi \) and its adjoint \( \psi \) transform as prop. \( U(3) \) tensor operators. Then, in analogy with the above, coupled \( \mu \)-shell \( U(3) \rightarrow R(3) \) density operators can be defined,

\[
\rho^{(\mu \times \omega)}(k) = \sum_{i=1}^{\mu} \langle \phi_i^{(\mu \times \omega)}(k) \psi_i^{(\mu \times \omega)}(k) \rangle
\]

\[
= \sum_{S_1} \langle \hat{L} M : L \rangle \left( \langle \phi_i^{(\mu \times \omega)}(k) \hat{L} \rangle \langle \psi_i^{(\mu \times \omega)}(k) \rangle \right) S_1
\]

\[
\times \sum_{\omega \omega_\omega} \langle \hat{L} M : L \rangle \left( \langle \phi_i^{(\mu \times \omega)}(k) \hat{L} \rangle \langle \psi_i^{(\mu \times \omega)}(k) \rangle \right) \omega \omega_\omega
\]

\[
\times \hat{L} \big( \hat{L} M : L \big) \omega \omega_\omega
\]

(15)

In eq. (15) \( \omega_\omega \) is a running index used to distinguish between multiple occurrences of \( \langle \phi_i \langle \psi \rangle \rangle \) in the product \( \langle \phi \langle \psi \rangle \rangle \). It can be shown that if the reduced matrix elements \( \phi \) are defined in an \( \mu \)-particle space by

\[
\langle \phi_i^{(\mu \times \omega)}(k) \hat{L} \rangle = \langle \hat{L} \rangle M \langle \hat{L} \rangle \hat{L} \rangle
\]

\[
\times \sum_{\omega \omega_\omega} \langle \hat{L} M : L \rangle \left( \langle \phi_i^{(\mu \times \omega)}(k) \hat{L} \rangle \langle \psi_i^{(\mu \times \omega)}(k) \rangle \right) \omega \omega_\omega
\]

\[
\times \langle \hat{L} M : L \rangle \left( \langle \phi_i^{(\mu \times \omega)}(k) \hat{L} \rangle \langle \psi_i^{(\mu \times \omega)}(k) \rangle \right) \omega \omega_\omega
\]

(16)

then in the \( l \leq 1 \) particle space

\[
\langle \theta(k) \rangle \rho^{(\mu \times \omega)}(k) \theta(k) \rangle\bigg|_{\mu \times \omega}
\]

\[
= \delta_{\mu \omega} \delta_{l \omega} \delta_{l \omega} \delta_{l \omega} [dE_k(L) L] \bigg|_{\mu \times \omega}
\]

where the factor \( dE_k(L) \) denotes the dimensionality of the \( k \)-particle symmetric group.
representation \([\mathcal{X}_{\lambda}](x) = \left[ x_{i1} + p_{x1} + x_{i2} + p_{x2} \right] \) with \( x_{i1} + 2p_{x1} + 3y = k \). This symmetric group dimensionality factor enters because we have a fermion realization of the \( U(3) \) symmetry. That is, associated with \( U(3) \) is a complementary \( U(N/3) \) symmetry, if \( N \) is the degeneracy of the single-particle space, with the full group structure being \( U(N) \to U(N/3) \times U(3) \). An \( \mathcal{X}_{\lambda} \) symmetric \( k \)-particle state vector in \( U(N) \) is formed by summing over the \( \mathcal{X}_{\lambda} \) standard arrangements of \( k \) particles in the symmetric group irrep \( \mathcal{X}_{\lambda} \) associated with the \( U(3) \) symmetry multiplied by its conjugate \( \mathcal{X}_{\lambda}^* \) which is associated with the corresponding \( U(N/3) \) symmetry. In eq. (16) and (17) the tripled-barred matrix element signifies reduction with respect to both \( SU(3) \) and \( RG \). It should be clear from the construction that for a particular value of \( k \) these density operators form a complete set. In particular, if \( k \) is the particle rank of \( SU(3) \to R(3) \), tensorial character of a 
 operator is known, it has a complete representation in terms of the corresponding subset of the \( k \)-particle coupled density operators,

\[
\langle \mathcal{X}_{\lambda_{1}}(x_{1}), \ldots, \mathcal{X}_{\lambda_{k}}(x_{k}) | \mathcal{X}_{\lambda}(y) \rangle \propto \sum_{\mathcal{X}_{\lambda}(y)} \langle \mathcal{X}_{\lambda_{1}}(x_{1}) | \mathcal{X}_{\lambda_{1}}(x_{1}) \rangle \ldots \langle \mathcal{X}_{\lambda_{k}}(x_{k}) | \mathcal{X}_{\lambda_{k}}(x_{k}) \rangle \mathcal{X}_{\lambda}(y).
\]

When a density operator acts outside of its defining space it measures the number of ways the configuration it represents is realized in that larger space. This can be determined explicitly from the definitions given above and by using the available \( SU(3) \) coupling-recoupling technologies. Specifically, by uncoupling the vector product \( \psi \otimes \psi \) in \( \psi \), inserting between the uncoupled \( \psi \) and \( \psi \) a complete set of \((n-k)\)-particle states, coupling the \( k \) and \((n-k)\)-particle vectors to form \( n \)-

particle ones, and summing the resulting product of three \( SU(3) \) Wigner coefficients to produce the one \( SU(3) \) Racah and one \( SU(3) \) Wigner coefficient, one can show that the reduced matrix element of eq. (17) has the value

\[
\langle \mathcal{X}_{\lambda_{1}}(x_{1}), \ldots, \mathcal{X}_{\lambda_{k}}(x_{k}) | \mathcal{X}_{\lambda}(y) \rangle \propto \sum_{\mathcal{X}_{\lambda}(y)} \langle \mathcal{X}_{\lambda_{1}}(x_{1}) | \mathcal{X}_{\lambda_{1}}(x_{1}) \rangle \ldots \langle \mathcal{X}_{\lambda_{k}}(x_{k}) | \mathcal{X}_{\lambda_{k}}(x_{k}) \rangle \mathcal{X}_{\lambda}(y).
\]

In eq. (19) \( U \) is an \( SU(3) \) Racah coefficient and the sum is over all allowed \( x = (m-k) \) particle \( SU(3) \) symmetries and associated multiplicities such that \( (\lambda_{1}\mu_{1}) \times (\lambda_{2}\mu_{2}) \to (\lambda_{3}\mu_{3}) \) and \( (\lambda_{2}\mu_{2}) \times (\lambda_{3}\mu_{3}) \to (\lambda_{4}\mu_{4}) \). In all the above, it is tacitly assumed that the \( U(N/3) \) symmetry of \( \rho \) is scalar. Eq. (16) together with eq. (19) suffices for determining matrix elements of coupled \( U(3) \to R(3) \) density operators in arbitrary \( n \)-particle spaces.
5. Integrity-basis monomial series for density operators

In general, if \( H \) is an operator of maximum particle rank \( k \) which conserves \( G \) and is a \( K \) scalar, then in the fundamental space it can be expanded as a \( G \rightarrow K \) integrity-basis monomial function of degree \( k \).

\[
H = \sum_{0 \leq s < k} \sum_{i=1}^{m} \lambda(i) L(i)^s \otimes O_i \tag{20}
\]

Here \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s) \) and \( k = (k_1, k_2, \ldots, k_s) \); so the restriction on \( s \) and \( k \) is the constraint relation of eq. (11). For the \( U(3) \rightarrow R(3) \) algebra, \( s = 6 \) and \( k = (1, 2, 2, 3, 3, 4) \) corresponding to a set of integrity-basis scalars \( (O_1) = 1, 2, 2, 3, 3, 4 \) corresponding to a set of integrity-basis scalars \( (O_2) = 0, C_0, E, C_3, X_0, X_4 \). The number of sets \( \lambda \) for any specific \( k \) is just the running total of the numbers in column 4 of table 1; that is,
1, 2, 5, 10, 15, 41,..., respectively, for $k = 0, 1, 2, 3, 4, 5,...$. This is the number of $k$-coefficients, eq. (20), required for a complete specification of $\rho$ in the $G \rightarrow K$ model space. A catalog of scalar operators is given for $k \leq 4$ in table 2.

An interaction can also be specified by giving its defining space matrix elements, which is equivalent to giving its expansion in terms of density operators, see eq. (12). The advantage of the integrability basis method, eq. (20), is that the evaluation of the matrix elements of $H$ when it acts outside its defining space is greatly simplified, reducing essentially to the evaluation of matrix elements of the required nomenclature scalars. As both sets of operators are complete, one can expand each in terms of the other. The expansion coefficients for the $\tau$ in terms of integrability basis scalars $u = \sigma u$ is given in table 3 and the inverse $\sigma = \tau^{-1} \rho$ in table 4 for $k \leq 4$ which, by the above counting, forms a $19 \times 19$ system. The transformation matrices $m$ and $m^{-1}$, were determined by assuming an expansion of one set in terms of the other and then solving the set of simultaneous equations obtained upon evaluating matrix elements in the defining space.

The structure of the transformation matrix reflects the particle-number cutoff. It is not block diagonal in $k$ for each $\rho$ involves, in general, a sum over all multinomials of degree less than or equal to $k$. The same is true for the inverse transformation, that is, each $\rho$ can be represented in terms of a sum over density operators of particle rank less than or equal to $k$. Note that we have used (1) in place of $n^k$ for this is a more natural form. It is the propagator of $k$-particle information when the $k$-particle matrix elements all have the same value $1/k$. It measures the number of $k$-particle clusters that can be realized in an $n$-particle space. Indeed, this is the origin of the binomial factor in eq. (49). The other factors is that expression serve to accommodate subtleties of the group geometry.

Of the 19 multinomial scalars for $k \leq 4$ theory only $X_1$, $nX_2$, and $X_4$, labeled $z_{10}(-1)$, $z_{11}$, and $z_{12}$, respectively, in the tables, have nonsimple matrix elements; see eqs. (9) and (10). From table 3 we see that $z_{10}(-1)$ occurs in the expansion of $\rho_{13} = X_1$, all of which have $SU(3)$ tensor character (22), and $z_{11}$ occurs in the expansion of $\rho_{12} = X_2$, with the tensor character of $\rho_{12}$ being (44). So a total of 7 of the 19 density operators have nonsimple parts. It is interesting to note that the density operator of highest $SU(3)$ tensorial character, $\rho_{13}$ with $\Lambda(x_{13}) = (44)$, expands in terms of $\rho_{13} = X_1$, $\rho_{12} = X_2$, $C_{12}$, $C_{12}^*$, $C_{12}E$, $C_{12}^*E$, and $X_4$. On the other hand, the $k = 4$ density operators with $\Lambda(x_{13}) = (22)$ all involve both $X_1$ and $X_2$.

Turning to table 1, where the $\rho_5$'s are expanded in terms of the $\rho_5$'s, one finds $\Lambda(x_{13}) = (00)$ and (22) for the tensorial character of $X_0$, $nX_2$, and $X_4$. These nonsimple scalars do not have $\Lambda(x_{13}) = (44)$ tensor parts; the (44) tensor enters only for $E^2$. Note that, as expected, the tensor character of any combination of $U(3)$ invariants is (00).
<table>
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The $\lambda$s are given by $\lambda = \sum \alpha_i c_i$. See Table 2 for the structure of the operators $\alpha_i$ and $c_i$.

Now consider what happens when U(1) is embedded in a higher group structure. For example, in the $d$ shell the fundamental representation of U(1) labelled [1] is realized as the symmetrically coupled tensor product of two SU(3) representations, labelled [20]. The 36 generators of U(3) formed by coupling the fundamental representation with its adjoint breaks up into SU(3) tensors of trivial character [20], [11] [generators of the SU(3) subgroup of U(6)], and [00] [the number operator]. Clearly one can realize an SU(3) → R(3) symmetry basis set in terms of the generators of this SU(3) subgroup of U(6). Just as in the p-shell one the $X_s$ and $X_a$ operators so formed would be of maximum particle ranks 3 and 4, respectively. The same thing occurs for higher shells under the mapping [1] → [N]. The particle rank of each realization of the SU(3) → R(3) set of symmetry-basis operators is the same. In particular, the model interaction

$$H = ax^2 + bx + cx + d(x^2),$$

(21)
which has been shown to be highly successful in several studies, including dst, 5p, and gds-shell applications, has three- and four-body parts \( \sigma^{(3)} \).

This situation changes under basis truncation by group symmetries. There is then the possibility that the action of an integrity-basis operator can be realized more economically, that is, by an operator of lower particle rank. This possibility arises whenever the tensorial character of the operator under consideration is the same as some other generic operator of lesser particle rank that also acts in the model space. This is not in conflict with a standard effective interaction theory for which the particle rank is raised as a result of an expansion in terms of powers of the original operator. Because the truncation is by group symmetries, one can use the tensorial character of an operator as well as its particle rank in seeking to duplicate its action in a subspace of the full space. Below we will show how this occurs for the extreme case when a d-shell model space is truncated to a single representation of \( \text{SU}(3) \). A less severe truncation would be, for example, to
consider the set of SU(3) representations belonging to a single representation of the higher symmetry group, e.g. U(6) in the di-shell case. That will be considered separately as part of another study.

Interest in the truncation to a single representation of SU(3) derives from the fact that theoretical calculations of spectra and both inter- and intra-band E2 transition rates for 24Mg and various rare earth and actinide nuclei show that such a truncation, though very severe, is meaningful. 20, 21 Also, it should help to establish the fact that using an integrity-basis interaction, in particular the H of eq. (21), is not just parameter-fitting with phenomenological forces of high particle rank but rather is in reality a substitution of simpler operators of higher particle rank for more complicated ones of lower rank, with the SU(3) = R(3) tensorial properties, and hence their action in the model space, being preserved.

Within a single representation of SU(3) the set of 16 k ≤ 4 integrity basis multireflections really reduce to 4 independent ones, i.e. X1, X2, X3, and (7)3. This accounts for the form chosen for the H of eq. (20). The others are either those

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<td>1 2 3 5</td>
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The Ωk are given where k = \sum m_j^3/2. See Table 2 for the structure of the operators \( Ω \) and \( Ω_k \).
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The tensors of the 4 noninvariant scalar operators have already been identified, apart from a trivial scalar part, as three independent (22) tensors and one of rank (44); see table 4. Whereas in the \(s\)-shell these can only be squared by including all particle ranks \(k \leq 4\), in the \(d\) and higher shells there are at least three of independent (22) and (44) tensors at the \(k \leq 2\) level. 

As a specific example, consider \(\rho_{14}\), which table 3 shows can be expanded in terms of \(C_{2}\), \(C_{2}^{2}\), \(C_{2}^{3}\), \(C_{2}^{4}\), \(C_{2}^{5}\), and \(C_{2}^{6}\). In \(SU(3)\) tensorial character is \([40] \times [40] \rightarrow [40] \times [40] \rightarrow [40] \times [40] \times [40] \). Within a single representation of \(SU(3)\), matrix elements of the latter only differ by a multiplicative factor from those of \(\rho_{14}\). Hence the four-body operator \(\rho_{14}\) duplicates the action of a more complicated...
2-body d-shell tensor when the latter acts within a single representation of SU(3). This suggests that when truncation is achieved using group symmetries, particle-rank and tensorial considerations should be complementary ingredients of a renormalization theory. In particular, the use of high-rank particle operators may be a matter of convenience and not necessity. Perhaps the simplest example of this is an expansion of the energy as a power series in $\ell^2$. As for $\rho_{3\lambda}$ above, terms through $\ell^3$ can be realized by a (perhaps complicated) 2-body d-shell interaction if its action is restricted to a single irrep of SU(3). When truncating by group symmetries, particle-rank considerations are only a part of the story.

### Table

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6. Typical results for $\ell \leq 4$ interactions

To illustrate typical results that can be achieved with a $\ell \leq 4$, U(3) → R(3) integrity basis interaction we diagonalized the $\mathcal{H}$ of eq. (21) in the $\ell = 0, 1, 2$ irrep of SU(3). A nonlinear least-squares routine was used to determine best-fit values for the parameters $a, b, c, d$ of $\mathcal{H}$ for five different spectra. Results for the last, a fit to low-lying levels ($\ell < 14$ MeV) of $^{24}$Mg, are shown in fig. 1.
The spectrum of $\ell^2$ and $(\ell^2)^2$ are of course well known. In terms of a traditional rotor description, the coefficient $a$ of $\ell^2$ is just $1/2I$ where $I$ is the moment of inertia. For such an $H$, a plot of $E$ versus $(\ell+1)$ is a straight line with slope $a$. Adding a $(\ell^2)^2$ term to $H$ produces an $E$ versus $(\ell^2+1)$ curve which for $d$ positive (negative) bends upwards (downwards) from the linear form. This can be interpreted as a centrifugal stretching (intensifying) phenomena. With no $X_1$ and $X_2$ dependence in $H$, all states of a given $I$ are degenerate. To distinguish between $K$-bands $h$ and/or $c$ must be nonzero.

To test the ability of the model to accommodate $K$-band shifts, we carried out two "theoretical" experiments. Starting with the simple rotor spectrum produced by $H = d\ell^2$ in the $(\ell\mu) = (84)$ irrep of SU(3) with $a = \frac{1}{2}$ MeV, we artificially induced a linear dependence on $K$, $H = \ell^2 \pm |K|$, so $\Delta K = E_I - E_I' = \pm |K|$ and $\Delta K = E_I - E_{I-1} = \pm 2$ MeV. A least-squares fit was then generated to the linearly shifted spectra of $H$ using the $H$ of eq. (11). The values in MeV of the parameters $(a, b, c, d)$ of $H$ as determined by the fitting procedure are $(0.2184, 0.1192, 0.004600, 0.0000002599, 0.000001878)$ for the $\Delta K = \pm |K|$ cases, respectively. As $0.2184 = \frac{1}{4} + 0.0517$ and $0.1150 = \frac{1}{4} - 0.0517$, one
not only has that the $h$, $c$ and $d$ parameters for the two cases are equal in magnitude but opposite in sign, but so are changes, $\delta a$, in the inertia parameter. Furthermore, in each case the calculated $\chi^2$ deviation from the ideal shifted spectrum was determined to be the same ($\approx 2.2$ MeV$^2$). In fitting the spectra, a weight of ten was assigned to the two bandheads as compared to one for all the other levels.

We then repeated the calculation fitting to an ideal shifted spectrum with $dK = 2\Delta K^2$. That is, we artificially introduced a quadratic dependence on $K$, $H = |\frac{1}{2}d| + \Delta K^2$, so $\Delta K_1 = \pm 1$ MeV as before but $\Delta K_2 = \pm 4$ MeV. The values for the parameters ($a$,$b$,$c$,$d$) of $H$ determined by fitting to the spectra of $H'$ are $0.1646$, $0.1688$, $0.007697$, $0.0006524$, $0.0002394$ for the $\Delta K > 0$ and $\Delta K < 0$ cases, respectively. As $0.1646 = \frac{1}{3} - 0.0021$ and $0.1688 = \frac{1}{3} + 0.0021$, one again has that the $a$, $b$, $c$ and $d$ parameters are equal in magnitude and opposite in sign for the two cases. The $\chi^2$ ($\approx 1.8$ MeV$^2$) were again determined to be equal. Note,
however, that this \( g^2 \) value is an order of magnitude smaller than the previous one. The same weighing was applied in each case.

The results of these calculations suggest that the \( X_3 \) and \( X_4 \) interactions are equally effective in producing both normal (\( \Delta K = 0 \)) and inverted (\( \Delta K = 1 \)) orderings of \( K \)-band rotational sequences. These are, however, some important differences in details of the cases considered. As already noted, a lower \( g^2 \) can be achieved if a spectrum displays a quadratic dependence on \( K \) rather than a linear one. Perhaps not surprisingly, a \( K^2 \) dependence in \( H \) can be accommodated with very little distortion of the rotational spectrum. This is not so for a linear \( K \)-dependence. In particular, for the \( \nu = 1 \) case considered we found that the \( K = 2 \) band, \( \ell = 6 \) and \( \ell = 7 \) states moved together (apart) for \( \Delta K < 0 \). Requiring a linear \( \Delta K \) shift in excess of about \( +2 \) MeV was found to actually produce an inversion of those levels. Such effects should be expected. They can be rectified but the solution may not reside within the parameter domain of a \( k \leq 4 \) interaction.

Best-fit results for the \( ^{24}\text{Mg} \) case using a \( k \leq 4 \) single irreducible theory are shown on the far right in fig. 1. Notice how accurately theory reproduces experiment. A weight of \( 5 \) was assigned to the two \( L = 2 \) states as compared to one for all the other levels. This was done to emphasize the monoch of inertia and the \( K \)-band splitting features of the \( ^{24}\text{Mg} \) spectrum. The best-fit parameter values are \((a, b, c, d) = (0.02519, -0.02352, -0.001662, -0.00004380)\). This set is most similar to the \( \Delta K = \pm 1 \) results reported above. The ratio of the coefficients multiplying the \( X_1 \) and \( X_3 \) terms in \( H \) is the pertinent measure. The \( a/c \) ratio is about \(-12 \) for the \( \Delta K = \pm 1 \) case and \(-88 \) for \( \Delta K = \pm 3 \) \( ^{24}\text{Mg} \). The ratio is about \(-14 \).

The fact that the magnitude of \( b \) is about three times greater than for the \( \Delta K = \pm 1 \) case is also consistent for the \( ^{24}\text{Mg} \) \( K \)-band splitting is \( \approx 7 \) MeV. Additional evidence indicating that the \( K \)-dependence of the effective \( H \) for \( ^{24}\text{Mg} \) is dominated by a quadratic term is that a level-distortion problem does not occur in the calculated results even though \( \Delta K \) is large compared to \( a \). Note that theory does not reproduce the deviation of the \( K = 2 \) band \( 3^1 \) and \( 4^1 \) states from the rotational \( (2\ell + 1) \) sequence. To reproduce such detail would require higher-degree polynomial terms in the model \( H \). Such structure lies beyond that available within a single irrep of \( SU(3) \) using a simple \( k \leq 4 \) integrity basis.

The \( \text{Sp}(3,R) \) calculation for \( ^{24}\text{Mg} \) (recorded in ref. 2) also used a \( k \leq 4 \) intensity basis \( H \) in the \((\mu = 0)\) on-shell irrep. The best-fit strengths for the intensity-basis operators \( X_1, X_3 \), and \( (L^2)^2 \) in the two cases are very similar. (Our definition of \( X_3 \) differs by a constant multiplicative factor from the one used in the \( \text{Sp}(3,R) \) calculation. To compare coefficients the \( a_3 \) of ref. 2) must be multiplied by \( \sqrt{10} \). The normalizations of \( L^2, X_3 \), and \( (L^2)^2 \) are the same for both.) The principle difference between the two is found in the moment-of-inertia parameter, which has a value of 0.255 here and 0.194 in the \( \text{Sp}(3,R) \) calculation. The reduction is understandable because although both models use the single
d-shell imp $(\nu)$ = (84), the Sp(3, R) calculation incorporates couplings to core-excited configurations as dictated by the quadrupole degree of freedom of the geometrical Bohr-Mottelson model. The representation mixing this generates allows one to achieve proper β(E2) values for both intranuclear and intraband transitions without introducing an effective charge or compromising on the quality of fit obtained with the $k \leq 4$ integrity basis interaction in the 60o space. This mixing is generated by the $a_4 \rightarrow \beta^2$, $a_3 \rightarrow \beta^2 \cos \gamma$ and $(a_2)^2 \rightarrow \beta^2$ terms of the Sp(3, R) hamiltonian.

7. Conclusion

General results have been given for $\hat{W}_x$ elements of the U(3) - R(3) integrity basis operators. They suffice for evaluating the matrix elements of any rotationally invariant U(3)-preserving interaction. In particular, we have given detailed results for the $k \leq 4$ density operators expanded as multilinear functions of the integrity basis set. The inverse of that expansion, which has also been given, allows one to identify the tensorial character of the integrity basis operators. Together, the results illustrate the economy of integrity basis forms. In the U(3) - R(3) case for example, there are but two nontrivial $k \leq 4$ integrity basis operators whereas in density matrix form there appears to be seven.

The theory was used to illustrate how K-band splitting can be achieved within a single representation of SU(3) with a $k \leq 4$ interaction. This includes both normal ($\Delta K > 0$) and inverted ($\Delta K < 0$) K-band orderings. However, as shown for the $(\nu) = (84)$ case, a linear shifting of K-bands is normally accompanied by a distortion of the ordering of L-states from that of a simple rotor. When this happens, structure beyond that available using a $k \leq 4$ interaction in a single representation of SU(3) will probably be required. For $^{24}$Mg excellent results are achieved using a $k \leq 4$ even though the $K = 2$ bandhead lies nearly 3 MeV above the $K = 0, I = 2$ level (interax parameter $\alpha = 1.21, \beta = 0.23$ MeV). Success in the $^{24}$Mg case can be traced to a quadratic dependence on $\beta$ in the effective $H$.

We used the $k \leq 4$ expansions to illustrate how focusing on the particle rank of an interaction to the exclusion of tensorial considerations can yield very misleading conclusions when a space is partitioned and truncated by group symmetries. In particular, though it may be convenient to introduce operators of high particle rank to achieve relatively simple effects, e.g. $(L^2)^2$ which has 4-body parts in a subspace of the full space such an operator can often be realized more economically, i.e. in terms of operators of lower particle rank. As an example, we noted that within a single d-shell representation of SU(3) an L^2 behavior can be realized with a $k \leq 2$ body interaction. Though this will be considered in greater depth elsewhere, it suggests the need for a symmetry adapted effective interaction
theory in which the tensorial character of an operator is treated on par with its par matrix.

From this development it should also be clear that the $k$ vector of a $G \rightarrow K$

*group* structure is of fundamental importance. The dimensionality $\nu$ of $K$ is
determined by the number of integrity-basis operators and the value of its
components specify their degrees. And the counting associated with the
$0 \leq \nu \leq K$ constraint produces a proper cataloging of the essential pieces of
information in a $\nu$th-degree theory using the $G \rightarrow K$ geometry. As $K$ is fixed by the
*group* structure, this dot product constraint can also be viewed as defining a region
in the $\nu$-dimensional $K$-space within which the interaction resides.

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