Symmetry algebra of the anisotropic harmonic oscillator with commensurate frequencies

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Abstract: The symmetry algebra of the 3-dimensional quantum anisotropic harmonic oscillator Hamiltonian $H$ with commensurate frequencies is shown to be $sl(3)$. Each singlet eigenstate of $H$ carries a single irreducible representation of $sl(3)$. However, distinct eigenstates can yield equivalent $sl(3)$ representations. The dynamical algebra is the non-compact symmetric algebra $sl(3,R)$. For $n=1$, the anisotropic Hamiltonian is relevant to deformed nuclei.

1. Introduction

The three-dimensional anisotropic harmonic oscillator defines the intrinsic state of rotating deformed nuclei in the Nilsson model [1]. For the recently discovered superdeformed nuclear states [2-5], the ratio of the axis length of the nuclear quadrupole sphere is approximately 2:1. Hence, the oscillator frequencies $\omega_x: \omega_y: \omega_z$ are in the ratio 2:2:1. In this paper, the symmetry algebra of the 3-dimensional anisotropic harmonic oscillator Hamiltonian with commensurate frequencies is shown to be $sl(3)$. However, in contrast to the isotropic oscillator, $sl(3)$ irreducible representations occur multiply in the anisotropic case. For example, the 3:1 superdeformed prolate sphere yields a doubling of the $sl(3)$ irreducible representations. The symmetry algebra for the two-dimensional case was determined by Juhasz and Hie [9] in the classical picture, and by Denkov [10] and Loutre, Mashinsky and Wolf [11] in the quantum picture.

2. Symmetry algebra

Let $a_i$ and $a_i^\dagger$ denote the usual oscillator creation and annihilation bosons in the $i$th Cartesian direction,

$$[a_i, a_j^\dagger] = \delta_{ij}.$$  \hspace{1cm} (1)

Set the number operator $\hat{n} = a_i a_i^\dagger$. Consider the $m$-dimensional anisotropic oscillator Hamiltonian with commensurate frequencies,

$$H = \sum_{i=1}^m \frac{\hat{p}_i^2}{2m} + \sum_{i=1}^m \omega_i \hat{n}_i$$  \hspace{1cm} (2)

where $\omega_i$ are positive integers.

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A typical symmetry operator for $H$ annihilates $p$ quanta in the $\ell$th direction and creates $p$ quanta in the $\ell$th direction. However, although the operators $D_0 \equiv (a_0^\dagger)^p$ commute with the anisotropic oscillator Hamiltonian, they do not close under commutation. Indeed, the commutator $[D_0, D_0]_+ \equiv a^{\dagger}_{0}$ is another operator of degree $|p_+ + p_- + p_0 - 2|$. Except for the isotropic special case ($p_+ = p_- = 1$ and/or $p_0 = 1$), successive commutators produce new polynomial symmetry operators of increasingly higher degree in the bosons. Therefore, the symmetry algebra generated by the $D_\ell$ is infinite-dimensional unless all $p_\ell = 1$, i.e., the Hamiltonian is isotropic and the symmetry algebra is $u(m)$.

In order to achieve a finite-dimensional symmetry algebra, it is necessary to define new bosons appropriate to the anisotropic oscillator. Consider first the one-dimensional case with the usual oscillator bosons $a^\dagger$ and $a$. Define the operators

$$a^\dagger = \tilde{a}^{-1/2} a^\dagger,$$

$$a = (\tilde{a} + 1)^{-1/2} a$$

with $\tilde{a} = a a^\dagger$.

When acting upon the oscillator state $|n\rangle$

$$a^\dagger |n\rangle = |n + 1\rangle,$$

$$a |n\rangle = \begin{cases} |n - 1\rangle & n > 0 \\ 0 & n = 0 \end{cases}$$

For each positive integer $p$, define the number operator modulo $p$ by

$$[\hat{N}/p]|n\rangle = [n/p]|n\rangle$$

where $[n/p]$ denotes the whole integer part of the ratio $n/p$. Define the $p$-step creation and annihilation operators

$$A(p)^\dagger = (\hat{N}/p)^{1/2} (a^\dagger)^p$$

$$A(p) = a^\dagger (\hat{N}/p)^{1/2}$$

These multiboson operators have been applied to squeezed photon states [12, 13]. Their action on the orbital-angular oscillator basis is given by

$$A(p)^\dagger |n\rangle = ([n/p] + 1)^{1/2} |n + p\rangle$$

$$A(p) |n\rangle = [\sqrt{n/p}]^{1/2} |n - p\rangle$$

Hence, $A(p)^\dagger$ creates $p$-quanta and $A(p)$ destroys $p$-quanta,

$$[\hat{N}, A(p)^\dagger] = p A(p)^\dagger$$

$$[\hat{N}, A(p)] = -p A(p).$$

The modulo-$p$ number operator is given by

$$[\hat{N}/p] = A(p)^\dagger A(p).$$

The crucial result for this paper is that the $p$-step creation and annihilation operators satisfy boson commutation relations

$$[A(p), A(p)^\dagger] = 1.$$
Symmetry algebra of the harmonic oscillator

It is interesting to note that, if the $p$-step operators were to be defined with the simple ratio $A/p$ instead of the whole number operator $[A/p]$, then the redefined operators would satisfy the boson commutation relation (9b) everywhere except when acting upon the first $p$ vectors $|n\rangle$, $0 < n < p$.

Returning now to the $m$-dimensional problem, it is evident that the $m^2$ operators

$$C_j = A_j(p^j)A_j(p^j) + A_j,$$  

(11)

commute with the anisotropic oscillator Hamiltonian, where $A_j(p)\dagger$ and $A_j(p)$ denote the $p_j$-step creation and destruction boson operators acting upon the $j$th Cartesian coordinate.

Since the $p_j$-step operators are bosons, the symmetry operators $C_j$ span the unitary Lie algebra $u(m)$,

$$[C_j, C_k] = \delta_{jk}C_j + \delta_{jk}C_k.$$  

(12)

3. Highest weight vectors

A $u(m)$ highest weight vector in $P^0(\mathbb{R}^m)$ must simultaneously be an eigenvector of $C_j$, $1 \leq i \leq m$, and be annihilated by the $m(m - 1)/2$ raising operators $C_{ij}$, $1 < i < j \leq m$. Thus, a highest weight vector is given by placing all excess quanta in the first Cartesian direction.

$$|n, (q)\rangle = |n_1, q_1, 0, \ldots, 0\rangle$$  

(13)

for $n = 0, 1, 2, \ldots$ and $(q) = (q_1, q_2, \ldots, q_m)$ is a sequence of $m$ integers with $0 < q_i < p_i$.

Clearly, $C_j|n, (q)\rangle = 0$ for $i < j$, since then there are at most $(p_j - 1)$ quanta in each of the $j$ directions, $1 < i < m$. Moreover, $|n, (q)\rangle$ is a simultaneous eigenvector of the Cartan operators

$$C_j|n, (q)\rangle = n_j\omega\langle n, (q)|.$$  

(14)

Therefore, the irreducible $u(m)$ representation generated from the highest weight vector $|n, (q)\rangle$ has the weight $(n, 0, \ldots, 0)$. Furthermore, each such irreducible $u(m)$ representation occurs with multiplicity $\Pi_{j \neq \lambda} p_j$.

Is $u(m)$ the maximal symmetry algebra? If all the integers $p_i$ are relatively prime, $(p_i, p_j) = 1$ for every pair $i \neq j$, then $u(m)$ is maximal and each eigenspace of the anisotropic oscillator spans a single irreducible representation. To demonstrate this, suppose by contrast that an eigenspace of $H$ is the direct sum of two or more irreducible representations of $u(m)$. Each such irreducible subspace contains a highest weight vector, which must be of the form $|n, (q)\rangle$. If two such highest weight vectors, $|n, (q)\rangle$ and $|n', (q')\rangle$ are from the same eigenspace of the anisotropic oscillator, then

$$n' + \sum_{j \neq \lambda} q_j/p_j = n + \sum_{j \neq \lambda} q_j/p_j.$$  

(15)

If both sides of this equation are multiplied by $\Pi_{j \neq \lambda} p_j$, then

$$(n' - n) \prod_{j \neq \lambda} p_j + \sum_{j \neq \lambda} (q_j - q_j) \prod_{k \neq j} p_k = \frac{(n' - n) \prod_{j \neq \lambda} p_j}{p_j}.$$  

(16)

Since the $p_j$ are relatively prime, $p_j$ does not divide $\Pi_{j \neq \lambda} p_j$. Hence, although the left-hand side of (16) is an integer, the right-hand side is not integral unless $q_j - q_j = 0$. Thus, $(q_j - q_j) = 0$ and, by (15), $n' = n$. 


4. Superdeformed nuclei

Because of its application to superdeformed prolate nuclei, the particular case of

$$H = \hat{a}_1 + \hat{a}_1^+ \hat{d}_0$$

(17)

is interesting to evaluate explicitly. Here the $\rho_i$ are relatively prime and the maximal symmetry algebra is $su(3)$. The new boson destruction operators are given in terms of the usual oscillator bosons by

$$\hat{A}_1 = a_1$$
$$\hat{A}_2 = a_2$$
$$\hat{A}_3 = a_1 \hat{a}_1^{3/2} a_2 \hat{a}_2^{-1/2} [\hat{d}_0 / 2]^{1/2}.$$ 

(18)

The highest weight vectors are just

$$|a\rangle \otimes |0\rangle \otimes |0\rangle$$

$$|a\rangle \otimes |0\rangle \otimes |1\rangle$$

(19)

for $n = 0, 1, 2, \ldots$; the corresponding su(3) irreducible representations have identical weights $(\Lambda, \mu) = (n, 0)$. In figure 1, the su(3) character of the anisotropic oscillator energy levels is indicated. In general, the decomposition of the anisotropic $(n, 0)$ irreducible representation as the vector span of isotropic states is

$$(n, 0) = \text{span} \{ \text{isotropic} \ (n+1, 0, \ldots) \} \quad E = n$$

$$(n, 0) = \text{span} \{ \text{isotropic} \ (n+1, 0, \ldots) \} \quad E = n + 1/2$$

(20)

where the subscript on the isotropic $su(3)$ representation indicates the number of vectors contributing to the anisotropic level $(n, 0)$. Hence, the number of spanning isotropic vectors at the level $E = n + 1/2$

$$\frac{1}{2} \left( n + 1 - i \right) = \left( n + 1 \right) \left( n + 2 \right)$$

(21)

**Figure 1.** The energy spectrum of the anisotropic three-dimensional harmonic oscillator $H = \hat{a}_1 + \hat{a}_1^+ \hat{d}_0 / 2$ is drawn. The levels are labeled by their $su(3)$ quantum numbers $(\Lambda, \mu)$. Their pairing in terms of isotropic oscillator states is indicated by the heavy lines.
which is the familiar dimension formula for the su(3) representation \((a, 0)\). Note that each anisotropic level is a mixture of even and odd parity states. If \(E = n\), then the number of positive parity states is \(((n/2) + 1)^2\). If \(E = n + 1\), then the number of negative parity states is \(((n/2) + 1)^2\).

5. Dynamical algebra

The dynamical symmetry algebra for the anisotropic oscillator with commensurate frequencies whose reciprocal are relatively prime is the non-compact real symplectic Lie algebra \(sp(m, R)\). In addition to the maximal compact subalgebra \(u(m)\), the real symplectic algebra is generated by the raising and lowering operators

\[
\alpha_y = [A_y, A_y^\dagger, p_y],
\]

\[
\beta_y = [A_y, A_y^\dagger, p_y].
\]

(22)

Every irreducible unitary representation of \(sp(m, R)\) is infinite-dimensional. They are generated by the successive application of \(\alpha_y\) to the \(u(m)\) representation spaces whose highest weight vectors are annihilated by the lowering operators \(\beta_y\), namely

\[
|n = 0, \{q\}\rangle
\]

\[
|n = 1, \{q\}\rangle.
\]

(23)

Therefore, the entire infinite-dimensional space \(L^2(\mathbb{R}^m)\) is the direct sum of two irreducible unitary representations of \(sp(m, R)\) generated from the \(u(m)\) representations \((0, 0, \ldots, 0)\) and \((1, 0, \ldots, 0)\), each of which occurs with multiplicity \(2n\).

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References