Shape Variables and the Shell Model

O. Castaños **, J.P. Draayer, and Y. Leschber
Department of Physics and Astronomy, Louisiana State University,
Baton Rouge, Louisiana, USA

Received September 1, 1987

The irreducible representation labels $\lambda$ and $\mu$ of the SU(3) shell model are related to the shape variables $\beta$ and $\gamma$ of the collective model by invoking a linear mapping between eigenvalues of invariant operators of the two theories. All but one parameter of the theory is fixed if the shell-model result is required to reproduce the collective-model geometry. And for one special value of the remaining free parameter there is a simple linear relationship between the eigenvalues, $\lambda_\nu$, of the quadrupole matrix of the collective model and the SU(3) representation labels:

\[
\lambda_1 = (-2 + \mu)/3, \quad \lambda_2 = -(2 + 2\mu + 3)/3, \quad \lambda_3 = (2 + \mu + 3)/3.
\]

The correspondence between hamiltonians that describe rotations in each theory is also given. Results are shown for two cases, $^{24}\text{Mg}$ and $^{184}\text{Er}$, to demonstrate that the simplest mapping yields excellent results for both energies and transition rates. For $\lambda$ and/or $\mu$ large, the $|3/2>\leftrightarrow|1/2>$ correspondence introduced here reduces to the simple shell-model result.

PACS: 21.60. - n; 02.20. + b

1. Introduction

The purpose of this paper is to give a shell-model interpretation to the shape variables of the collective model. This is done by establishing a linear mapping between eigenvalues of the invariant operators of the rotational model, $\text{Tr}(Q^2)$ and $\text{Tr}(Q^3)$, where $Q$ is the mass quadrupole operator, and the eigenvalues of $C_2$ and $C_3$, the second and third order Casimir invariants of SU(3). The latter is the symmetry group of the isotropic harmonic oscillator which is a reasonable zeroth-order approximation to the shell-model potential. Since everything follows from this costruc-

---

* Supported in part by a grant from the U.S. National Science Foundation
** Permanent address: Centro de Estudios Nucleares, UNAM, Apdo. Postal 70-343, México, D.F., México
shows in the next section, in the present case the correspondence is particularly simple: the dynamical symmetry group of the quantum rotor and SU(3) are related to one another by group contraction. This means that in a certain well-defined limit there is a linear relationship between the generators of the groups of these two theories. Since the correspondence between the generators of the groups is linear, so is the constraining relation among the eigenvalues of the invariant operators. The latter follows from the former because the invariants operators of the two theories are polynomial functions of like order in the generators of their respective groups. Then it is natural to require that the linear correspondence between the eigenvalues of the invariants of the two theories hold for all the cases, not just in the contraction limit. This matter is taken up in Sect. 3.

Values for the parameters of the mapping is the topic of Sect. 4. In particular, it is shown that four of the five parameters that enter into the theory can be fixed by requiring the shell-model formulation to reproduce the collective-model geometry. Specifically, two of the five parameters must be zero if particle-hole configuration is to correspond so the \( g = \sqrt{3} \) transformation of the collective model. An additional two are fixed by requiring the shell-model results to reproduce the full range of values for \( \lambda, 0 \leq \lambda \leq 1 \). It is also shown that for one special value of the remaining parameter there is a simple linear relationship between the eigenvalues, \( \lambda_0 \), of the collective quadrupole matrix and the SU(3) representation labels \( \lambda \) and \( \mu \). In the Sect. 3 this relationship is used in determining the parameters of a shell-model hamiltonian that reproduces the dynamics of the triaxial quantum rotor. Results for energies of the lowest two bands in \(^{208}\text{Pb}\) and \(^{132}\text{Sn}\) and some interband transition rates are presented to illustrate the effectiveness of the theory.

To reiterate, the requirement that there be a mapping between the parameters of the invariants operators of two theories that are used to describe the same quantum phenomena is simply asking that there be features that are common to all the basis states of one theory with features that are common to all the basis states of the other theory. As already pointed out, this is a very simple and natural requirement that is a derivative of the conservation laws of classical mechanics. In this article this proposition is given a group theoretical interpretation and used to establish a connection between microscopic and macroscopic theories of nuclear rotational motion. As is suggested in the conclusion, it seems reasonable to expect that in other cases where the symmetry groups of two theories are related to one another by contraction one may also gain new insight and understanding of the physics under consideration through similar analyses.

2. Operators of the Theories and their Algebras

The rotational model operators are \( L_x \) and \( Q' \), the angular momentum and mass quadrupole moment, respectively:

\[
L_x = -3/2 \int \rho(r) r \times \phi \phi^* dr,
\]

\[
Q' = \int \rho(r) r^2 Y_2(\Omega) dV,
\]

where \( \rho(r) \) is the nuclear density and the integration is over the whole nuclear volume. The corresponding shell-model quantities are \( L \) and \( Q' \),

\[
\lambda_0 = \sum_{i=1}^{A} \lambda_i \lambda_i',
\]

\[
Q_0' = \frac{4}{3} \sum_{i=1}^{A} r_i Y_2(\Omega_i) = \frac{1}{3} \sum_{i=1}^{A} \gamma_i Y_2(\Omega_i),
\]

where \( A \) denotes the number of nucleons of the system and we use units in which \( h = 1 \), the mass of the nucleon \( m \), and the harmonic oscillator frequency \( \omega \) are all unity. It is important to note that whereas the \( L \)s of the theories are the same, the \( Q' \)s are not. The form chosen for \( Q' \) insures that its action is restricted to a single major shell of the oscillator. (The shell-model equivalent of \( Q' \) has matrix elements coupling major shells that differ by \( \Delta n = \pm 2 \) oscillator quanta.) Within a single major shell of the oscillator the matrix elements of \( Q' \) and \( Q' \) are the same. The form \( Q' \) is used because we are interested in giving a standard (0,0) shell-model interpretation to the shape variables \( \beta \) and \( \gamma \) of the collective model. As has been shown, the effect of shell-mixtures can be accommodated by renormalizing the (0,0) results [2].

The dynamical symmetry group of the quantum rotor is the semidirect product \( T_2 \times SO(3) \), where \( T_2 \) is generated by the five components of \( Q' \) and \( SO(3) \) is the angular momentum group [2]. The corresponding Lie algebra is the semidirect sum \( t_2 + so(3) \) which is defined by the commutation relations

\[
[\gamma_{ij}, \lambda_0] = -\frac{1}{2} \gamma_{ij} \mu^{ij} \lambda_0,
\]

\[
[\gamma_{ij}, Q'] = -\frac{1}{2} \gamma_{ij} \mu^{ij} Q',
\]

\[
[Q', Q'] = 0.
\]

The Casimir operators of a Lie algebra are the independent functions of its generators that commute with
all the generators. For the $t_4 + so(3)$ algebra these invariant operators are

$$\text{Tr}((Q^2)_{12}) = \frac{1}{6}(Q^2 \times Q^2)_0 = \frac{1}{6} Q^4 \cdot Q^2, \quad (2.4 a)$$

$$\text{Tr}((Q^2)_{13}) = -\frac{1}{12}(Q^2 \times Q^2)_0 = 3(Q \cdot Q \times Q^2). \quad (2.4 b)$$

The eigenvalues of these operators are given in terms of the quantum labels $\lambda_1, \lambda_2, \lambda_3$ which characterize the irreducible representations (irreps) of $T_2 \times SO(3)$ by

$$\text{Tr}((Q^2)_{12}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad (2.5 a)$$

$$\text{Tr}((Q^2)_{13}) = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 = 3(\lambda_1 \lambda_2 \lambda_3). \quad (2.5 b)$$

The $\lambda_i$ are expectation values of the quadrupole matrix, $Q^2$, in its body-fixed, principal-axes system: $\langle Q^2 \rangle_{\text{body}} = \lambda_1 \delta_{1,0}, \lambda_2 \delta_{2,0}, \lambda_3 \delta_{3,0}$. The last form (2.5b) follows because by definition, see (2.11), $Q^2$ is traceless, that is, $\lambda_1 + \lambda_2 + \lambda_3 = 0$.

The Lie algebra generated by the $L^2$ and $Q^2$ is $su(3)$ which is given by

$$[L_+^\gamma, L_\pm] = -\frac{1}{2}(1, \mu, 1) \Gamma_{\pm \mu} L_{\pm \mu}, \quad (2.6 a)$$

$$[L^\gamma_+ Q^\gamma - \frac{1}{2}(1, \mu, 2) \Gamma_{\pm \mu} Q^\gamma_\pm, \quad (2.6 b)$$

$$[Q^\gamma_+ Q^\gamma - \frac{1}{2}(1, \mu, 2) \Gamma_{\pm \mu} Q^\gamma_\pm, \quad (2.6 c)$$

The invariant operators of the $su(3)$ algebra are $C_2$ and $C_4$,

$$C_2 = \frac{1}{6}(Q^2 \times Q^2)_0 - \frac{1}{2}(L \times L^2) - \frac{1}{6} Q^4 \cdot Q^2 + \frac{1}{2} L^2, \quad (2.7 a)$$

$$C_4 = \frac{1}{36}(Q^2 \times Q^2)_0 - \frac{1}{4}(L \times L^2)^2 + \frac{15}{4}(L \times L^2)_0. \quad (2.7 b)$$

The eigenvalues of these invariants are given in terms of the irreducible representation (irrep) labels $\lambda$ and $\mu$ of $SU(3)$ by

$$\langle C_{2} \rangle = \frac{1}{2}(\lambda_1 + \lambda_2)^2 + \lambda_1 \lambda_2 + 3(\lambda_1 \lambda_2 \mu_3), \quad (2.8 a)$$

$$\langle C_{4} \rangle = \frac{1}{2}(\lambda_1 + \lambda_2)^2 + 3(\lambda_1 \lambda_2 \mu_3) + 2(2 \lambda_1 + \lambda_2 + \mu_3). \quad (2.8 b)$$

One can see that the $t_4 + so(3)$ algebra is a contraction of the $su(3)$ algebra by noting that for $Q^2_{12}$ large compared to $\langle L^2 \rangle$, (2.6c) reduces to (2.3). Since $Q^2_1 = -6C_2 - 3I$, this is is how large angular momentum states via large dimension representations. This argument can be made mathematically rigorous in the following way [3]. First of all define normalized quadrupole operators: $q^\gamma Q^\gamma (C_j)_\mu$. Now consider the commutation relations of the $L_{\pm \mu}$ and the $Q^\gamma_{\pm \mu}$. Since $C_2$ commutes with the generators of $su(3)$, they automatically satisfy (2.6a) and (2.6b). On the other hand, in (2.6c) the $L$ on the right-hand-side gets replaced by $\langle L^2 \rangle$. So for $C_4$ values that are small compared to $\langle C_0 \rangle$ the commutator goes to zero, that is, (2.6c) reduces to (2.3a). Therefore, one has that $su(3) \rightarrow su(2)$ for $\langle L^2 / \langle C_2 \rangle \rightarrow 0$.

### 3. Shape Variables and Representation Labels

The eigenvalues of the quadrupole matrix of the collective model satisfy the reduced cubic equation [4]

$$l_1 - q l_2 - r = 0, \quad (3.1)$$

where

$$q = \frac{1}{2}\text{Tr}((Q^2)_{12}), \quad (3.2 a)$$

$$r = \frac{1}{2}\text{Tr}((Q^2)_{13}). \quad (3.2 b)$$

The roots of this cubic equation can be given in the form [5]

$$l_1 = -2(\epsilon/3)^2 \cos(\theta/2 - \pi/2) \quad (3.3 a)$$

with $\theta = \pi - \frac{1}{2} \sqrt{q^3 - 3r}$.

The conditions for there to be three real solutions are the following:

$$q \geq 0, \quad (3.4 a)$$

$$q^2 \geq 2r^3. \quad (3.4 b)$$

Inequality (3.4a) is always satisfied.

A parametrization of the principal moments can be given in terms of the shape variables $\beta$ and $\gamma$ introduced by Bohr and Mottelson [6].

$$l_{3} = \frac{1}{2}\sqrt{(4\pi)^2 (3\beta \cos(\pi/2 - \pi/3))}, \quad (3.5)$$

Details on how one arrives at this particular form are given in the Appendix. In (3.5), $A$ is the number of nucleons and $\mu^2$ is the dimensionless mean square radius: $\langle r^2 \rangle = \nu_{cm}^2 \mu^2 A^{1/3}$ where $\nu_{cm} \approx 0.87$, which is an experimentally determined number [7]. For this
parametrization of the nuclear shape the invariants of the rotor are given by
\begin{equation}
\langle \mathcal{T}(\hat{Q}^2) \rangle = (3/8)\pi \alpha \beta^2 \beta^2
\end{equation}
\begin{equation}
\langle \mathcal{T}(\hat{Q}^3) \rangle = (5/16) \mathcal{L}_2 (\mathcal{L}_2^3 + 1/3 \mathcal{L}_1^2) \beta d_2 \cos(3 \theta)
\end{equation}
where the fact that \(1/\sin^2((2n-1/2)\beta) = 1/2\) and
\begin{equation}
1 = \cos^2(\beta) - 2 \cos(\beta) + 1/4 \cos(3 \beta)
\end{equation}
have been used. It is important to note that with this parametrization, inequality (3.4b) is automatically satisfied.

We now turn to a consideration of the consequences of the requirement that the eigenvalues of the invariants of the rotor algebra map onto the eigenvalues of the invariants of the \(s(3)\) algebra. The mapping can be established, in general, by requiring that
\begin{equation}
\langle \mathcal{T}(\hat{Q}^2) \rangle = \langle F(C_2, C_4) \rangle
\end{equation}
\begin{equation}
\langle \mathcal{T}(\hat{Q}^3) \rangle = \langle G(C_2, C_4) \rangle
\end{equation}
where \(F\) and \(G\) are arbitrary functions of the \(s(3)\) Casimir operators, (3.7). However, comparing (3.4) and (3.7) and being guided by the group contraction mechanism, one is led to propose the following linear forms for \(F\) and \(G\):
\begin{equation}
F(C_2, C_4) = a_1 C_2 + a_2 C_4
\end{equation}
\begin{equation}
G(C_2, C_4) = a_1 C_2 + a_2 C_4 + a_3 C_2
\end{equation}
These forms also ensure that the particle rank of the operators is preserved across the equal signs in (3.7). Substituting (3.9) and (3.8) into (3.3) (3.1) one finds that
\begin{equation}
q = 1
\end{equation}
\begin{equation}
r = 1
\end{equation}
\begin{equation}
I = 1
\end{equation}
\begin{equation}
\sin(\beta) = \cos(\beta)
\end{equation}
\begin{equation}
\cos(\beta) = \cos(\beta)
\end{equation}

Note that this transformation (3.4 \(\rightarrow\) 3.5) preserves the equilibrium relationship among the \(a_3 \leq a_2 \leq a_1 \leq I \). To see how these considerations can be used to fix the mapping parameters \(a_1\), \(a_2\), and \(a_3\), one must first recall that since \(\beta = 3\pi/2\), the \(\cos(\beta) = -\sin(\beta)\). Hence, the transformation implies an interchange of \(a_1\) and \(a_2\), and \(a_3\) and \(a_4\). Indeed, this choice makes the transformation consistent with the shell-model operation of particle-hole conjugation. The parameters \(a_1\) and \(a_2\) must satisfy the constraint relation, \(\beta = 3\pi/2\), for all values of \(\lambda\) and \(\mu\). That is,
\begin{equation}
\sin(\beta) = \cos(\beta)
\end{equation}

And finally, comparing (3.9) and (3.10) we arrive at the general relationship between the shape variables \(\beta, \gamma\) and the SU(3) isospin labels \(\lambda, \mu\):
\begin{equation}
\beta = (4L)^{-1/2} (6 \pi/5) a_1 (C_2 + 3z_a C_4)
\end{equation}
\begin{equation}
\gamma = (4L)^{-1/2} (6 \pi/5) a_2 (C_2 + 3z_a C_4)
\end{equation}

4. Parameter Specification

In terms of the collective model has the range \(0 \leq r \leq \infty\). The \(r = 0\) geometry specifies a prolate shape, \(\gamma = \pi/2\) and oblate configuration, and \(\gamma = \pi/2\) the most asymmetric case. The choice \(2 \leq \lambda \leq 3 \lambda\) is made to select one of the \(21\) equivalent geometries that can be realized through a permutation of axis labels. A reflection across the \(y = -z\) plane carries a prolate shape into an oblate one, leaves the most asymmetric configuration unchanged, etc. Specifically, under the transformation \(\gamma \rightarrow \gamma = \pi/2\) one finds that
\begin{equation}
\lambda_1 = -\lambda_1
\end{equation}
\begin{equation}
\lambda_2 = -\lambda_2
\end{equation}
\begin{equation}
\lambda_3 = -\lambda_3
\end{equation}
\begin{equation}
\lambda_4 = -\lambda_4
\end{equation}
\begin{equation}
\lambda_5 = -\lambda_5
\end{equation}
\begin{equation}
\lambda_6 = -\lambda_6
\end{equation}

\(\lambda_7 = -\lambda_7\).

Note that this transformation (3.4 \(\rightarrow\) 3.5) preserves the equilibrium relationship among the \(a_3 \leq a_2 \leq a_1 \leq I \). To see how these considerations can be used to fix the mapping parameters \(a_1\), \(a_2\), and \(a_3\), one must first recall that since \(\beta = 3\pi/2\), the \(\cos(\beta) = -\sin(\beta)\). Hence, the transformation implies an interchange of \(a_1\) and \(a_2\), and \(a_3\) and \(a_4\). Indeed, this choice makes the transformation consistent with the shell-model operation of particle-hole conjugation. The
the Q's of the two theories would differ from one another in the contraction limit.

There is only one parameter, \(a_2\), that remains unspecified. It is easy to show that it must be greater than or equal to 3/4 (if 3.12) is to hold for all values of \(\lambda\) and \(\mu\). Arguments regarding the geometry and/or scale impose no further constraints. One is therefore left with the following question: is there a natural choice for \(a_2\) or do the infinity of allowed values (3.7) to (3.9) yield equally acceptable solutions? The answer to the last part of the question is apparently yes so long as \(a_2\) is small relative to \(\langle C_2 \rangle\). This conclusion is based on a comparison of results for numerous examples of the type given in the next section.

The answer to the first part of the question can also be answered in the affirmative, that is, there is a natural choice: \(a_2 = 2/3\). For this special value, and only for this value, there is a linear relationship between \(\lambda_0\) and \(\lambda\) and \(\mu\). Specifically:

\[\phi^2 = (\pi a_2 (4 \gamma^2)^{-1} (A + 1, 2 \lambda + \mu + 2 \mu + 3 \mu + 3) \), \quad (4.6a)\]
In the contraction limit these results see in agreement with the microscopic collective model, see [5].

The relationship between $\beta$, $\gamma$ and $(\lambda, \mu)$ is shown schematically in Fig. 2 on a traditional $(\beta, \gamma)$ or polar plot with $\beta$ the radius vector and $\gamma$ the azimuthal angle. The cartesian components of $\beta$ are given by

$$ k_1 \beta = -k \beta \cos(\beta) \gamma + (\lambda + \mu + 3) \beta, $$

(4.6a)

$$ k_2 \beta = -k \beta \sin(\beta) \gamma = (\mu + 1) \beta. $$

(4.6b)

From this it is easy to deduce the value of $k^2 = (5/9)(\lambda^2 + \mu^2)$, see (4.6a). This shows that each $(\lambda, \mu)$ corresponds to a unique value for the exit $(\beta, \gamma)$. Since $\lambda$ and $\mu$ are representation labels of SU(3) they can only be non-negative integers. Indeed, for any particular nucleus there is a finite (relatively small) set of allowed $(\lambda, \mu)$ values. Turning things around, one has that the particle nature of nuclei dictates a discrete set of allowed values for $(\beta, \gamma)$. This notion is contrary to the liquid drop picture of nuclear structure which takes $\beta$ and $\gamma$ to be continuous variables. In particular, in light of these findings it appears that the whole concept of a potential energy surface in nuclear physics is pointless since for a given state under investigation, the cartesion components of $\beta$ are given in (4.7).

$$ \beta = \tan^{-1} \left( \frac{3(\lambda + 1)}{\lambda + 2} \right). $$

$$ \gamma = \tan^{-1} \left( \frac{3(\lambda + 1)}{\lambda + 2} \right). $$

5. Shell Model Reproduction of Quantum Rotor Dynamics

The hamiltonian of the asymmetric or triaxial quantum rotor is given by

$$ H_{\text{rot}} = A_1 I_1^2 + A_2 I_2^2 + A_3 I_3^2, $$

(5.1)

where $I_3$ is the projection of the total angular momentum of the rotor on the z-axis body-fixed symmetry axis and $A_i$ is the corresponding inertia parameter. This system enjoys a prominent place in the history of physics for it was one of the first problems addressed using the new quantum mechanics. Early researcher [10] who worked on the theory include Dennison (1920), Reiche and Raubendacher (1920), Kronig and Rabe (1927), Mandelbrot (1927), Wigner (1927), Westfall (1929), Kramer and Heitmann (1929), and Klein (1929). Indeed, algebraic methods similar to those being used today were first introduced by Casimir (1931) in his thesis on the subject. "Rotation of a Rigid Body in Quantum Mechanics". Some excellent reviews of the theory of the quantum rotor are available [11]. In this section we will give the form of a special hamiltonian, constructed out of products of SU(3) generators that are SO(3) scalars, that reproduces the dynamics of the quantum rotor. Since rotational motion is championed by the collective model and SU(3) is a fundamental shell-model symmetry, for theory establishes a means for linking macroscopic and microscopic theories of nuclear rotational phenomena.

The theory is really quite simple [12]. The rotor hamiltonian, (5.1), is first rewritten in a frame independent form by introducing three special rotational scalars:

$$ L_x^2 = \sum \frac{3}{2} l_x^2 + l_2^2 + l_3^2, $$

(5.2a)

$$ X_3^2 = \sum \frac{3}{2} l_x^2 + l_2^2 + l_3^2, $$

(5.2b)

$$ X_4^2 = \sum \frac{3}{2} l_x^2 + l_2^2 + l_3^2. $$

(5.2c)

In (5.2) the $L_x$ and $Q_x$ are the cartesian form of the $L_x$ and $Q_x$ introduced in (2.1). Note that the last expression gives both $L_x$ and $Q_x$ as functions of $l_x$, $l_2$, and $l_3$.

The equations of motion give an expression for $l_x$ in terms of $l_x$, $X_3$, and $X_4$:

$$ l_x^2 = (l_1 l_2 l_3)^2 + \left( (l_1 l_2 l_3) X_3 + (l_2 l_3) X_4 / (l_2 + l_2 l_3 + l_3) \right)^2. $$

(5.3)
Substituting this result for the $i^2 \gamma_i$ operators into (5.1), one obtains

$$H_{\text{rot}} = aL^2 + bX_3 + cX_4,$$

(5.4)

where the parameters $a$, $b$, and $c$ depend on the inertia parameters and the eigenvalue of $Q^c$.

$$a = \sum_i A_i, \quad b = \sum_i B_i, \quad c = \sum_i C_i,$$

(5.5a)

$$A_i = \lambda_i, \quad B_i = \lambda_i^2, \quad C_i = \lambda_i,$$

(5.5b)

$$D_i = 2\lambda_i + \lambda_i^2.$$

(5.5c)

This is an exact rewriting of the rotor hamiltonian; the eigenstates of the two are the same. Though the first form, (5.1), is the simplest and displays the fact that $H_{\text{rot}}$ is invariant to rotations by $\pi$ about the principal axes and reflections across symmetry planes perpendicular to these axes, a set of operations which generate a multiplication of the Vierergruppe or $D_3$ (in the Schoenflies notation), the second form, (5.4), points directly to the $SU(3)$ image of this $T_2 \times SO(3)$ hamiltonian.

It follows from (5.4) above as an almost trivial corollary that the $SU(3)$ image of the rotor hamiltonian is

$$H_{\text{rot}} = aL^2 + bX_3 + cX_4.$$  

(5.6)

Note that the $X_3$ and $X_4$ of (5.4) have been replaced by $X_3$ and $X_4$, respectively. The $X_4$ has the same functional form as the $X_3$, see (5.2), but with $Q^c$ replaced by $Q^c$. The parameters $a$, $b$, and $c$ are given by (5.4) but with the $A_i$ replaced by their expression in terms of $\lambda_i$ and $\Lambda_i$ see (4.5). When this is done, the claim is that $H_{\text{rot}}$ acting in the $(A, \Lambda)$ representation of $SU(3)$ reproduces the dynamics of the $H_{\text{rot}}$.

This is an exact assertion because $SU(3)$ is a compact group while $T_2 \times SO(3)$ is noncompact. This means, for example, that the irreps of $SU(3)$ are finite dimensional whereas those of $T_2 \times SO(3)$ are infinite. In particular, for $H_{\text{rot}}$, the number of levels with angular momentum $J \leq L + 1$, each itself being $2L + 1$ degenerate due to rotational invariance, while for a representation of $SU(3)$ the number of levels is given by

$$d(A, \Lambda, \Lambda) = \binom{J + \Lambda}{J + \Lambda - L} = \binom{J + \Lambda + 1}{J + \Lambda} - \binom{J + 1}{J + \Lambda + 1},$$

(5.7)

where the solid bracket denotes the greatest integer function and is to be taken to be zero when its argument is negative. Furthermore, because of the Vierergruppe symmetry $H_{\text{rot}}$ matrices can be brought into block diagonal form with each block transforming irreducibly under the action of the generators of $D_3$, displaying no apparent additional symmetry. The resolution of these differences has been discussed elsewhere so the detailed arguments will not be repeated here [13]. Suffice it to say that by studying the transformation properties of the eigenstates of $H_{\text{rot}}$ one can show that the odd-even character of $\lambda$ and $\mu$ tags the $D_3$ symmetry and for $L < \min(\Lambda, \Lambda + 1)$ the dimensionality agrees. Indeed, as the large $\Lambda$ and $\Lambda$ limit one can show that the two theories are identical, even the off-diagonal matrix elements of the $X^\nu$ and $X^\nu$'s agree.

The surprising and important result that comes out of all of this is that near equivalence of the theories is achieved, provided the prescription outlined above is followed, even for small representations of $SU(3)$ like those commonly encountered in nuclear physics, representations that are far from the contraction limit. This is by no means obvious; there is no simple analytic proof. The best way to see that this is so is by examples [14]. We have chosen to show two, $^{28}$Si and $^{168}$Er, for which the leading shell-model configuration has $(2, \mu) = (8, 0)$ and $(30, 0)$, respectively. The experimental spectra, a fit with $H_{\text{rot}}$ from which the inertia parameters were determined, and the results of a diagonalization of $H_{\text{rot}}$ with $a$, $b$, and $c$ determined using (5.5) and the $A_i$ values determined by (4.5) are given in Figs. 3 and 4. Slightly better agreement with experiment can be obtained in each case by fitting $b$ and $c$ of $H_{\text{rot}}$ without going through $H_{\text{rot}}$, but since our purpose here is to show the near equivalence of results for $H_{\text{rot}}$ and $H_{\text{rot}}$ these are not shown. Notice however that the excitation spectra of $H_{\text{rot}}$ and $H_{\text{rot}}$ really are. Remember, $H_{\text{rot}}$ is a shell-model interaction which describes the interaction between the $A$ and $\Lambda$ and there are no free parameters in the mapping: given the $A_i$ of $H_{\text{rot}}$ the $a$, $b$, and $c$ of $H_{\text{rot}}$ are determined once $A$ and $\Lambda$ are specified. The agreement extends to electromagnetic transition rates between the eigenstates. This is shown in Fig. 5 where inhomogeneous $E2$ transitions to the ground state of the rotor are compared with the corresponding $SU(3)$ predictions.

There are two interesting observations one can make regarding $H_{\text{rot}}$ and $H_{\text{rot}}$. First is that apart from the $L^2$ term, $H_{\text{rot}}$ is the usual $SO(3)$ scalar Hamiltonian of degree four that one can build out of the generators of $SU(3)$. In fact, the set of five operators $\{C_1, C_1, L_2, X_3, X_4\}$ form the $SU(3) \rightarrow SO(3)$ identity basis [15]. This means that $SO(3)$ scalar operator built of generators of $SU(3)$ can be given as a polynomial function in these five variables. Since $C_1$ and $C_1$ are a multiple of the identity within any irrep of $SU(3)$, they do not appear
in \( H_{SU3} \). They are important in a full shell-model analysis, however, because they serve to order the various allowed representations of SU(3) relative to one another. In this regard, they play a vital role in a shell-model analysis of the potential energy surface concept, the topic of a future study. The second matter is that \( H_{SU3} \) has 3- and 4-body parts. Indeed, so does \( H_{ROT} \) though it appears \( \chi \) to when written in its principal-axis form (5.1). This is contrary to shell-model folklore that accepts the proposition that a Hamiltonian with 4-body, \( k \leq 2 \) terms should suffice. While the latter may seem reasonable when viewed in terms of a \( jj \)-coupled shell-model formulation, the theory presented here suggests that the simplest way to generate rotations is by admitting special (LS-coupled) 3- and 4-body forms.

Before leaving this section it seems necessary to comment on a feature of the dynamics that is very important to gaining a deeper understanding of the structure of nuclei but of no real consequence to the current discussion, namely, the nature of the flow when nuclei spin. This is buried in the relationship of the \( A_n \)'s and the \( \lambda_n \)'s, a topic we have not addressed. If nuclei rotate as rigid bodies there is one relationship while if the flow is irrotational there is another. For real nuclei the flow assumes neither of these simple limits. Since \( \chi \) and \( \mu \) are fixed by shell-model considerations and the \( A_n \) come from experiment, what has been developed here can be used to study the flow question. Indeed, this is a topic of another investigation that is underway. This problem serves to illustrate the type of question one can address using the \( H_{ROT} \rightarrow H_{SU3} \) connection established in this paper.

6. Conclusion
A simple one-to-one relationship between the shape variables \((\beta, \gamma)\) of the collective model and the SU(3) irrep labels \((I, \mu)\) has been established. This was done by specifying a linear mapping between eigenvalues of the invariant operators of the collective model and the SU(3) shell-model. Justification for this model operators \( H_{ROT} \) provided through a comparison and analysis of generators of the groups of the two theo-
An essential feature to the argument was that $t_2 	imes s_0(3)$, the algebra of the dynamical group of the triaxial quantum rotor, is a contraction of $su(3)$. The results suggest that in other cases where two groups enter whose algebras are related to one another through contraction, a similar story relating macroscopic and microscopic pictures of quantum phenomena might be found.

For any particular nucleus there is a finite set of allowed $(\lambda, \mu)$ values. The mapping established in this paper can therefore be viewed as placing a constraint on the parameter space of the collective model. For example, since the minimum energy configuration of $^{24}Mg$ is dominated by the $(\lambda, \mu)=(8, 4)$ irrep, one expects the collective model to have a potential energy minimum localized around $(\beta, \gamma) = (0.10, 20)$. Of course, since $SU(3)$ is not an exact symmetry there is representation mixing and consequently the $(\lambda, \mu)=(\beta, \gamma)$ mapping only imposes a soft constraint on the allowed values for $(\beta, \gamma)$. Nonetheless, the full $(\beta, \gamma)$ parameter space is clearly not available because the particle nature (fermions) of the quantum system rules out many configurations. This is just one example of what can learn form the theory.

Our purpose has been to give a 0 to shell-model realization of collective-model dynamics. Major shell model adventures are an important part of the whole scheme of things in nuclear physics [15]. In particular, to get E2 transitions strengths correct without using effective operators one must include couplings to giant resonance states. The microscopic collective model $\Sigma p(|0) + G(|0)$ includes these important modes. Our results show that the collective and shell-model pictures are more than complementary discrinations of diverse nuclear phenomena, they are compatible theories of nuclear structure.
Appendix

The collective model characterizes a nucleus in terms of shape and orientation variables. The low-lying structure is assumed to be due to vibrations and rotations of the nuclear matter distribution. A parametrization is gained by specifying the nuclear surface in terms of an expansion in spherical harmonics about a spherical equilibrium shape \( \mathcal{R}_0 \),

\[
\mathcal{R}(\theta, \phi) = \mathcal{R}_0 + \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \phi).
\]

(1)

The mass multipole moment operators of the theory are defined by

\[
\mathcal{Q}^2 = \frac{3}{4} \frac{A}{c} \int \int \rho (r, \phi) r^2 d\Omega,
\]

(2)

where \( \rho \) is a normalization constant and we assume a constant nuclear density \( \rho_0 = 3A/(4\pi R_0^3) \). Carrying out the radial integration and substituting the expression given in (1) for \( \mathcal{R} \), one finds that

\[
\mathcal{Q}^2 = \frac{3}{4} \frac{A}{c} \int Y_{lm}(\theta, \phi) \mathcal{R}_0^l \sum_{m=-l}^{l} Y_{lm}^*(\theta, \phi) d\Omega = \frac{3}{4} \frac{A}{c} \mathcal{R}_0^l (\ell + 3) \sum_{m=-l}^{l} a_{lm} (\ell + 3) a_{lm} + O(\ell^2),
\]

(3)

where \( O(\ell^2) \) denotes terms of order two or higher in \( \ell/n \).

The mass quadrupole operator is obtained by taking \( \ell = 2 \) in (3) and using the fact that \( a_2 = \sqrt{16\pi/5} \),

\[
\mathcal{Q}_2 = -\frac{3}{5} \frac{A}{c} \mathcal{R}_0^2 a_{20}.
\]

(4)

where square of and higher order terms are neglected. The constant \( \mathcal{R}_0 \) can be given in terms of the mean-square-radius \( \mathcal{Q}^2 \) of the nucleus \([8, 7]\),

\[
\mathcal{R}^2 = (5/3) A \left( \sum_{l=0}^{\infty} a_{lm}^2 \right)^{1/2}.
\]

(5)

If the nucleons are moving in the central potential of a harmonic oscillator, that is,

\[
H_0 = \sum_{i=1}^{N} (\mathbf{p}_i^2 + \mathbf{r}_i^2)/2\hbar,
\]

(6)

then it is straightforward to show that

\[
\mathcal{R}_0 = (3/4) A N_0.
\]

(7)

where \( N_0 \) is the eigenvalue of \( H_0 \).

These results can be used to obtain a relationship between \( \mathcal{Q}_2 \) and the quadrupole deformation parameters:

\[
\mathcal{Q}_2 = \frac{3}{5} \mathcal{N}_0 a_{20}.
\]

(8)

In the body-fixed, principal-axes frame these are given by

\[
\mathcal{Q}^{(x,y,z)}_2 = \frac{1}{2\sqrt{5}} N \mathbf{e} \cdot \mathbf{b} \cos(\gamma),
\]

(9a)

\[
\mathcal{Q}^{(x,y,z)}_{2\mathbf{b}} = 0,
\]

(9b)

\[
\mathcal{Q}^{(x,y,z)}_{2\mathbf{b}} = \frac{1}{2\sqrt{5}} N \mathbf{e} \cdot \mathbf{b} \sin(\gamma).
\]

(9c)

Then through the relationship between cartesian and spherical forms for the mass quadrupole operator one has that

\[
\lambda_1 + \lambda_2 - \frac{4}{3} \mathcal{Q}^{(x,y,z)}_2 = 0,
\]

(10a)

\[
\lambda_1 = \lambda_2 = -\frac{4}{3} \mathcal{Q}^{(x,y,z)}_2.
\]

(10b)

Here we have used the fact that in the body-fixed, principal-axes frame the cartesian components of \( \mathcal{Q}_2 \) are given by \( \lambda_j \cdot \mathcal{Q}^{(x,y,z)}_{2\mathbf{b}} = \lambda_j a_{20} \). From these results one can immediately deduce the form given in (3.5) for \( a_{20} \) in terms of \( \beta \) and \( \gamma \).

References

North, E.: Math. Ann. 77, 99 (1918);

O. Castaños
Centro de Estudios Nuclear
UNAM
Aptdo. Postal 70-543
Mexico, D.F. México

J.P. Drainer, Y. Leducer
Department of Physics and Astronomy
Louisiana State University
Baton Rouge, LA 70803-4001
USA