SU(4) \supset SU(2) \otimes SU(2)

1. INTRODUCTION

In 1937 Wigner\(^1\) pioneered work that established SU(4) as a group of major importance in nuclear structure studies. Its basis, the charge independence of nuclear forces, followed from an observed approximate fourfold degeneracy of nuclear energy levels. The result was the introduction of a nuclear distinguishing isospin quantum number which was combined with that of ordinary spin in the development of a spin-isospin supermultiplet theory. Group-theoretically, it corresponds to a state labeling scheme based upon the spin-isospin reduction SU(4) \supset SU(2) \otimes SU(2).

In general, a complete specification of states in the supermultiplet scheme requires six labels in addition to those of the irreducible representation (IR) of SU(4). The direct product SU(2) \otimes SU(2) provides only four; two additional labels are needed. Techniques that can be used to resolve the multiplicity have been proposed by several authors.\(^2\) In particular, Moultzky and Niehfl have given a recipe for the construction of two operators whose eigenvalues may be used to complete the labeling. Labels obtained in this manner do not, however, exhibit any obvious symmetry properties, nor do they correspond in any way to know quantities of physical interest. In addition, the labels are not necessarily rational numbers.

A mathematically more convenient reduction is the natural or Gell-Mann\(^3\) chain U(4) \supset U(2) \otimes U(2). In this case, the IR labels of U(3), U(2), and U(1) provide the required six labels. Unfortunately, the reduction is unmotivated. Nevertheless, since calculations are simpler within such a framework, the scheme has been used to calculate quantities of physical interest which depend only upon the IR labels of SU(4). An example in point is that of the SU(4) unitary coupling coefficients (U functions) given by Hecht and Pang.\(^4\)

The purpose of the present paper is to state and prove the existence of another solution to the SU(4) \supset SU(2) \otimes SU(2) multiplicity problem, one in which the two additional labels chosen is to furnish an integer or half-integer characterization of the multiplicity that exhibits spin-isospin symmetry properties. The technique used is one of spin-isospin projection; it parallels closely Elliott\(^5\)'s resolution of the multiplicity problem in the SU(3) \supset SU(2) reduction. The simplifications associated with the U(4) \supset U(2) \otimes U(2) \otimes U(1) reduction are incorporated into the scheme via coefficients which relate the projected SU(4) \supset SU(2) \otimes SU(2) basis states to those labeled according to the U(4) \supset U(2) \otimes U(1) chain.

To establish notation, Sec. 2 is devoted to a brief review of SU(4) operator and state labeling techniques. In Sec. 3 a discussion of SU(4) spin-isospin degeneracy diagrams is given, and a new rule for determining the number of occurrences of a spin-isospin pair (ST) in a given IR of SU(4) is derived. In Sec. 4 the projection hypothesis is stated, and the completeness of the states so defined is proved. In Sec. 5 an expression is obtained for the coefficients which relate the projected basis states to those labeled according to the canonical U(4) \supset U(2) \otimes U(2) \otimes U(1) reduction; general expressions for SU(4) \supset SU(2) \otimes SU(2) coupling coefficients and tensorial matrix elements in terms of the corresponding U(4) \supset U(2) \otimes U(2) \otimes U(1) quantities are also given.

2. BASIC NOTATION

A. Infinitesimal Generators

The 16 infinitesimal generators of U(4) are given in terms of nuclear spin-charge creation and annihilation operators by

\[ A_{uv} \approx \sum_{ij} c_{ij}^{uv} \phi_{ij}, \quad (2.1) \]

\[^{1}\text{Phys. Rev. 47, 1009 (1935).}\]
\[^{2}\text{M. Grodzins and C. Negele, Phys. Rev. 126, 886 (1962).}\]
\[^{3}\text{H. A. Kramers, Phys. Rev. 57, 1363 (1940); 60, 389 (1941).}\]
\[^{4}\text{S. Hecht and V. Pang, Phys. Rev. 126, 891 (1962).}\]
where \( N \) denotes the full set of space quantum numbers. The \( A_m \) satisfy the \( \text{SU}(4) \) commutation relations

\[
[A_{\alpha}, A_{\beta}] = i \delta_{\alpha \beta} A_{\alpha} - h_{\alpha \beta} A_{\beta}.
\]

(2.2)

The operators \( N \) are a set of 18 mutually commuting operators for the group \( \text{SU}(4) \). If \( \alpha = 1, 2, 3, \) and \( 4 \) represent the spin-spectroscopic quantum numbers \( m_\alpha \) and \( m_5 \) in the sense

\[
(1) = \{+_{1/2}, +_{1/2}\}, \quad (2) = \{+_{3/2}, -_{1/2}\},
\]

\[
(3) = \{-_{1/2}, +_{1/2}\}, \quad (4) = \{-_{3/2}, -_{1/2}\}.
\]

(2.3)

and then the \( \text{SU}(4) \) generators can be expressed in terms of \( \text{SU}(2) \otimes \text{SU}(2) \) as

\[
A_m = \frac{1}{2}(a_m^- - a_m^+ - a_m^0), \quad B_m = \frac{1}{2}(a_m^- + a_m^+ - a_m^0),
\]

\[
C_m = \frac{1}{2}(a_m^- + a_m^+ + a_m^0), \quad D_m = \frac{1}{2}(a_m^- + a_m^+ + a_m^0),
\]

(2.4)

where \( a_m^- \), \( a_m^+ \), \( a_m^0 = \frac{1}{2}(a_m^- + a_m^+) \) are the \( \text{SU}(2) \) generators. The \( A_m \) and \( B_m \) satisfy the reality conditions

\[
A_{\alpha} A_{\beta} = A_{\beta} A_{\alpha}, \quad B_{\alpha} B_{\beta} = B_{\beta} B_{\alpha}, \quad C_{\alpha} C_{\beta} = C_{\beta} C_{\alpha}, \quad D_{\alpha} D_{\beta} = D_{\beta} D_{\alpha}.
\]

(2.5)

The commutation properties of \( S, T, \) and \( E \) follow from the commutation properties of \( A_m \) given by Eq. (2.2).

B. Irreducible Representations

Gel'fand patterns of the type

\[
|G_\text{IR}| = \begin{pmatrix}
    h_1 & h_2 & h_3 & h_4 \\
    h_2 & h_5 & h_6 & h_7 \\
    h_3 & h_6 & h_5 & h_2 \\
    h_4 & h_7 & h_2 & h_1
\end{pmatrix}
\]

(2.5)

furnish a complete set of labels for the basis states of an \( \text{IR} \) of \( \text{SU}(4) \). The \( h_\alpha \) are integers and satisfy the Young tableau or betweenness conditions

\[
h_\alpha \geq h_{\alpha+1}, \quad h_\alpha \geq 0.
\]

(2.6)

Replacing each \( h_\alpha \) by \( h_\alpha - h_\alpha \) leads to the corresponsive basis state for \( \text{SU}(4) \), it differs from the \( \text{SU}(4) \) state by at most an \( h_\alpha \)-dependent phase factor.

Other characterizations for the \( \text{IR} \) 's of \( \text{SU}(4) \) include the set of three numbers \( (\lambda_1, \lambda_2, \lambda_3) \) given by

\[
\lambda_1 = h_4 - h_2, \quad \lambda_2 = h_3 - h_1, \quad \lambda_3 = h_3 - h_4 + h_2 - h_1.
\]

(2.7)

For some states with \( \lambda_1 = \lambda_2 = \lambda_3 = 0 \), the \( \text{SU}(4) \) configuration may be expressed as the \( \text{SU}(3) \) and \( \text{SU}(2) \) representations. The \( \text{SU}(3) \) configuration parameters can be expressed as relating the \( \text{SU}(3) \) and \( \text{SU}(2) \) representations. The \( \text{SU}(3) \) configuration parameters can be expressed as relating the \( \text{SU}(3) \) and \( \text{SU}(2) \) representations.
Note that for (a) \((1,1,0)\)-odd even \(K_0\) and \(K_2\) are half-integral with \(K_0\) differing from \(K_2\) by twice an integer, for (b) \((1,0,0)\)-odd odd \(K_0\) and \(K_2\) are integral with \(K_2\) differing from \(K_2\) by twice an integer plus one, and for (c) \((1,0,0)\)-even odd \(K_0\) and \(K_2\) are half-integral with \(K_0\) differing from \(K_2\) by twice an integer plus one, and for (d) \((1,0,0)\)-even even \(K_0\) and \(K_2\) are integral with \(K_2\) differing from \(K_2\) by twice an integer. That is, the odd-even characteristics of \(K_0\) and \(K_2\) furnish a complete characterization of distinct symmetry types for the \((K_0,K_2)\)-values associated with \((G)\).

3. SPIN-ISOSPIN MULTIPLECTIES

Racah\textsuperscript{10} has given a relatively simple algebraic formula for determining the multiplicity \(N_{\Delta_4}(A|a_k)\) of \((ST)\)-values in an IR \((A|a_k)\) of \(SU(4)\). Some simplifications in the result follow from the investigations of Kretzschmar\textsuperscript{19} and Perelomov and Popov.\textsuperscript{21} In each case the expressions given are based upon the Littlewood rule\textsuperscript{18} which allows \(N_{\Delta_4}(A|a_k)\) to be related to a sum over terms of the type \(N_{\Delta_4}(A|a_k)\), where the IR’s \((A|a_k)\) have particular simple multiplicity structures. In this section an expression for \(N_{\Delta_4}(A|a_k)\) is given which involves a sum over terms of the type \(N_{\Delta_4}(A|a_k)\) where the \((ST)\)-values are related to the \((ST)\)-values in a very simple way. Since Racah’s expression for \(N_{\Delta_4}(A|a_k)\) is quite transparent, the result is particularly convenient for study of the origin of the \((ST)\)-multiplicities and leads quite naturally to a rule for the projection numbers of Sec. 4.

A Degeneracy Diagram

A spin-isospin degeneracy diagram for the IR \((A|a_k)\) of \(SU(4)\) is a regular lattice of points \((ST)\) each of which is labeled by the numerical value of \(N_{\Delta_4}(A|a_k)\), the multiplicity of the pair \((ST)\) is the IR \((A|a_k)\). Figure 4 of Sec. 4 gives examples. The spin-isospin symmetry property \(N_{\Delta_4}(A|a_k) = N_{\Delta_4}(A|a_k)\) corresponds to reflection symmetry in the \(S = T\)-plane. The conjugate properties of \(SU(4)\) imply that \(N_{\Delta_4}(A|a_k) = N_{\Delta_4}(A|a_k)\). A systematic study of \(SU(4)\) spin-isospin degeneracy diagrams can therefore be limited to a consideration of those IR’s of \(SU(4)\) for which \(A\geq 2\) and within such IR’s those \((ST)\)-values for which \(S = T\).

Figure 1 illustrates features common to all \(SU(4)\) spin-isospin degeneracy diagrams. The heavy solid curve \(E\rho(A|a_k)\) is, in the terminology of Perelomov and Popov,\textsuperscript{22} the enveloping polygon for the spin-isospin degeneracy diagram associated with \((A|a_k)\) IR of \(SU(4)\). It circumscribes all \((ST)\)-values.
The results for $E_P(\lambda_1, \lambda_2)$ suggest that $N_{UTP}(\lambda_1, \lambda_2)$ may be simply related to $N_{UTP}(\lambda_2, \lambda_1)$ and, furthermore, that the classification scheme (a) $(\lambda_2, \lambda_1)$-(odd, even), (b) $(\lambda_1, \lambda_2)$-(odd, odd), (c) $(\lambda_1, \lambda_2)$-(even, odd), and (d) $(\lambda_2, \lambda_1)$-(even, even) may furnish a complete characterization of distinct $N_{UTP}(\lambda_1, \lambda_2)$, and hence $N_{UTP}(\lambda_1, \lambda_2)$ symmetry types. To test the hypothesis, a quantitative study of the numerology of related degeneracy diagrams has been made (e.g., see Fig. 4 in Sec. 4). In terms of $N_{UTP}(\lambda_1, \lambda_2) = N_{UTP}(\lambda_2, \lambda_1)$, $U = T + S$, and $V = T - S$, the result of the investigation, with $V \geq 0$, is that

$$N_{UTP}(\lambda_1, \lambda_2) = N_{UTP}(\lambda_2, \lambda_1) := \lambda_2 \geq 0, \lambda_1 \geq 0,$$

$$U' = U - \lambda_1.$$

$$V' = \max \left\{ V - \lambda_1, \left\lfloor \frac{V}{2} \right\rfloor \right\}.$$

where $\lambda_2 \geq 0$ for cases (a), (b), and (c) and, for case (d),

$$A_{UTP}(\lambda_1, \lambda_2) = \begin{cases} 1, & \lambda_2 > U \geq V, U - \lambda_2 \text{ even,} \\ -1, & \lambda_2 > U \geq V, U - \lambda_2 \text{ odd.} \\ 0, & \text{otherwise.} \end{cases}$$

The formula is recursive and therefore may be iterated to yield

$$N_{UTP}(\lambda_1, \lambda_2) := \sum N_{UTP}(\lambda_1, \lambda_2)$$

$$U' = U - m.$$ (3.3)

$$V' = \max \left\{ V - m, \left\lfloor \frac{V - m}{2} \right\rfloor \right\}.$$ (3.4)

which is applicable to all four cases (a)-(d). In terms of $N_{UTP}(\lambda_2, \lambda_1)$, Eq. (3.3) has the form

$$S \leq T.$$ (3.5)

$$\Delta \hat{N}_{UTP}(\lambda_1, \lambda_2) := \bar{N}_{UTP}(\lambda_1, \lambda_2)$$

$$- N_{UTP}(\lambda_2, \lambda_1)$$

$$\Delta \hat{N}_{UTP}(\lambda_1, \lambda_2) := \bar{N}_{UTP}(\lambda_1, \lambda_2)$$

$$- N_{UTP}(\lambda_2, \lambda_1).$$ (3.6)

The function $\varphi(t)$ is given by

$$\varphi(t) = |t^2|/|t|, \quad t \neq 0,$$

$$= 0, \quad t = 0.$$ (3.7)

where the boldface brackets indicate the greatest integer contained in the argument.

Define

$$\Delta \hat{N}_{UTP}(\lambda_1, \lambda_2) := \bar{N}_{UTP}(\lambda_1, \lambda_2)$$

$$- N_{UTP}(\lambda_2, \lambda_1).$$ (3.8a)

$$\Delta \hat{N}_{UTP}(\lambda_1, \lambda_2) := \bar{N}_{UTP}(\lambda_1, \lambda_2)$$

$$- N_{UTP}(\lambda_2, \lambda_1).$$ (3.8b)

$$\Delta \varphi(c) = \varphi(t) - \varphi(t - 1).$$ (3.8c)
Then, to prove Eq. (3.1), it is sufficient to demonstrate the equivalence of

\[
\Delta N_{\mathcal{U} \otimes \mathcal{V}}(\lambda_1 \lambda_2) = \Delta n_{\mathcal{U} \otimes \mathcal{V}}(\lambda_1 + \lambda_2)
\]

\[
= \Delta n_{\mathcal{U} \otimes \mathcal{V}}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)
\]

\[
= \Delta \Phi(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 1 - \lambda - 1)
\]

\[
= \Delta \Phi(\lambda_1 + \lambda_2 - \lambda - 1)
\]

\[
= \Delta \Phi(U - 1)
\]

(3.9)

and

\[
N_{\mathcal{U} \otimes \mathcal{V}}(\lambda_1 \lambda_2 \lambda_3 \lambda_4)
\]

\[
= \Delta n_{\mathcal{U} \otimes \mathcal{V}}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 1 - \lambda - 1)
\]

\[
= \Delta \Phi(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 1 - \lambda - 1)
\]

(3.10)

For \(\lambda_1 \lambda_2 \lambda_3 \lambda_4\) (even, even) and \(\lambda_1 > U \geq V\), the factor \(\Delta \Phi(\lambda_1 + \lambda_2 - \lambda - 1)\) must, of course, be added to 

\[N_{\mathcal{U} \otimes \mathcal{V}}(\lambda_1 \lambda_2 \lambda_3 \lambda_4).\]

Consider the following special cases:

Case 1: \(U \geq V \geq \lambda_1\).

Case 2: \(U \geq \lambda_2 \geq V\).

(a) \(V - \lambda_2 = -2n\)

(b) \(V - \lambda_2 = -2n - 1\).

Case 3: \(\lambda_1 \geq U \geq V\).

(a) \(V = U + 2n + 1\):

(1) \((\lambda_1 \lambda_2)\) (odd, even);

(2) \((\lambda_1 \lambda_2)\) (even, odd);

(b) \(U = 2n + 1\):

(1) \((\lambda_1 \lambda_2)\) (odd, odd);

(2) \((\lambda_1 \lambda_2)\) (even, even).

For case 1 the result is trivial since \(U = U - \lambda_1\), \(V = V - \lambda_2\) makes \(\Delta N_{\mathcal{U} \otimes \mathcal{V}}(\lambda_1 \lambda_2 \lambda_3 \lambda_4)\) and \(N_{\mathcal{U} \otimes \mathcal{V}}(\lambda_1 \lambda_2 \lambda_3 \lambda_4)\) identical functions in \(\Phi\). In both (a) and (b) of case 2 an application of the result \(\Delta \Phi(\lambda_1 + \lambda_2 - \lambda - 1) = 0\) for any integer and \(\Phi\) being 0 or 1 simplifies \(\Delta N_{\mathcal{U} \otimes \mathcal{V}}(\lambda_1 \lambda_2 \lambda_3 \lambda_4)\) to

\[
\Delta \Phi(\lambda_1 + \lambda_2 + 1 - \lambda)
\]

(3.11)

For (b1) the result is

\[
\Delta N_{\mathcal{U} \otimes \mathcal{V}}(\lambda_1 \lambda_2 \lambda_3 \lambda_4)
\]

\[
= \Delta \Phi(U + 1)\]

(3.12)

\[
= \Delta \Phi(U - 1)
\]

(3.13)

For (b2), which is the desired result.

4. SPIN-ROSPIN PROJECTION

The additional quantum numbers that are required to resolve the twofold multiplicity associated with the reduction \(SU(4) \otimes SU(2) \otimes SU(2)\) may be shown in a variety of ways. The solution proposed by Mohini-

sky and Negele is not necessarily the most convenient because of the algebraic difficulties inherent with the corresponding eigenvalue problem. In this section the existence of another solution to the multiplicity problem is stated and proved. It is based upon spin-
rospin projection techniques in which the \(\{L_1, L_2, L_3, L_4\}\) amplitudes associated with the states \(G_{I_1}\) furnish the required labels.

A. Projection Hypothesis

A projection operator for a state of total angular momentum \(J\) with projection \(M\) may be expressed in Hill-Wheeler integral form as

\[
P_{J M} = (2J + 1)^{-1} \int \mathcal{D} \mathcal{O} \mathcal{D} \mathcal{O}' R_{J M}(\mathcal{O}, \mathcal{O}')
\]

(4.1)

where \(\mathcal{D} \mathcal{O}\) is an \(R(3)\) rotation matrix and \(\mathcal{D} \mathcal{O}\) is an \(R(3)\) rotation operator.

\[
R_{J M}(\mathcal{O}) = e^{i \theta J_3} \mathcal{O} \mathcal{O}' e^{i \theta J_3}
\]

(4.2)

The integral is over Euler angles (\(\alpha, \beta, \gamma\)). From this definition it follows that

\[
P_{J M}^* P_{J M} = \delta_{J' J} \rho_{M' M}\]

(4.3)

\[
P_{J M}^* = P_{J - M}
\]

(4.4)

where \(P_{J M}\) indicates the Hermitian conjugate of \(P_{J M}\). Cases of interest in the present analysis are those for which \(J\) is either the spin \(S\) or the isospin \(T\) of Eq. (2.4).
Since eigenstates of the total spin and isospin operators may be obtained from a state (G) by simply applying the projection operators \(P_{S_{z},I_{z}}\) and \(P_{I_{z}}\), we define

\[
|G_{K_{0},M_{S},M_{I},P_{S_{z},I_{z},E}}\rangle = P_{S_{z},I_{z}}|G_{K_{0},M_{I}}\rangle.
\] (4.5)

The complete \(G\) symbol has been retained in the projected ket as a reminder of the \(G\)\(1\) state from which it was derived; only the IR labels \(K_{0}\) however, remain valid state labels. In many cases the \(|G_{K_{0},M_{S},M_{I},P_{S_{z},I_{z},E}}\rangle\) will turn out to be identically zero. It remains to specify the \(G\) and pairs \((K_{0},K_{p})\) with their corresponding \((ST)\) values for which projected states span the IR space.

The Projection Hypothesis

The projected states

\[
|G_{K_{0},K_{p},M_{S},M_{I},P_{S_{z},I_{z},E}}\rangle = P_{S_{z},I_{z}}|G_{K_{0},M_{I}}\rangle (G_{K_{0}}, \quad (4.6))
\]

with \(|G_{K_{0}}\rangle\) the \(G\)\(1\) states for which the operator \(L_{\mu}\) assumes its maximum \((\lambda_{1} \geq \lambda_{2})\) or minimum \((\lambda_{1} < \lambda_{2})\) eigenvalue, span the \((S_{z} I_{z})\) IR space of \(SU(3)\) if each integer \((\lambda_{1} + \lambda_{2}\) even) or half-integer \((\lambda_{1} + \lambda_{2}\) odd) pair \((K_{0},K_{p})\) satisfying

\[
K_{0} + K_{p} = \max (\lambda_{1}, \lambda_{2}) - 2p,
\]

\[
K_{0} - K_{p} = \min (\lambda_{1}, \lambda_{2}) - 2q,
\]

\[
0 \leq p \leq \min (\lambda_{1}, \lambda_{2})/2,
\]

\[
\kappa \leq q \leq \min (\lambda_{1}, \lambda_{2}),
\]

\[
\kappa = 0, \quad K_{0} + K_{p} \neq 0,
\]

\[
\kappa = \min (\lambda_{1}, \lambda_{2})/2, \quad K_{0} + K_{p} = 0, \quad (4.7)
\]

is associated with the \((ST)\) values

\[\sigma \geq \tau: \quad (ST) = (\sigma + \mu, \tau + v),\]

\[0 \leq \mu \leq h_{1},\]

\[0 \leq v < -\sigma - \tau + \lambda_{1} - \mu, \quad (4.8a)\]

\[\sigma \leq \tau \neq 0: \quad (ST) = (\sigma + \mu, \tau + v),\]

\[0 \leq \mu \leq h_{1},\]

\[0 \leq v \leq -\sigma - \tau + \lambda_{1} - \mu, \quad (4.8b)\]

\[\sigma = \tau = 0: \quad (ST) = (\lambda_{1} - 2p - \mu, \nu),\]

\[0 \leq \mu \leq [\lambda_{1}/2],\]

\[0 \leq v \leq \lambda_{1} - 2 - 2p, \quad (4.8c)\]

where \(\sigma = |K_{0}|\) and \(\tau = |K_{1}|\). The projections \(M_{S}\) and \(M_{I}\) assume the usual values \(-S \leq M_{S} \leq S\) and \(-S_{z} \leq M_{I} \leq S_{z}\).

The proof of the hypothesis will be made in two steps. First, the value of \(N_{ST}(\lambda_{1},\lambda_{2},\mu,\nu)\) predicted by the rule will be shown to be precisely that derived in Sec. 3. And, secondly, the assumption that there exists a function belonging to the IR space but orthogonal to the projected states will be shown to lead to a contradiction. Before proceeding, however, we first consider in more detail the structure of the rule as given by Eqs. (4.7) and (4.8).

Since the \(G\)\(1\) state labels \(|G_{K_{0}}\rangle\) are eigenstates of \(S_{z}\) and \(T_{z}\) the \((K_{0},K_{p})\)-pairs of Eq. (4.7) are necessarily a subset of the allowed \((K_{0},K_{p})\)-pairs given by Eq. (2.9). The choice made (see Fig. 2) is not, however, unique; other possibilities exist. For example, simply replacing each \((K_{0},K_{p})\)-pair of Eq. (4.7) by \((-K_{0},-K_{p})\) inversion in the \((K_{0},K_{p})\)-plane provides an equally acceptable set of projection numbers. It is also true that any partial inversion in the \((K_{0},K_{p})\)-plane provides an acceptable set of projection numbers. The essential feature of any such choice is that only one of the pairs, \((K_{0},K_{p})\) or its inversion \((-K_{0},-K_{p})\), be included. Inclusion of both pairs leads to states which are not linearly independent. The choice made by Eq. (4.7) is therefore one of conventions; its simplifying feature is that it maximizes the number of \((K_{0},K_{p})\)-pairs contained within \(SU(3)\).

In some applications it is convenient to know the rule corresponding to Eq. (4.7) for projection from \(|G_{K_{1}}\rangle\) if \(K_{0} < K_{1}\) and from \(|G_{K_{1}}\rangle\) if \(K_{0} \geq K_{1}\). It can be obtained from Eq. (4.7) by simply interchanging the max-min specifications. It follows that the rules for determining the \((K_{0},K_{p})\)-pairs for projection from \(|G_{K_{1}}\rangle\) and \(|G_{K_{1}}\rangle\) without regard to the relationship of \(K_{0}\) and \(K_{1}\) are given by the following:

projection from \(|G_{K_{1}}\rangle:\)

\[
K_{0} + K_{p} = \lambda_{1} - 2p,
\]

\[
K_{0} - K_{p} = \lambda_{1} - 2q,
\]

\[0 \leq p \leq [\lambda_{1}/2],\]

\[
\kappa \leq q \leq \lambda_{1},
\]

\[
\kappa = 0, \quad K_{0} + K_{p} \neq 0,
\]

\[
\kappa = [\lambda_{1}/2], \quad K_{0} + K_{p} = 0, \quad (4.9a)
\]

projection from \(|G_{K_{1}}\rangle:\)

\[
K_{0} + K_{p} = \lambda_{1} - 2p,
\]

\[
K_{0} - K_{p} = \lambda_{1} - 2q,
\]

\[0 \leq p \leq \lambda_{1},\]

\[
\kappa \leq q \leq \lambda_{1}/2,
\]

\[
\kappa = 0, \quad K_{0} + K_{p} \neq 0,
\]

\[
\kappa = \lambda_{1}/2, \quad K_{0} + K_{p} = 0. \quad (4.9b)
\]

Figure 2 illustrates the result schematically. The dashed curve \((K_{0} + K_{p} = \mu_{0}\) not allowed) and the broken curve \((K_{0} + K_{p} = 0\) allowed) divide the \((K_{0},K_{p})\)-pairs of Eq. (2.9) into two sets equivalent under
inversion; the pairs for which $K_0 + K_0 = 0$ are by convention the projection numbers of Eq. (4.8). In any case the spectrum of $(ST)$-values given by Eq. (4.8) depends only upon $\sigma$ and $\pi$ and is therefore independent of the $(K_0K_0)$-rule chosen as long as all $(K_0K_0)$-pairs belonging to the Gell-Mann state $(\omega)$ under consideration, but not equivalent under inversion, are included in the rule specification.

Figure 3 illustrates Eq. (4.8) by giving the spectrum of $(ST)$-values associated with a given $(K_0K_0)$-pair for the cases $\sigma = \tau$, $\pi = \pi'$, and $\sigma = \pi = 0$. The schematics of the figure are such that the $(ST)$-value labeled by the same symbol are those derived from the same $(K_0K_0)$-pair. In the examples shown, $\lambda_0 = 4$. For $\sigma < \tau$, both $(K_0K_0) = (\sigma\pi)$ and $(K_0K_0) = (\pi\sigma)$ have been given. In the case $\sigma < \tau$, note that except for $(ST) = (\sigma + \lambda_0 - \pi, \pi + \lambda_0)$, $0 \leq \sigma \leq \lambda_0$ for each $(ST)_{K_0K_0}$ (labeled by $\pm$) there exists the transpose set $(ST)_{K_0K_0}$ (labeled by $\mp$). The asymmetry can be removed for $\lambda_0 = 0$ by selecting $(ST) = (\tau + \lambda_0 - \pi, \pi)$, $0 \leq \pi \leq \lambda_0 = 1/2$, to $(\sigma\pi)$ and $(ST) = (\tau + \lambda_0 - \pi + \pi, \pi)$ it cannot be removed. For $\lambda_0 = 0$, however, the asymmetry associated with $(ST) = (\pi + \lambda_0, \pi + \lambda_0)$ cannot be removed. The choice made by Eqs. (4.1) is therefore again one of convention. Its simplifying feature is manifest in the form of Eqs. (4.8a) and (4.8b). For $\sigma = \tau = \pi'$, an asymmetry only exists if $(K_0K_0) = (\pi', \pi')$. It is related to the fact that the transpose of $(ST)_{K_0K_0}$ is not allowed because $(K_0K_0)$ is related to $(K_0, -K_0)$ by inversion. The singularity of the point $(K_0K_0) = (0, 0)$ is manifest in the form of Eqs. (4.8c).

The eight degeneracy diagrams of Fig. 4 illustrate in complete detail the result of associating $(ST)$-values as prescribed by Eqs. (4.8) with the $(K_0K_0)$-pairs defined by Eqs. (4.7). The examples shown correspond to symmetry types (a) $(\lambda_0\lambda_0)$-even, even), (b) $(\lambda_0\lambda_0)$-odd, odd), (c) $(\lambda_0\lambda_0)$-even, odd), (d) $(\lambda_0\lambda_0)$-odd, even) for two cases, $\lambda_0$ zero and $\lambda_0$ such that the degeneracy of $S = T = \pi'$ is a maximum. On each degeneracy diagram the $(K_0K_0)$-lattices corresponding to Eqs. (4.7) is given its outline form. Note that for symmetry types (a) and (b) the $(K_0K_0)$-lattices are rectangular $(K_0 + K_0 \neq 0$ not allowed). The corresponding degeneracy diagrams reflect a maximum degree of regularity. For symmetry type (c) and (d) the $(K_0K_0)$-lattices are not rectangular $(K_0 + K_0 = 0$ allowed). Nevertheless, since symmetry type (c) is equivalent to symmetry type (a) under conjugation $(\lambda_0 \alpha_{\lambda_0})$, degeneracy diagrams of type (c) also possess a maximum degree of regularity. For symmetry type (d), however, the singularity of the point $(K_0K_0) = (0, 0)$ is an inherent feature which propagates an irregularity into the multiplicities of the $(ST)$-values associated with $(ST) = (0, 0)$ by $\lambda_0 \leq 2\lambda_0$ regular lattice displacements.

B. Completeness of the Projected States

First of all, consider the multiplicity $N_{ST}(\lambda_0\alpha_{\lambda_0})$ of $(ST)$-values predicted by Eqs. (4.8). As can be seen from Fig. 3, the basic structure of the rule is one of triangulation. That is, the $(ST)$-values associated with each $(K_0K_0)$-pair for $\lambda_0 \geq 0$ are simply those

\[ \sum_{\lambda_0} N_{ST}(\lambda_0\alpha_{\lambda_0}) \geq \sum_{\lambda_0} \frac{1}{2} \sum_{\lambda_0} (\lambda_0 + 1) \quad (\lambda_0 \leq 2\lambda_0) \]

In the examples shown, $\lambda_0 = 4$.
(ST) values contained within the envelope of isosceles right triangles built by b_4 regular lattice displacements from the (ST) values associated with (K_rK_p) for \lambda_b = 0. The one exception, \((K_rK_p) = (00)\), admits only the subset of these (ST) values for which \(S + T\) differs from \(b_4\) by twice an integer (\(U = b_4\) even). It therefore follows that the \((K_rK_p)\)-pairs that contribute to \(N_{ST}^{\lambda_b}(\lambda_r\lambda_p)\) are the \((K_rK_p)\)-pairs that contribute to the \(N_{ST}^{\lambda_b}(\lambda_r\lambda_p)\) related to \(N_{ST}^{\lambda_b}(\lambda_r\lambda_p)\) in the same way as the \(N_{ST}^{\lambda_b}(\lambda_r\lambda_p)\) related to \(N_{ST}^{\lambda_b}(\lambda_r\lambda_p)\). That is, \(N_{ST}^{\lambda_b}(\lambda_r\lambda_p)\) satisfies Eqs. (3.4). It remains to prove that

\[ N_{ST}^{\lambda_b}(\lambda_r\lambda_p) = N_{ST}^{\lambda_b}(\lambda_r\lambda_p) \]

Consider Eqs. (4.8) for the special case \(\lambda_b = 0:\)

\[ \sigma \geq \tau: (ST) = (\sigma, \tau, \sigma) \quad 0 \leq \tau < \sigma - \tau; \quad (4.10a) \]

\[ \sigma \leq \tau: (ST) = (\sigma, \tau, \sigma) \quad 0 \leq \mu \leq \tau - \sigma. \quad (4.10b) \]

Then \(N_{ST}^{\lambda_b}(\lambda_r\lambda_p)\) is equal to the number of \((K_rK_p)\)-pairs given by Eqs. (4.7) for which \((ST)\) is contained in the set given by Eqs. (4.10):}

\[ S > T: N_{ST}^{\lambda_b}(\lambda_r\lambda_p) = \text{number of } (K_rK_p) \text{-pairs for which } \sigma = S, \tau \leq T; \quad (4.11a) \]

\[ S \leq T: N_{ST}^{\lambda_b}(\lambda_r\lambda_p) = \text{number of } (K_rK_p) \text{-pairs for which } \sigma \leq S, \tau = T. \quad (4.11b) \]

The algebraic formulation is straightforward; it leads directly to the result that \(N_{ST}^{\lambda_b}(\lambda_r\lambda_p) = N_{ST}^{\lambda_b}(\lambda_r\lambda_p)\) and hence \(N_{ST}^{\lambda_b}(\lambda_r\lambda_p) = N_{ST}^{\lambda_b}(\lambda_r\lambda_p)\). On the degeneracy diagrams of Fig. 4 the \((K_rK_p)\)-lattices corresponding to Eqs. (4.7) have been included. By using Eq. (4.11) the result can be verified for each of the four cases (a) \((\lambda_r\lambda_p)\)=(odd, even), (b) \((\lambda_r\lambda_p)\)=(even, odd), (c) \((\lambda_r\lambda_p)\)=(even, even), and (d) \((\lambda_r\lambda_p)\)=(even, odd).

To complete the proof of the projection hypothesis, an adaptation of the method first given by Efimov for the A(2) \(\rightarrow\) R(3) reduction and subsequently used by Williams and Pucey in considering the R(3) \(\rightarrow\) R(3) reduction problem will be used. It proceeds by reduction ad absurdum. That is, the consequence of assuming that the projected states do not
span the IR space is shown to be a contradiction. Explicitly, suppose there exists a function \( \rho(S, M, T) \) belonging to the IR but orthogonal to all the \( G_{K,K} S M_{2} S M_{3} T M_{4} \).

\[
(\rho(S, M, T) G_{K,K} S M_{2} S M_{3} T M_{4}) = 0. \tag{4.13}
\]

Since \( N_{\mu}^{\rho}(a_{\alpha}, \lambda_{\beta}) = N_{\mu}^{\rho}(a_{\alpha}, \lambda_{\beta}) \), the only nontrivial implications of such an assumption are those which follow for \( 0 = S, M_{2} = M_{4}, T = T, \) and \( M_{3} = M_{3}, \) namely,

\[
(\rho(S, M, T) G_{K,K} S M_{2} S M_{3} T M_{4}) = 0. \tag{4.13}
\]

As is shown below, Eq. (4.13) implies that

\[
(\rho(S, M, T) G_{K,K} S M_{2} S M_{3} T M_{4}) = 0. \tag{4.14}
\]

where \( \theta \) is an arbitrary element of \( SU(4). \) By definition of irreducibility, functions of the type \( \rho(S, M, T) G_{K,K} S M_{2} S M_{3} T M_{4} \) span the IR space. Hence a contradiction exists; the hypothesis that there exists a function \( \rho(S, M, T) \) belonging to the IR which is orthogonal to all the \( G_{K,K} S M_{2} S M_{3} T M_{4} \) is false. It follows that the \( G_{K,K} S M_{2} S M_{3} T M_{4} \) span the IR space.

The argument given above hinges upon a proof that Eq. (4.13) implies Eq. (4.14). For this, note that the operator \( \rho(S, M, T) \) being an element of \( SU(4) \) implies that it can be expressed as a power series in the generators of the group. Furthermore, note that the commutation properties of the generators imply that the order of the generators within each term of such an expansion can be chosen in any desired manner. Then we define

\[
\begin{align*}
\delta_{\mu}^{\nu} &= \{S_{\mu}, E_{\nu}\}, \\
\delta_{\mu}^{\nu} &= \{S_{\mu}, T_{\nu}\}, \\
\delta_{\mu}^{\nu} &= \{T_{\mu}, E_{\nu}\}. 
\end{align*}
\tag{4.15}
\]

and consider the case of projection from \( G_{K,K} \). It is convenient to devide the generators into the two sets

\[
\begin{align*}
A &= \{A_{1}, A_{2}, A_{3}, A_{4}\}, \\
B &= \{B_{1}, B_{2}, B_{3}, B_{4}\}, \\
A &= \{A_{1}, A_{2}, A_{3}, A_{4}\}, \\
B &= \{B_{1}, B_{2}, B_{3}, B_{4}\}, \\
E_{1} &= A_{1}, \quad E_{2} = A_{2}, \tag{4.16a}
\end{align*}
\]

When a generator of the set \( A \) operates on \( G_{K,K} \), the result is either another intrinsic state of the same type \( (E_{0}, S, T_{2}, E_{1}, E_{3}, E_{4}, E_{5}, \ldots), (E_{1}, T, \ldots), (E_{2}, \ldots) \) or zero \( (E_{1}, T, \ldots) \). Generators of the set \( B \) do not reproduce intrinsic states but are operators which act only in the direct product space \( SU(2) \otimes SU(2) \). Express 0 in the form

\[
0 = \sum \gamma_{K,K} E_{1} \pm \gamma_{K,K} E_{2} \tag{4.17}
\]

where the \( \gamma_{K,K} \) are constants and \( \gamma_{K,K} \) are products of generators of the type \( A \) and \( E_{1} \), respectively. Then consider

\[
(\rho(S, M, T) G_{K,K} S M_{2} S M_{3} T M_{4}) = \sum \gamma_{K,K} (\rho(S, M, T) G_{K,K} S M_{2} S M_{3} T M_{4}). \tag{4.18}
\]

Each factor \( \gamma_{K,K} \) acting to the right changes at most \( K_{P} \) and \( K_{P} \) and the \( \gamma_{K,K} \) factors acting to the left change at most \( M_{S} \) and \( M_{S} \). Therefore,

\[
(\rho(S, M, T) G_{K,K} S M_{2} S M_{3} T M_{4}) = \sum \gamma_{K,K} (\rho(S, M, T) G_{K,K} S M_{2} S M_{3} T M_{4}) = 0. \tag{4.19}
\]

The equivalent proof for the case of projection from \( G_{K,K} \) follows by merely replacing the \( E_{1}, E_{2}, \gamma_{K,K}, \gamma_{K,K} \) operators of set \( A \) by the operators \( E_{1}, E_{2}, \gamma_{K,K}, \gamma_{K,K} \). 5. TRANSFORMATION BRACKETS

Although the projection numbers \( K_{P} \) furnish an integral or half-integral solution exhibiting spin-isospin symmetry properties for the \( S(4) \otimes SU(2) \otimes SU(2) \) multiplicity problem, the projected states are not normalized nor are they necessarily orthogonal on the \( K_{P} \) and \( K_{P} \) labels. The difficulties associated with the nonorthornormality of the projected states can be resolved, however, if an expression for the coefficients (transformation brackets) which relate the projected states to the orthonormal Gell/Mann basis vectors is known. This section is devoted to deriving such an expression. The method used is similar to that developed in Ref. 16, where the analogous problem in the \( SU(3) \otimes R(3) \) reduction was considered, it is based on the results of Moshinsky and Chadov for the matrix elements of the permutations \( (1, 2), (2, 3), \) and \((3, 4)\) between the \( U(4) \) basis states \( (\alpha). \)
A. The Expression

Since the Gel'fand basis vectors \( G \) for a given IR of the representation space, an arbitrary projected state \( |G_K,SM_J,K_J,T_M,F_M> \) belonging to the IR may be expanded in terms of the \( |G> \) as

\[
|G_K,SM_J,K_J,T_M,F_M> = \sum \left< G | G_K,SM_J,K_J,T_M,F_M \right> |G> \tag{5.1}
\]

where it is to be understood that \( K_J = K_J. \) The \( \left< G | G_K,SM_J,K_J,T_M,F_M \right> \) in Eq. (5.1) are the transformation brackets which relate the \( U(4) \to SU(2) \) scheme of Sec. 4 to the Gel'fand \( U(4) \to U(3) \to U(2) \to U(1) \) scheme. By definition of the projected states, we have

\[
\left< G | G_K,SM_J,K_J,T_M,F_M \right> = \left< G | P_{ME}^{K_J} P_{F_M}^{T_M} P_{SM}^{J} P_{K_J} \right> |G> \tag{5.2}
\]

Therefore, an expression for the

\[
\left< G | G_K,SM_J,K_J,T_M,F_M \right> = \int dO_{2,2} \frac{dO_{2,2}}{(2\pi)^4} |G>(2T + 1) \times \left< g_{2,2} D_{T}^{22} \left( \Omega_2 \right) |g_{2,2} \right> R_{2,2}\left( \Omega_2 \right) R_{2,2}\left( \Omega_2 \right) |G> \tag{5.3}
\]

are known. Note that the inverse of the transformation matrix defined by Eq. (5.1) is only guaranteed to exist if the \( G_K,SM_J,K_J,T_M,F_M \) are restricted to the projected basis vectors \( G_K,SM_J,K_J,T_M,F_M \) defined in Sec. 4 by the projection hypothesis. An expression for the \( \left< G | G_K,SM_J,K_J,T_M,F_M \right> \) follows as a special case of the general result for \( \left< G | G_K,SM_J,K_J,T_M,F_M \right> \).

For notational convenience let

\[
\langle G | = \left\langle \begin{array}{ccc} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{array} \right> \tag{5.4}
\]

The infinitesimal generators of \( SU(2) \) corresponding to \( U(2) \) in the chain \( U(4) \to U(3) \to U(2) \to U(1) \) are given by

\[
\mathbf{J}_r = \left( \begin{array}{ccc} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad = \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{array} \right) \tag{5.5}
\]

where

\[
J_z = J_z \pm J_{z1} \tag{5.6}
\]

Then, for

\[
\mathbf{R}(\Omega) = e^{i\phi \mathbf{J}_x} e^{i\theta \mathbf{J}_z} e^{i\psi \mathbf{J}_y} \tag{5.7}
\]
The transformation brackets of Eq. (5.2) are then given by

\[ (G' | G) K_{0} S M_{0} K'_{2} T M'_{2} \]  

(5.15)

An expression for the matrix Mbps(KJM) can be obtained by using the completeness of the orthonormal set of states (G) and Eq. (5.7) to put Mps(KJM) into the form

\[ \sum_{G'} \sum_{G_{2}} (G' | G_{2}) (G_{2} | G) (G_{2} | G_{1}) (G_{1} | G) \]

(5.15)

× \{ \sum K_{0} S M_{0} K'_{2} T M'_{2} \} K_{0} S M_{0} K'_{2} T M'_{2} .

(5.15)

The property brackets \( (n - 1, a) | G \), \( n = 2, 3, 4 \), required for an evaluation of Eq. (5.17), have been given by Moshinsky and Charon; they are equivalent to special unitary recoupling coefficients for the groups U(1), U(2), and U(3), respectively. Note that \( (n - 1, a) \) operating on \( G \) changes only the \( K_{0} \), for which \( \beta = n - 1 \) and those in such a manner that the result is zero unless \( w_{1} = w_{a} \). The apparent 6 \times 6 = 36-fold sum over the \( G_{2} \) in Eq. (5.17) is therefore in actual fact at worst a sixfold sum. The result so given by Eq. (5.17) may, however, be the most convenient for the purposes of machine coding since the summations over \( G_{2} \), \( G_{1} \), and \( G_{0} \), \( G_{2} \) are matrix multiplications involving the permutation matrices. The remaining summation over \( G_{1} \) then involves simply the product of two Clebsch-Gordan coefficients and one element from each of the matrix products.

It is to be noted that the transformation brackets are equivalent to normalization and overlap integrals of the projected states. This may be seen by considering

\[ (G' | G_{2} K_{0} S M_{0} K'_{2} T M'_{2}) | (G | G_{2} K_{0} S M_{0} K'_{2} T M'_{2}) \]

(5.18)

\[ = (G' | G_{2} K_{0} S M_{0} K'_{2} T M'_{2}) | (G | G_{2} K_{0} S M_{0} K'_{2} T M'_{2}) \]

(5.18)

B. The Application

In general, the transformation brackets\( ^{n} A (G' | G_{2} K_{0} S M_{0} K'_{2} T M'_{2}) \) relate the set of nonorthogonal basis vectors \( (G_{2} K_{0} S M_{0} K'_{2} T M'_{2}) \) to the set of orthogonal basis vectors \( (G') \) and are therefore the elements of a nonorthogonal matrix \( A \). The inverse expansion of the \( G \) in terms of the \( (G_{2} K_{0} S M_{0} K'_{2} T M'_{2}) \) exists, and the coefficients \( B(G_{2} K_{0} S M_{0} K'_{2} T M'_{2}) | G \) can be obtained by inverting the appropriate \( A \) matrix. An equivalent but perhaps somewhat simpler evaluation of these coefficients can be obtained by considering directly the expansion

\[ G = \sum_{K_{0} S M_{0} K'_{2} T M'_{2}} \langle K_{0} S M_{0} K'_{2} T M'_{2} | G \]

(5.19)

\[ \times \langle K_{0} S M_{0} K'_{2} T M'_{2} | K_{0} S M_{0} K'_{2} T M'_{2} \rangle \]

(5.19)

Then

\[ \langle K_{0} S M_{0} K'_{2} T M'_{2} | G \rangle = \sum_{K_{0} S M_{0} K'_{2} T M'_{2}} \langle K_{0} S M_{0} K'_{2} T M'_{2} | G \]

(5.19)

\[ \times \langle K_{0} S M_{0} K'_{2} T M'_{2} | K_{0} S M_{0} K'_{2} T M'_{2} \rangle \]

(5.19)

\[ \times \langle K_{0} S M_{0} K'_{2} T M'_{2} | G \rangle \]

(5.19)

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In general, the transformation brackets

\[ A (G' | G_{2} K_{0} S M_{0} K'_{2} T M'_{2}) \]

(5.18)

relate the set of nonorthogonal basis vectors \( (G_{2} K_{0} S M_{0} K'_{2} T M'_{2}) \) to the set of orthonormal basis vectors \( (G') \) and are therefore the elements of a nonorthogonal matrix \( A \). The inverse expansion of the \( G \) in terms of the \( (G_{2} K_{0} S M_{0} K'_{2} T M'_{2}) \) exists, and the coefficients \( B(G_{2} K_{0} S M_{0} K'_{2} T M'_{2}) | G \) can be obtained by inverting the appropriate \( A \) matrix. An equivalent but perhaps somewhat simpler evaluation of these coefficients can be obtained by considering directly the expansion

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(5.19)

\[ \times \langle K_{0} S M_{0} K'_{2} T M'_{2} | K_{0} S M_{0} K'_{2} T M'_{2} \rangle \]

(5.19)

Then

\[ \langle K_{0} S M_{0} K'_{2} T M'_{2} | G \rangle = \sum_{K_{0} S M_{0} K'_{2} T M'_{2}} \langle K_{0} S M_{0} K'_{2} T M'_{2} | G \]

(5.19)

\[ \times \langle K_{0} S M_{0} K'_{2} T M'_{2} | K_{0} S M_{0} K'_{2} T M'_{2} \rangle \]

(5.19)

\[ \times \langle K_{0} S M_{0} K'_{2} T M'_{2} | G \rangle \]

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\[ A (G' | G_{2} K_{0} S M_{0} K'_{2} T M'_{2}) \]

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relate the set of nonorthogonal basis vectors \( (G_{2} K_{0} S M_{0} K'_{2} T M'_{2}) \) to the set of orthonormal basis vectors \( (G') \) and are therefore the elements of a nonorthogonal matrix \( A \). The inverse expansion of the \( G \) in terms of the \( (G_{2} K_{0} S M_{0} K'_{2} T M'_{2}) \) exists, and the coefficients \( B(G_{2} K_{0} S M_{0} K'_{2} T M'_{2}) | G \) can be obtained by inverting the appropriate \( A \) matrix. An equivalent but perhaps somewhat simpler evaluation of these coefficients can be obtained by considering directly the expansion

\[ G = \sum_{K_{0} S M_{0} K'_{2} T M'_{2}} \langle K_{0} S M_{0} K'_{2} T M'_{2} | G \]

(5.19)

\[ \times \langle K_{0} S M_{0} K'_{2} T M'_{2} | K_{0} S M_{0} K'_{2} T M'_{2} \rangle \]

(5.19)

Then

\[ \langle K_{0} S M_{0} K'_{2} T M'_{2} | G \rangle = \sum_{K_{0} S M_{0} K'_{2} T M'_{2}} \langle K_{0} S M_{0} K'_{2} T M'_{2} | G \]

(5.19)

\[ \times \langle K_{0} S M_{0} K'_{2} T M'_{2} | K_{0} S M_{0} K'_{2} T M'_{2} \rangle \]

(5.19)

\[ \times \langle K_{0} S M_{0} K'_{2} T M'_{2} | G \rangle \]

(5.19)
coefficients in the expansion of \( G_{\alpha \beta} S M_{\mu} G_{\mu \rho} T M_{\rho} \) and in terms of the \( G_{\alpha \beta} S M_{\mu} G_{\mu \rho} T M_{\rho} \). Using this result, we can determine a unique solution for the
\[
\mathcal{R}(G_{\alpha \beta} S M_{\mu} G_{\mu \rho} T M_{\rho})(G)
\]
from the set of simultaneous equations
\[
\sum_{\sigma} A(G | G_{\alpha \beta} S M_{\mu} G_{\mu \rho} T M_{\rho}) = \delta_{\alpha \delta} \delta_{\beta \epsilon} \delta_{\mu \lambda} \delta_{\rho \epsilon} \delta_{\sigma \delta} 
\]
(5.23)

In a fashion similar to that demonstrated in detail in Ref. 16 for the SU(3) \( \otimes \) SU(2) case, quantities of physical interest which depend upon the SU(4) \( \otimes \) SU(2) \otimes SU(2) labels can be expressed in terms of the corresponding quantities labeled according to the canonical \( U(3) \otimes U(2) \) scheme by means of the \( \alpha \) and \( \beta \). For example, for the SU(4) \( \otimes \) SU(2) \( \otimes \) SU(2) coupling coefficients defined by

\[
\rho(G_{\alpha \beta} S M_{\mu} G_{\mu \rho} T M_{\rho}) = \sum_{\kappa \lambda \mu \nu} C(G_{\alpha \beta} S M_{\mu} G_{\nu \lambda} T M_{\mu \nu}) \times |G_{\alpha \beta} S M_{\mu} G_{\nu \lambda} T M_{\mu \nu}| (5.24a)
\]

\[
|G_{\alpha \beta} S M_{\mu} G_{\nu \lambda} T M_{\mu \nu}| = C(G_{\alpha \beta} S M_{\mu} G_{\nu \lambda} T M_{\mu \nu}) \times \rho(G_{\alpha \beta} S M_{\mu} G_{\nu \lambda} T M_{\mu \nu}) (5.24b)
\]

it can be shown that

\[
C(G_{\alpha \beta} S M_{\mu} G_{\nu \lambda} T M_{\mu \nu}) = \sum_{S_1 M_1 S_2 M_2 T_1 T_2} C(S_1 M_1; S_2 M_2 | T_1 T_2) \times |G_{\alpha \beta} S M_{\mu} G_{\nu \lambda} T M_{\mu \nu}| (5.25a)
\]

\[
|G_{\alpha \beta} S M_{\mu} G_{\nu \lambda} T M_{\mu \nu}| = \sum_{S_1 M_1 S_2 M_2 T_1 T_2} \rho(G_{\alpha \beta} S M_{\mu} G_{\nu \lambda} T M_{\mu \nu}) \times C(S_1 M_1; S_2 M_2 | T_1 T_2) (5.25b)
\]

where \( \rho \) is a label that distinguishes multiple occurrences of a given IR of \( G_1 \) in the reduction of the direct product \( G_1 \otimes G_1 \). In Eqs. (5.25a, 5.25b), \( |G_{\alpha \beta} S M_{\mu} G_{\nu \lambda} T M_{\mu \nu}| \) and \( \rho(G_{\alpha \beta} S M_{\mu} G_{\nu \lambda} T M_{\mu \nu}) \) are \((44 \otimes 44 \otimes 12) \otimes (12 \otimes 12)\) Wigner coefficients, and the \( |T_1 T_2| \) are ordinary SU(2) Wigner coefficients.

Similarly, consider the SU(4) \( \otimes \) SU(2) \( \otimes \) SU(2) Wigner coefficients defined by

\[
\mathcal{R}(G_{\alpha \beta} S M_{\mu} G_{\mu \rho} T M_{\rho})(G)
\]

(5.26)
where \( T(G) \) is the corresponding \( U(3) \supseteq U(2) \supseteq U(1) \) tensor defined by

\[
[A_{ij}, T(G)] = \sum_{G'} C_{ij}^{G'} A_{ij} T(G').
\]

(5.27)

The \((K_{6}, K_{8}) \) quantum-numbers resolve the \( SU(4) \supseteq SU(2) \supseteq SU(2) \) tensorial multiplicity in precisely the same manner as described in Sec. 3 for the \( SU(4) \supseteq SU(2) \supseteq SU(2) \) basis states. It can then be shown that

\[
|\langle G_2G_3 | G_4G_5 | G_6G_7 | G_8G_9 \rangle \rangle \supseteq \langle G_2G_3 | G_4G_5 | G_6G_7 | G_8G_9 \rangle \rangle
\]

(5.28)

where \((G_2G_3 | G_4G_5) (G_6G_7 | G_8G_9) \) is the reduced matrix element of \( T(G) \) corresponding to the state \((G_2) \).

The particularly elegant feature of all such relationships is that a knowledge of the \( d^13_2 \) and \( d^17_2 \) allows completely general expressions for \( SU(4) \supseteq SU(2) \supseteq SU(2) \) quantities to be expressed in terms of a subset of the corresponding \( U(4) \supseteq U(3) \supseteq U(2) \supseteq U(1) \) quantities [e.g., all \( SU(4) \supseteq SU(2) \supseteq SU(2) \) coupling coefficients are determined in terms of \( U(4) \supseteq U(3) \supseteq U(2) \supseteq U(1) \) Wigner coefficients for which one set of labels corresponds to the operator \( E_{PQ} \) having either its maximum or minimum eigenvalue]. Furthermore, the problems associated with phase conventions and multiplicity relate simply and directly to the corresponding problems in the canonical scheme.

6. DISCUSSION

The fact that a many-nucleon wavefunction can be decomposed into a projector of its space and its spin-isospin parts allows the techniques developed in this paper to be applied quite independently of any specific spatial considerations. A common principle, however, is that dealing with well-model-calculated up to and through the first half of the \( Z = 1d \) shell. For such nuclei the most promising theoretical tool for that spatial part of the wavefunction in the Elliott \( St(3) \supseteq R(3) \) classification. For this reason the techniques developed in Ref. 16 together with those of the present paper furnish expressions which can be used to simplify as well as extend such theoretical investigations.

The simplifications are, of course, in calculational technique in that the \( SU(3) \supseteq R(3) \) and \( SU(4) \supseteq SU(2) \supseteq SU(2) \) transformation brackets reduce the difficulties inherent in the physically significant labeling schemes, but not present in the corresponding canonical labeling schemes, to forms which can be machine coded. Nevertheless, the solution furnished by the transformation brackets to the problems associated with the nonorthonormality of the projected states is indirect and not necessarily the most convenient (or purposes of machine-code matrix element calculations). The difficulty is that the \( SU(4) \supseteq SU(3) \supseteq SU(2) \) coupling coefficients of Ref. 16 and the \( SU(4) \supseteq SU(2) \supseteq SU(2) \) coupling coefficients of the present paper are not Wigner coefficients. That is, the coupling coefficients do not represent the scalar product of orthornormalized coupled and uncoupled basis states.

By orthonormalizing separately within each \( L \) and \( ST \)-multiplet according to a symmetric recipe [e.g., see Ref. 19], the transformations which orthonormalize the \( St(3) \supseteq R(3) \) and \( SU(4) \supseteq SU(2) \supseteq SU(2) \) basis states can be given in simple algebraic form as the ratios of normalization and overlap integrals. Since such integrals are equivalent to transformation brackets, the problems associated with the nonorthonormality of the projected states can be resolved. And, in particular, they can be resolved in a form convenient for machine coding while still maintaining all the simplifications associated with the projective process. In fact, the \( St(3) \supseteq R(3) \) and \( SU(4) \supseteq SU(2) \setminus SU(2) \) transformation brackets can be incorporated directly into programs which calculate the transformation brackets. The result is then \( SU(2) \supseteq R(3) \) and \( SU(4) \supseteq SU(2) \setminus SU(2) \) transformation brackets which relate physically significant orthonormal basis states to the corresponding canonical basis states. Within such a framework the \( SU(2) \supseteq R(3) \) coupling coefficients of Ref. 13 and the \( SU(4) \supseteq St(3) \setminus SU(2) \) coupling coefficients of the present paper become Wigner coefficients, and hence standard algebraic techniques introduced by Racah~supplementary text~ can be applied to simplify matrix element calculations.

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⁷ SU(4) conjugation properties are defined and discussed more fully by Brunet and Remikoff, Ref. 2.
⁸ The supermultiplet quantum numbers \( PP' \) specify the IR of \( O(6) \) which is locally isomorphic to \( SU(4) \) according to the standard Gel'fand labeling scheme.
⁹ For notational convenience, states of the type \((2,8)\) will be denoted simply by \( |G_2\rangle \), an arrow being added to the \( E \) when a distinction between types \((a)\) and \((b)\) is required. The \((K_R K_T)\)-label will only be included when it is necessary to distinguish between \( |G_2\rangle \) states of different spin–isospin projections.
¹⁰ G. Racah, Rev. Mod. Phys. 21, 494 (1949).
¹⁴ A regular lattice displacement is one which increases either \( S \) or \( T \) by one unit.
¹⁵ The Löwdin–Shapiro form for the projection operator may also prove to be useful in some applications; for example, see Yu. F. Smirnov, "Projection Operators for Semisimple Lie Groups", Proceedings on the International Conference on Clustering Phenomena in Nuclei, Bochum, Germany, 1969 (International Atomic Energy Agency, Vienna, 1969).
¹⁹ To avoid possible confusion, the notational change

\[ A(G') \langle GK_SM_KTM_T | GK_SM_KTM_T \rangle \equiv \langle G' | GK_SM_KTM_T \rangle \]

expressing the transformation brackets as the elements of a matrix \( A \), with rows labeled by the \( h_{\delta \beta} \) and columns labeled by the \( (K_SM_KTM_T) \)-values, will be adopted in this section.