Reconsideration of enhancement of $sd$ dominance in interacting boson models

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The enhancement of $sd$ dominance in interacting boson models for nuclei is shown to be due primarily to the relatively small single-particle energy gap between the $s$ boson and the $d$ boson compared to larger single-particle energy gaps between the $s$ and $d$ bosons and bosons with $l \geq 4$ when a repulsive pairing interaction between the boson pairs is introduced.

In Ref. [1], an exactly solvable model for bosons that interact through a repulsive pairing force was studied. In the model, the boson single-particle energies are taken to be proportional to the angular momentum quantum number $\epsilon_l = l$. This means that the boson single-particle energies used in Ref. [1] increase with increasing values of the angular momentum. Though such a choice for boson single-particle energies does not correspond to a realistic situation, it is an interesting example for the purpose of getting a better understanding of the dominance of the $s$ and $d$ bosons in boson models for nuclei. An important conclusion of Ref. [1] is that the repulsive pairing interaction is a robust mechanism for reinforcing $sd$ dominance in boson models of nuclei. In this paper, this problem is revisited. The results show that the enhancement of $sd$ dominance in an interacting boson model for nuclei with the Hamiltonian adopted in Ref. [1] is due mainly to the gap structure in the single-particle energy spectrum.

First of all, we introduce the generators of the SU$_{(1,1)}$ algebra as

$$A_i^+ = \frac{1}{2} \sum_m (-)^{l-m} b_{im}^+ b_{1-m}^+,$$

$$A_i^- = \frac{1}{2} \bar{n}_i + \Omega_i,$$

where $\Omega_i = 2l + 1$ and $b_{im}^+(b_{1m})$ is an $l$-boson creation (annihilation) operator. Then, up to a constant, one can use these operators to construct the standard interacting boson pairing model Hamiltonian as

$$\hat{H} = \sum_l \epsilon_l \hat{n}_l^+ + \frac{g}{2} \sum_{l \neq l'} A^+_l A^+_{l'},$$

where $C$ is a normalization constant, $L$ and $M$ are quantum numbers of the total angular momentum and its third component, respectively, $\zeta$ is an additional quantum number used to distinguish distinct eigenstates with other quantum numbers the same, and the lowest weight state $| \nu;LM \rangle = | \nu_0, \nu_1, \ldots, \nu_p;LM \rangle$ satisfies

$$A_{2\mu}^+ | \nu;LM \rangle = 0$$

for $\mu = 0, 1, 2, \ldots, p$. By using an infinite-dimensional algebraic expansion, the functional $A^+(x)$ can be determined [2],

$$A^+(x) = \sum_{\mu=0}^p \frac{1}{1 - \epsilon_{2\mu} x} A_{2\mu}^+, \quad (5)$$

where the spectral parameters $x_i$ satisfy

$$\frac{2}{x_i} = \frac{1}{2} \sum_{\mu=0}^p \frac{g(\nu_0 + \Omega_{2\mu})}{1 - \epsilon_{2\mu} x_i} - \sum_{j \neq i} g x_j$$

for $i = 1, 2, \ldots, k$, where $\Omega_{2\mu} = \frac{1}{2} \Omega_{2\mu}$. The eigenenergy of Eq. (2) is given by

$$E^{(i)}_k = \frac{2}{\sum_{i=1}^k x_i} + \sum_{\mu=0}^p \epsilon_{2\mu} x_i. \quad (7)$$

Some sample solutions are considered in Ref. [1]. Since the authors choose $\epsilon_l = l$, the boson single-particle energies are ordered as $\epsilon_0 < \epsilon_2 < \epsilon_4 < \cdots < \epsilon_{2p}$. This assumption, namely, that bosons with higher angular momentum lie higher in energy, is common to most applications. It means, at least in low-energy regimes, that higher angular momentum boson degrees of freedom can be neglected, an assumption that greatly reduces the dimension of the corresponding configuration space. In particular, this assumption seems to apply for real nuclei and justifies the inclusion of only $s$ and $d$ bosons in most such studies [3–5], although $g$-boson degrees of freedom need to be considered in some situations [6–8].

In contrast with Ref. [1], however, we contend that it is this choice of single-particle energies that leads to the dominance of $s$ and $d$ pairs in interacting boson models. To expand on this, first consider the limit with all boson single-particle energies assumed to be the same, $\epsilon_0 = \epsilon_2 = \cdots = \epsilon_p$. DOI: 10.1103/PhysRevC.68.014308XX PACS number(s): 21.60.Fw, 21.60.Ev
In the original \( s \) and \( d \) interacting boson model, this corresponds to the \( O(6) \) \( \gamma \)-unstable limit \([5]\). Although this limit also does not correspond to a realistic scenario, and therefore does not have direct relevance to the microscopic foundation of interacting boson model, it is a useful limit that shows how important the single-particle energy gaps are in leading to \( sd \) dominance. Bethe ansatz \((3)\) cannot be used in this limit because of condition \((4)\). However, it can be shown that in this case the pairing vacuum satisfies the equation

\[
\sum_{\mu=0}^{p} A_{2\mu}|q,\nu;LM\rangle = 0,
\]

where \( \Sigma_{\mu=0}^{p} A_{2\mu} \) is a generalized pair and \( q \) is a quantum number that is related to the total boson number \( N \) by \( N = 2k + 2q + \Sigma_{\mu=0}^{p} n_\mu \) with \( k = \lceil (N - \sigma)/2 \rceil \) generalized pairs and \( \sigma = 2q + \Sigma_{\mu=0}^{p} n_\mu = N - 2, N - 4, \ldots \). In this case the eigenenergies are

\[
E_\sigma = \epsilon N + \frac{g}{8}\{N[N+(p+1)(2p+1)-2]-\sigma(p+1)\
\times(2p+1)-2\}\]

Actually, the equal energy assumption, \( \epsilon_0 = \epsilon_1 = \cdots = \epsilon_{2p} = \epsilon \), leads to the \( O((p+1)(2p+1)) \) limit of the interacting boson system, where \( \sigma \) is the quantum number of \( O((p+1)(2p+1)) \). For example, when \( \epsilon_0 = \epsilon_2 \), one gets the \( O(6) \) limit of the \( U(6) \) \( sd \) interacting boson model \([5]\), and when \( \epsilon_0 = \epsilon_2 = \epsilon_4 \) one obtains the \( O(15) \) limit of the \( U(15) \) \( sd \) interacting boson model \([6]\). These two limiting cases have been used to describe realistic nuclear systems \([5,6]\). In this case the quantum number \( \sigma \) for ground state of Eq. \((2)\) is equal to \( N \), which is free of the generalized pair, \( \Sigma_{\mu=0}^{p} A_{2\mu} \). However, the eigenfunctions of Eq. \((2)\) are still not determined because there are many different \( O((p+1)(2p+1)) \supset O(5) \times O(9) \times \cdots \times O(4p+1) \) reductions. In the following, we will show some typical examples for the \( p = 6 \) case, namely

\[
O((p+1)(2p+1)) = O(91) \supset O(L_1) \times O(L_2) \cdots \supset O(5) \times O(9) \times \cdots \times O(85)
\]

with \( L_1 = 1, 6, 15, 28, 45, 66 \), and \( L_1 + \cdots + (p+1)(2p+1) = 91 \), which correspond to the \( O(91) \supset O(90) \), \( O(91) \supset O(6) \times O(85) \), \( O(91) \supset O(15) \times O(76) \), \( O(91) \supset O(28) \times O(63) \), \( O(91) \supset O(45) \times O(56) \), and \( O(91) \supset O(66) \times O(25) \) reductions, respectively. In the \( O(91) \supset O(90) \) case, there is no \( s \) boson operator contained in the generators of \( O(90) \), while in the \( O(91) \supset O(6) \times O(85) \) case the generators of \( O(6) \) are \( s \) and \( d \) boson operators and those of \( O(85) \) are the remaining bosons operators, etc. Because there is no multipole-multipole force involved in Hamiltonian \((2)\), one cannot choose a specific basis from these possible reductions; that is, the eigenfunction of \((2)\) can be any one or a linear combination. Because wave functions of these special cases can be constructed explicitly, it is interesting to explore whether different choices of the bases will affect the percentages of bosons with different angular momenta in the ground state. In all these cases ground state wave functions can be written as

\[
|N, \sigma=N, q, \nu; L=0 M=0\rangle = \mathcal{N}|2q\rangle|\nu 00\rangle,
\]

where the normalization constant is

\[
\mathcal{N} = \sqrt{\frac{(L_1 + 2 \tau_1 + 2q - 2)!!(L_2 + 2 \tau_2 + 2q - 2)!!(2q)!!(L_1 + L_2 + 2 \tau_1 + 2 \tau_2 + 2q - 4)!!
\times (L_1 + 2 \tau_1 - 2)!!(L_2 + 2 \tau_2 - 2)!!(L_1 + L_2 + 2 \tau_1 + 2 \tau_2 + 4q - 4)!!
\times (L_1 + L_2 + 2 \tau_1 + 2 \tau_2 + 4q - 4)!!
\times (L_1 + L_2 + 2 \tau_1 + 2 \tau_2 + 4q - 4)!!
\times (L_1 + L_2 + 2 \tau_1 + 2 \tau_2 + 4q - 4)!!}
\]

and

\[
\mathcal{N} = \sqrt{\frac{(L_1 + 2 \tau_1 + 2q - 2)!!(L_2 + 2 \tau_2 + 2q - 2)!!(2q)!!(L_1 + L_2 + 2 \tau_1 + 2 \tau_2 + 2q - 4)!!
\times (L_1 + 2 \tau_1 - 2)!!(L_2 + 2 \tau_2 - 2)!!(L_1 + L_2 + 2 \tau_1 + 2 \tau_2 + 4q - 4)!!
\times (L_1 + L_2 + 2 \tau_1 + 2 \tau_2 + 4q - 4)!!
\times (L_1 + L_2 + 2 \tau_1 + 2 \tau_2 + 4q - 4)!!
\times (L_1 + L_2 + 2 \tau_1 + 2 \tau_2 + 4q - 4)!!}
\]

FIG. 1. (Color online) Occupation probabilities for bosons with angular momentum \( l = 0, 2, 4, 6, 8, 10, 12 \) for the ground state of a system of five boson pairs and maximum angular momentum \( l = 12 \) when all the single-particle energies are the same in six different phases: (a) \( s \)-dominant phase, (b) \( d \)-dominant phase, (c) g-dominant phase, (d) \( ig \)-dominant phase, (e) \( mki \)-dominant phase, and (f) \( o \)-dominant phase.
with $k_1 + k_2 = q$,\

\[
Q(2q) = \sum_{k_1 k_2} (-1)^{k_2} \frac{(L_1 + 2 \tau_1)!!(L_2 + 2 \tau_2 - 2)!!}{k_1!k_2!(L_1 + 2 \tau_1 + 2k_1 - 2)!!(L_2 + 2 \tau_2 + 2k_2 - 2)!!} A^+_{k_1}(L_1) A^+_{k_2}(L_2)
\]

(13), one can calculate ground state occupation probabilities $\rho_l$ for bosons with different angular momentum $l$ according to

\[
\rho_l = \langle \hat{n}_l \rangle / N.
\]

(15)

In Fig. 1, we show the occupation probabilities for the ground state of a system of ten equal-energy bosons with an
angular momentum cutoff of $l = 12$ for six different reductions with $L_1 = 1, 6, 15, 28, 45, 66$, respectively. The results show that the values of the occupation probabilities for bosons with different angular momentum vary drastically from one reduction to the other. For $L_1 = 1$ corresponding to $O(91) \supset O(90)$, the ground state is $s$-boson dominant; when $L_1 = 6$ corresponding to $O(91) \supset O(6) \times O(85)$, the ground state is $d$-boson dominant, but there is 14.5% $s$-boson content. When $L_1 = 15$ corresponding to $O(91) \supset O(15) \times O(76)$, the ground state is $g$-boson dominant, but there is obvious $d$-boson content in the ground state. It is therefore clear that there is a hidden phase for the boson system with interactions described by Hamiltonian (2), in which $s$ and $d$ bosons dominate. We call $L_1 = 1$ the $s$ dominant phase, $L_1 = 6$ the $d$ dominant phase, etc. Because all single-particle energies are set to be equal and there is no other interaction considered, the phase of the system is not determined. In these limiting cases, contrary to the conclusion shown in Ref. [1] that repulsive pairing is a robust mechanism for reinforcing $sd$ dominance in the boson models, the pairing interaction does not affect the ground state behavior at all. It is obvious that the phase of the system becomes undetermined when the single-particle energy gaps disappear.

Since the first term in Eq. (2) can be written as $\Sigma \epsilon \hat{n}_s$, $\epsilon = n + \Sigma \epsilon (\epsilon - \epsilon) \hat{n}_s$, only differences in the single-particle energies change the nature of the system. To explore how these single-particle energy differences affect the system, in what follows we set $\epsilon = l$ for $l = 0$ and $l \geq 4$, which are the same as used in Ref. [1], and allow $\epsilon = \alpha$ to vary in the closed interval $[0, 2]$. Figure 2 shows the ground state occupation probabilities for $s$ and $d$ bosons and the complementary $sd$ depletion factors as a function of the pairing interaction strength $g$ with $k = 5$ and maximum angular momentum $l = 12$ for nine different $\alpha$ values. The $\alpha = 0$ ($g \neq 0$) limit corresponds to a $d$-dominant phase for the system, and there is a notable phase transition from the $d$-dominant phase to a $s$-dominant phase. The latter is a second order transition with the critical point around $\alpha \sim 0.0551$. When $\alpha = 0.0551$, the structure changes gradually with increasing $\alpha$ from a $d$-dominant ($\alpha = 0$) to $sd$-dominant configuration. It should be noted, however, that in this case the depletion factor increases faster with increasing $g$ than in other cases. The depletion reaches about 10% when $g = 2$. As pointed out in Ref. [1], this is due mainly to an enhanced scattering to a high-spin pair mechanism because the single-particle energy gaps between the $s$ and $d$ bosons and bosons with angular momentum $l \geq 4$ are relatively small.

It is important to note that $g$ cannot be taken to be zero for the case under consideration. In particular, the structure for $g = 0$ and $\alpha = 0$ cannot be determined, it could be $s$ dominant $[U(5) \text{limit}]$, $d$-dominant $[(6) \text{limit}]$, or a combination of the two. Furthermore, there is a first order transition that leads to a $sd$-dominant phase when $g$ is different from 0 when $\alpha = 0$. The phase at the $g = 0$ point for $\alpha \neq 0$ is $s$ dominant. In addition, there is an obvious first order phase transition from $s$ dominant ($g = 0$) to $sd$ dominant ($g \neq 0$) when $\alpha = 0$. In short, the phase diagram at the $g = 0$ point is not continuous. As a result, we only consider pairing interaction strength $g$ close to zero with three scenarios: $\alpha = 0$, 0.025, and 0.05 shown in Fig. 2. The structure is $sd$ dominant when $\alpha \sim 0.0551$. However, the phase gradually changes from $sd$ dominant to $s$ dominant with increasing values of $\alpha$. Also, the overall depletion factors are reduced as $\alpha$ increases. When $\alpha \leq 1$, this can be understood as enhanced scattering to higher-spin pairs with most going to $d$ boson pairs, thus decreasing the depletion factor. And when $\alpha > 1$, although there is enhanced scattering to higher order than $d$ pairs, the larger single-particle energy gap between the $s$ and $d$ bosons again results in a lowering of the depletion factor. Clearly, one can reduce contributions from bosons with higher angular momenta by simply increasing the energy gaps between the $s$ and $d$ bosons and bosons with angular momentum $l \geq 4$, that is, by setting $\epsilon = \beta l$ with $\beta > 1$ for $l \geq 4$. Figure 3 shows the ground state occupation probabilities for $s$ and $d$ bosons and the corresponding depletion factors as a function of $\beta$ with $k = 5$ and maximum angular momentum $l = 12$ for $\alpha = 2$, and the pairing interaction strengths $g = 1$ and $g = 2$, respectively. Figure 3 indicates that $s$ bosons and bosons with angular momentum $l \geq 4$ dominant the system when $\beta < 0.5$. There is a phase transition from $s$ and $l \neq 2$ dominant phase to $sd$ dominant phase around $\beta \sim 0.5$ for both $g = 1$ and $g = 2$ cases. After the critical point, the system gradually becomes $sd$ dominant. In both cases the depletion factor decreases as $\beta$ increases. It is obvious that $sd$ dominance is effectively enhanced with large $\beta$ values when $\alpha \neq 0$ is small.
All of this can be seen in another way when $g \neq 0$ by rewriting Hamiltonian (2) as

$$2\hat{H}/g = \sum_l \frac{2\epsilon_l}{g} \hat{a}_l \hat{a}_l^+ + \sum_{ll'} A_l^+ A_{l'}^+.$$

(16)

In this form it is very clear that increasing the pairing strength $g$ is equivalent to decreasing the energy gaps among bosons with different angular momenta. Because $\epsilon_l$ for $l \geq 4$ are large relative to $\epsilon_2 = 2$ and $\epsilon_0 = 0$, one can get some $sd$ dominance by increasing the pairing interaction strength $g$ as shown in Ref. [1]. However, the $sd$ dominance is far from being complete in this case. The $sd$ dominance can effectively be achieved by reducing the energy gap $\epsilon_2 - \epsilon_0$, and increasing the energy gap $\epsilon_l - \epsilon_l = \frac{1}{2}(\epsilon_2 - \epsilon_0)$ for $l \geq 4$ as shown in Fig. 3. However, in order to get the $sd$ dominance, one cannot set $\epsilon_0 = \epsilon_2$, which corresponds to a $d$-dominant phase, because of enhanced scattering to a high-spin pair mechanism as noted in Ref. [1].

In conclusion, the $sd$ dominance of the interacting boson model discussed in Ref. [1] is due primarily to the relatively small energy gap between the $s$ and $d$ bosons as compared to larger gaps between the $s$ and $d$ bosons and bosons with higher angular momentum. Whenever the single-particle gap between the $s$ boson and the $d$ boson is small relative to gaps between the $s$ and $d$ bosons and other bosons with higher angular momentum, there is a greater enhancement of $sd$ dominance.

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