Pseudospin and nuclear deformation

A.L. Blokhin\textsuperscript{a}, T. Beuschel\textsuperscript{a}, J.P. Draayer\textsuperscript{a}, C. Bahri\textsuperscript{b}

\textsuperscript{a} Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803, USA
\textsuperscript{b} Department of Physics, University of Toronto, Toronto, Ont. M5S 1A7, Canada

Received 10 September 1996
NUCLEAR PHYSICS A
Journal devoted to the experimental and theoretical study of the fundamental constituents of matter and their interactions

Editorial Board
Supervisory Editor: G.E. BROWN, Stony Brook
Executive Editor: C.L. Schwarz, Amsterdam

Associate Editors: G. Baym, Urbana; K. Dietrich, Munich; J.P. Elliott, Sussex; I.D. Garrett, Oak Ridge; M.N. Harakeh, Groningen; S.E. Koonin, Pasadena; W. Nörenberg, Darmstadt; A. Richter, Darmstadt; R.A. Sorensen, Carnegie-Mellon; I. Tanaka, RIKEN; D. Vautherin, Orsay; D.H. Wilkinson, Sussex

Associate Editors for Intermediate Energy: G. Baym, Urbana; H.A. Bethe, Cornell; J.-P. Blaizot, Saclay; T. Ericson, CERN; J.L. Friar, Los Alamos; A. Gal, Jerusalem; J.W. Harris, Berkeley; J. Hüfner, Heidelberg; K. Nakai, KEK; D.O. Riska, Helsinki; I. Sick, Basel; M. Soyeur, Saclay; H. Specht, GSI; J. Stachel, Heidelberg; R. Stock, Frankfurt; K. Yuzaki, Tokyo

Aims and Scope
Nuclear Physics A covers the domain of general nuclear physics together with intermediate energy and heavy-ion physics, and astrophysics. The emphasis is on original research papers. A number of carefully selected and reviewed conference proceedings are published as an integral part of the journal.

Abstracted/Indexeed in:

Subscription Information 1997
Volumes 612–627 are scheduled for publication in 64 issues. Publication frequency: weekly. Subscription prices are available upon request from the Publisher. A combined subscription to Nuclear Physics A – ISSN 0375-9474 (volumes 612–627), Nuclear Physics B – ISSN 0550-3213 (volumes 483–508) and Nuclear Physics B Proceedings Supplements – ISSN 0920-5632 (volumes 52–59) is available at a reduced rate. Subscriptions are accepted on a prepaid basis only and entered on a calendar year basis. Issues are sent by SAL (Surface Air Lifted) mail wherever this service is available. Airmail rates are available on request. For orders, claims, product enquiries (no manuscript enquiries) please contact the Customer Support Department at the Regional Sales Office nearest to you:

New York: Elsevier Science, P.O. Box 945, New York, NY 10159-0945, USA. Tel: (+1) 212-633-3730, Fax: (+1) 212-633-3680, E-mail: unixfo-f@elsevier.com
Amsterdam: Elsevier Science, P.O. Box 211, 1000 AE Amsterdam, The Netherlands. Tel: (+31) 20-485-3757, Fax: (+31) 20-485-3432, E-mail: nlinfo-f@elsevier.nl
Tokyo: Elsevier Science, 9-15, Higashi-Azabu 1-chome, Minato-ku, Tokyo 106, Japan. Tel: (+81) 3-5561-5033, Fax: (+81) 3-5561-5047, E-mail: kyo40035@niftyserve.or.jp
Singapore: Elsevier Science, No. 1 Temasek Avenue, #17-01 Milenia Tower, Singapore 039192. Tel: (+65) 434-3727, Fax: (+65) 337-2230, E-mail: asiainfo@elsevier.com.sg

Claims for issues not received should be made within six months of our publication (mailing) date.

Advertising Offices
International: Elsevier Science, Advertising Department, The Boulevard, Langford Lane, Kidlington, Oxford OX5 1GB, UK. Tel: +44 (0)1865 843505; Fax: +44 (0)1865 843976.
USA and Canada: Weston Media Associates, Dan Lipner, P.O. Box 1110, Greens Farms, CT 06456-1110, USA. Tel: +1 (203) 261 2500; Fax: +1 (203) 261 0101.
Japan: Elsevier Science Japan, Marketing Services, 1-9-15 Higashi-Azabu, Minato-ku, Tokyo 106, Japan. Tel: +81 3 5561 5033; Fax: +81 3 5561 5047.

US mailing notice – Nuclear Physics A (ISSN 0375-9474) is published weekly by Elsevier Science B.V., Molenweg 51, P.O. Box 211, 1000 AE Amsterdam, The Netherlands. The annual subscription price in the USA is US$ 6923 (valid in North, Central and South America only), including air speed delivery. Second class postage paid at Jamaica, NY 11431.

© The paper used in this publication meets the requirements of ANSI/NISO Z39.48-1992 (Permanence of Paper).

North-Holland, an imprint of Elsevier Science

Printed in The Netherlands

Abstract
An operator in a model of the interaction of a two-particle skimmer shell with a target nucleus is constructed. Preliminary calculations show that such a model should be capable of simulating the momentum distribution in the reaction. In the spherical shell model, the momentum distributions have the form of Pauli matrices, the ground state is a single-particle state, and the excited states are a superposition of single-particle states. The model predictions are compared with experimental data for the reaction 132Nd(132Nd, 90Nd)134Ce at 17 MeV. The results show that the model is able to reproduce the experimental data within the uncertainties of the measurements.

1. Introduction

Shell-model calculations have been performed for the reaction 132Nd(132Nd, 90Nd)134Ce at 17 MeV. The model includes the effects of spin-orbit coupling and the interaction between the two particles. The results show that the model is able to reproduce the experimental data within the uncertainties of the measurements. The model is able to predict the momentum distribution in the reaction and can be used to simulate the reaction.
Pseudospin and nuclear deformation

A.L. Blokhin a, T. Beuschel a, J.P. Draayer a, C. Bahri b

a Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803, USA
b Department of Physics, University of Toronto, Toronto, Ont. M5S 1A7, Canada

Received 10 September 1996

Abstract

An operator which effects the transformation to the pseudospin representation within the oscillator shell model at arbitrary triaxial deformation is constructed as a special projection of the momentum helicity operator. Since the exact transformation of many important operators cannot be performed in a closed analytical form, a procedure of approximate transformation is developed. In the spherical and asymptotic prolate limits the transforms thus derived reduce to the known exact results. The approximate transform of a modified Nilsson Hamiltonian is found to be almost indistinguishable from the "pseudo" Hamiltonian with the strongly reduced spin–orbit strength.

PACS: 21.10.Pc; 21.30.+y; 21.60.Cs

1. Introduction

Shell-model based microscopic studies of heavy nuclei (A \( \geq 100 \)) are simplified considerably with the help of the pseudo space-spin concept. This concept [1,2] assigns new spin and orbital momentum labels (so called pseudo labels) to single-particle levels in accordance with an observed near degeneracy of certain normal-parity eigenstates (pseudospin doublets) of the spherical nuclear mean field. The spin–orbit interaction in the pseudo representation is weak compared to the normal physical representation, and this makes pseudo LS-coupling the scheme of choice. The concept has been extended to incorporate axial deformations [3–5] which remove most of the degeneracy characteristic of the spherical mean field. In this case the pseudo labels are valid for projections of the spin and orbital momenta on the body-fixed symmetry axis, and the pseudospin doublets are easily distinguishable in the spectra of realistic single-particle Hamiltonians. Pseudospin symmetry, the goodness of which is governed by the weakness of pseudo spin–orbit interaction, has been successfully applied to various nuclear phenomena including, for example, superdeformation [3], identical bands [6,7], and
double beta decay [8]. This symmetry lies in the foundation of the pseudo SU(3) and pseudo symplectic models [9–11]. In the framework of these many-particle theories, shell-model calculations can be carried out for heavy deformed nuclei.

The pseudospin symmetry, clearly seen in results of mean-field calculations, has been viewed for more than two decades as primarily a tool for constructing an effective shell-model coupling scheme. An important exception was Ref. [12] where the issue of an operator for effecting a transformation to the pseudo representation was raised for the first time. Recently a number of articles appeared discussing an explicit form for such an operator in terms of single-nucleon variables [13–15] and its possible connection to the symmetries of nuclear interactions [16,17]. Once an explicit form of the transformation operator is identified, one can learn more about the origin of the pseudospin symmetry and check its goodness directly by transforming the Hamiltonian of an appropriate nuclear model. This test is preferable to the analysis of the single-particle spectra since the latter does not provide direct evidence for the weakness of the spin–orbit interaction in the presence of various other forces. If the spin–orbit strength in the transformed Hamiltonian is strongly reduced at any deformation, one can speak about pseudospin dynamical symmetry. In this case the LS-coupling schemes retain their significance for the construction of basis states.

A study of the goodness of pseudospin dynamical symmetry in triaxial nuclei is the subject matter of this paper. The analysis is carried out within the framework of the harmonic-oscillator shell model. An explicit form for the extended pseudospin transformation for arbitrary deformations is suggested and applied to some modifications of the triaxial Nilsson Hamiltonian.

It is worth noting that the shell-model pseudospin representation, which is adapted to the oscillator basis, has a precursor emerging in realistic mean-field and many-particle descriptions of nuclei. Realistic nuclear models require the momentum space helicity transformation for an effective reduction in the spin–orbit interaction strength and the normal—pseudo relabelling of the spin and orbital momenta [17]. However, the helicity transforms of the oscillator wavefunctions are no longer oscillator eigenfunctions, especially in the nuclear surface region where they acquire different asymptotics. The pseudospin transformation of the spherical oscillator functions can therefore be understood as the helicity transformation followed by a special projection back onto the oscillator basis. This operational definition is taken in the present paper, generalized for the triaxial case, and used as a basis for subsequent analyses.

The discussion starts from the relation between the helicity and pseudospin transformations of the basis functions in the spherical representation. Then, model single-particle Hamiltonians and their pseudo transforms in the spherical and asymptotic prolate cases are considered. Based on an explicit introduction of the triaxially deformed pseudospin operator and an approximate analytical evaluation of its action on the components of the model Hamiltonians, the functional form and spectra of the transformed Hamiltonians are analyzed. This analysis validates the microscopic “pseudo” models on both the single-particle and many-particle levels.

2. Helicity and pseudospin

The many-particle Hamiltonian of the system of A nucleons

$$\mathcal{H}_{\text{total}} = \prod_{n=1}^{A} p_n$$

where $p_n$ stand for the spin matrices of the $n$th particle, Hamiltonian that yields the many-particle and orbital momenta of a nucleon system. Analysis of the spin–orbit interactions in nuclear models is based on the interpretation of the Hamiltonians.

The pseudospin does not originate from, but simply is generated by, the real transformation that yields the effective Hamiltonian and orbital momenta. Analysis of the spin–orbit interactions in nuclear models is based on the interpretation of the Hamiltonians.

Consider the relation between the helicity and pseudospin in the spherical representation for the single-particle level

$$\psi_{nljhf}(r, \sigma)$$

where $n$ is the angular momentum and $l$ is the orbital momentum of $\sigma$, the unit vector under the helicity transformation.

$$\mathcal{H}\psi_{nljhf}(r, \sigma)$$

where

$$q = 2(l - j)$$

The analytical expressions for Eqs. (A.13) and
2. Helicity and pseudospin transformations

The many-particle helicity transformation can be written in a multiplicative form for the system of A nucleons:

$$\mathcal{H}_{\text{total}} = \prod_{i=1}^{A} \frac{p_i \cdot \sigma_i}{p_i}$$

(1)

where $p_i$ stand for the momentum, $p_i$ for its absolute value, and $\sigma_i = 2s_i$ for the Pauli spin matrices of the individual particles. This is the only microscopic unitary transformation that yields the normal-pseudo transformation rules for the single-particle spin and orbital momenta and simultaneously preserves the translational invariance of the nucleon system. Another important feature of this transformation is the effective reduction of the spin-orbit splitting in the single-particle energy spectra derived from the realistic internucleon forces. All these features, underscored in Ref. [17], are predicated upon the interpretation of the helicity operation as a microscopic pseudospin transformation.

The pseudospin transformation that is normally used in nuclear theory is different from, but simply related to, the helicity transformation. In fact, as discussed in detail in Ref. [17], the usual pseudospin transformation is a special modification of the helicity form, one that is convenient for oscillator-based, shell-model applications. The two operations are not, however, equivalent. The helicity transformation is fundamental at the nucleon level and applicable within any realistic many-particle approach to nuclear structure regardless of the choice of basis functions. The conventional pseudospin transformation—whose explicit form will be given in the next section—is an auxiliary construction which retains some significant features of its precursor but has the advantage of having a very simple representation in an oscillator basis. The following analysis shows the relation and distinction between these two transformations in more detail.

Consider the action of the helicity transformation on the shell-model wavefunctions in the spherical representation. The nature of the shell model reduces the problem to the single-particle level, and the relevant wavefunctions are written in the form [12, 18]

$$\psi_{nlj_{1}j_{2}}(r, \sigma) = iR_{n}(r)(Y_{l}(\sigma_{r}) \otimes \chi)_{j_{1}j_{2}}$$

(2)

where $n$ is the number of quanta, $l, j$ and $j_{2}$ denote orbital momentum, total angular momentum and its projection, respectively. $Y_{l}(\sigma_{r})$ is a spherical harmonic (in the direction of $\sigma_{r}$, the unit vector along $r$), and $\chi$ is a Pauli spinor. As shown in Appendix A, under the helicity operation the function (2) transforms as follows:

$$\mathcal{H}_{h} \psi_{nlj_{1}j_{2}}(r, \sigma) = iR_{n}^{q}(r)(Y_{l}(\sigma_{r}) \otimes \chi)_{j_{1}j_{2}}$$

(3)

where

$$q = 2(l - j), \quad \bar{l} = l - q, \quad \bar{n} = n - q.$$  

(4)

The analytical expressions for the radial functions $R_{nl_{+}}(r)$ and $R_{nl_{-}}(r)$ are given by Eqs. (A.13) and (A.14), and the value of the $n$ quantum number is chosen equal to
the shell number of the basis state providing the maximal contribution into the spherical oscillator basis expansion.

The radial dependence of the transformed functions differs from the radial dependence of the related shell-model functions. A direct comparison of the functions is shown in Fig. 1 for the case \( n = 5, l = 3 \) rather slowly in the nuclear matter, the closest oscillator function (see Eq. (A.19)). The functions are plotted in Fig. 2 where the behavior in the bulk of the total, with the Woods–Saxon potential, the above remains valid.

Therefore, the helicity representation is model-independent, approximately, via similar operation makes an expansion analysis [17]. Further, it can be understood as the expansion of the oscillator shell images of the single-particle transformation rules for the steps. First, the transformation, i.e. the \( R_{nlq}(r) \) factors of the nucleus is unitary in the entire \( n \) onto the states with the value of \( q \) (see Eq. (4)), regardless of \( q \); however, Eq. (4). Thus, the pseudo shell map onto the complete shell map onto the \( s_{1/2} \) while the \( h_{11/2} \) orbital.

The conventional pseudo but retains the unitary subspace before the transformation, and in description of the nucleus has been found to be deformed nuclei with some refinement, base brings calculated trans whole, the pseudospin relevant way of arriving the special advantage of the
Fig. 1 for the case $n = 5$, $l = 3$. The results show that the transformed functions decrease rather slowly in the nuclear surface region while in the bulk they behave practically like the closest oscillator function, either slightly compressed ($q = 1$) or dilated ($q = -1$) (see Eq. (A.19)). The relevant oscillator basis expansion coefficients for the same case are plotted in Fig. 2 which demonstrates that the dominant shell contribution is 80–85% of the total, with the higher shells only effecting the tail of radial function. Since the behavior in the bulk of more realistic single-particle functions (for instance, for the Woods–Saxon potential) is very similar to the behavior of their oscillator counterparts, the above remains valid for realistic mean-field models.

Therefore, the helicity transformation of the basis functions in the spherical representation is model-independent to a good extent and can be accomplished, at least approximately, via simple prescriptions. However, the strong nonlocality of the helicity operation makes an expression for the transformed mean field rather complex for standard analyses [17]. From this standpoint the conventional pseudospin transformation can be understood as a practical compromise, which allows, albeit in the framework of the oscillator shell model only, for a straightforward analytical construction of the images of the single-particle Hamiltonian and the basis states along with the correct transformation rules for the spin and orbital momenta.

The passage from the helicity transformation to the pseudospin one requires two major steps. First, the transformed functions are replaced by the closest oscillator functions, i.e. the $R_{nl}(r)$ factors in the r.h.s. of Eq. (3) change to $R_{nl}(r)$. Within the bulk region of the nucleus this operation is well approximated by a simple rescaling. The operation is unitary in the entire Hilbert space of the oscillator states and maps the states of shell $n$ onto the states which belong to both the $n - 1$ and $n + 1$ shells, depending upon the value of $q$ (see Eq. (4)). In the second step, $\tilde{n}$ is redefined to be equal to $n - 1$ regardless of $q$; however, the relabelling rule for the orbital momentum is still given by Eq. (4). Thus, the pseudospin transformation maps the normal-parity states of a given shell onto the complete shell of one quantum less; but in this case the defector states no longer have images. For instance, the $p_{1/2}$, $p_{3/2}$, $p_{5/2}$, $d_{3/2}$ and $h_{9/2}$ orbitals of the $n = 5$ shell map onto the $\tilde{s}_{1/2}$, $\tilde{d}_{3/2}$, $\tilde{d}_{5/2}$ and $\tilde{g}_{9/2}$ orbitals of the $\tilde{n} = 4$ shell, respectively, while the $h_{11/2}$ orbital has no image.

The conventional pseudospin transformation is convenient for practical applications but retains the unitarity property only within the normal-parity subspace. The latter subspace before the transformation corresponds to the entire space of states after the transformation, and in this sense the pseudospin scheme delivers only an effective description of the nuclear system for the low-energy region. Nevertheless, this description has been found to be very reasonable in predicting energy spectra and shapes of heavy deformed nuclei within the framework of a many-particle, shell-model approach [19]. Some refinement, based on a microscopic treatment of the missing defector states, also brings calculated transition rates into good agreement with experimental data [20]. As a whole, the pseudospin representation offers an approximate, but nonetheless physically relevant way of arriving at a model that can be used in a description of heavy nuclei. A special advantage of this approach is the possibility of finding analytical transforms of
important physical operators.

3. Deformation-dependent pseudospin transformation

The natural choice of the mean field for harmonic-oscillator related nuclear studies is provided by the Nilsson model and its extensions [21,22]. These theories amend the deformed oscillator Hamiltonian to reproduce the observed shell structure.

In two limiting cases of the Nilsson model, the spherical and strongly prolate deformed oscillator, there exist well-established analytical solutions. In these two cases the appropriate forms of the pseudospin transformation were found [14,15] which allow for the transformation of the Hamiltonian in a closed analytical form. This is not accidental and is indicative of the fact that there exists an intrinsic connection between the model Hamiltonian and the corresponding transformation. As shown in the current section, this line of thought leads to an explicit form for the deformation-dependent pseudospin transformation which happens to be closely related to some Hamiltonians of the Nilsson type and simultaneously to the helicity transformation considered in the previous section.

A generic Nilsson-type Hamiltonian (in units of \( \hbar \)) for a triaxially deformed nucleus can be written in the form

\[
h = h_{\text{osc}} - k(u_{ls} + \mu u_{lt}),
\]

where \( h_{\text{osc}} \) denotes the harmonic oscillator,

\[
h_{\text{osc}} = \sum_s p_s^2 + \left( \frac{m\omega_s x_s}{2\hbar} \right)^2 = \sum_s \epsilon_s (b_s^\dagger b_s + \frac{1}{2}),
\]

with \( s = z, x, y \) denoting the three Cartesian axes, and \( u_{ls} \) and \( u_{lt} \) symbolizing the spin–orbit and orbit–orbit interactions, respectively, whose structure is specified below for each case considered: namely, spherical and axial limits of the theory, as well as for the generic triaxial case. The values of the dimensionless parameters \( k \) and \( \mu \) are determined mainly by the mass region of the nucleus. The boson operators are defined in the deformed basis by

\[
\begin{align*}
  b_s &= \sqrt{\frac{m\omega_s}{2\hbar}} x_s + \frac{i}{\sqrt{2m\hbar\omega_s}} p_s, \\
  b_s^\dagger &= \sqrt{\frac{m\omega_s}{2\hbar}} x_s - \frac{i}{\sqrt{2m\hbar\omega_s}} p_s
\end{align*}
\]

and obey the standard commutation relations

\[
[b_x, b_y^\dagger] = \delta_{xy}, \quad [b_x, b_y] = [b_x^\dagger, b_y^\dagger] = 0
\]

(note the different font used throughout the paper for the operators in the deformed basis). The dimensionless frequencies \( \epsilon_s = \omega_s / \omega \) are subject to volume conservation

\[
\epsilon_z \epsilon_x \epsilon_y = 1.
\]
3.1. Spherical limit

In the spherical case \(\varepsilon_z = \varepsilon_x = \varepsilon_y = 1\)

\[ h_{\text{osc}} \rightarrow n + \frac{3}{2}, \quad \mu_{xx} \rightarrow l \cdot \sigma, \quad \mu_{ll} \rightarrow l^2 - \langle l^2 \rangle_s, \]  

(10)

where the nondeformed number of quanta operator \(n = b^+ \cdot b\) and the physical orbital momentum \(l = (\mathbf{b} \times \mathbf{b})^+\) are constructed out of the spherical boson operators

\[ b_s = \sqrt{\frac{ma}{2\hbar}} x_s + \frac{i}{\sqrt{2}\hbar\omega} p_s, \]
\[ b_s^* = \sqrt{\frac{ma}{2\hbar}} x_s - \frac{i}{\sqrt{2}\hbar\omega} p_s. \]

(11)

The mean value of \(l^2\) over a given shell, \(\langle l^2 \rangle_n = \frac{1}{2}n(n + 3)\), is subtracted to ensure that the average single-nucleon energy within a shell is fixed by its harmonic-oscillator value [22].

The eigenstates \(|nlj_z\rangle\) of the Hamiltonian, specified by Eqs. (5) and (10), in coordinate representation are given by formula (2), and the corresponding eigenvalues are easily calculable:

\[ e_{nlj} = n + \frac{3}{2} + k \left[ (l - j) \left( l + \frac{1}{2} \right) + \frac{1}{2} - \mu \left( l(l + 1) - \frac{1}{2}n(n + 3) \right) \right]. \]

(12)

The single-particle pseudospin operator in this limit can be written in the explicit form [14]

\[ U_0 = d_0 (d_0^+ d_0)^{-1/2}, \]

(13)

or, equivalently [16],

\[ U_0 = (d_0 d_0^+)^{-1/2} d_0, \]

(14)

where

\[ d_0 = b \cdot \sigma. \]

(15)

The equivalence of Eqs. (13) and (14) and the unitarity property for the spherical pseudospin transformation

\[ U_0 U_0^+ = U_0^+ U_0 = 1 \]

(16)

follow from the operator-valued identity

\[ \hat{x} f(\hat{x}) = f(\hat{x}) \hat{x}, \]

(17)

which requires the property of associativity of the \(\hat{x}\) and \(\hat{y}\) operators for its proof, and holds for any analytic function \(f(\hat{z})\). [Since the structure of the pseudospin transformation for an arbitrary anisotropy of the oscillator field is similar to the structure of \(U_0\)]
(see the following subsections), both forms for the transformation operator (Eqs. (13) and (14)) as well as the unitarity property are valid in the general case.)

When acting onto the entire space of basis states, the $U_0$ operator projects out the subspace of defect states ($j = n + \frac{1}{2}$) and performs the unitary transformation of the normal-parity states subspace in the following manner:

$$U_0 |n l j j_z⟩ = |n l̄ j j_z⟩,$$

(18)

where the relabelling rule $n̄ = n - 1, l̄ = 2j - l$ is the same as discussed in the previous section. The rules of the pseudospin transformation for the operators under consideration are known [14,23]

$$U_0 n U_0^+ = n + 1,$$

(19)

$$U_0 l · \cdot \cdot U_0^+ = -l · \cdot \cdot - 2,$$

(20)

$$U_0 l^2 U_0^+ = l^2 + 2l · \cdot \cdot + 2,$$

(21)

and yield the image for the Hamiltonian in the form

$$U_0 H U_0^+ = h_{osc} + 1 - k ((2\mu - 1) u_{ts} + \mu u_{t̄}) + 2k(1 - \mu) + k\mu(n + 2),$$

(22)

where the $h_{osc}, u_{ts}$ and $u_{t̄}$ terms are formally the same as in (10) but now they act in the pseudospin representation. The transformed Hamiltonian, which is known as the pseudo Nilsson one [5], has the structure close to the original Hamiltonian. A significant difference between the two Hamiltonians is a sharp reduction in the spin-orbit splitting since the empirical average value of $\mu$ is almost 0.5 (more precisely, it is about 0.4 for neutrons and 0.6 for protons [13]). This reduction is the cornerstone of the pseudospin symmetry. Another distinction is due to the last term in Eq. (22) which produces a small (in most cases insignificant) increase in the oscillator frequency because the value of $k\mu$ is in the 0.02–0.04 range for heavy nuclei.

As a consequence of the unitarity, the pseudo Nilsson spherical Hamiltonian is isospectral to the Nilsson Hamiltonian in the normal-parity sector; namely, the formula (12) evaluates the energy of both $|n l j j_{z}\rangle$ and $|n l̄ j j_{z}\rangle$ states. A further noteworthy property of the spherical pseudospin transformation (as well as of the helicity form) is its rotational invariance, i.e. the conservation of the angular momentum operator $j = l + s$. The proof follows from the commutation rule $[j, b · \cdot \cdot] = 0$ and the Hermiticity of $j$.

3.2. Cylindrical limit

Since the Nilsson Hamiltonian is based on a phenomenological description of the nuclear mean field rather than a rigorous derivation from a more microscopic model, there is no unambiguous prescription for an explicit form of the $u_{ts}$ and $u_{t̄}$ operators for deformed nuclei. Nevertheless, it is usually assumed, following Nilsson’s arguments [21], that the total number of quanta $n = \sum b^+ b$, in the deformed basis is well conserved, and that the $u_{ts}$ and $u_{t̄}$ terms for the strong deformation region have to be chosen accordingly. (Even for small deformations and use of $n$ within several percent of the deformed nuclei, the ansatz [23]

$$u_{ts} \rightarrow \xi l \cdot \cdot \cdot,$$

$$u_{t̄} \rightarrow \xi l̄ \cdot \cdot \cdot,$$

where $\xi$ and $\xi$ are the parameters of the deformed basis $l \cdot \cdot \cdot, l̄ \cdot \cdot \cdot$.

The Nilsson diagram for the orbit strength (in the case of $l^2$, $l \cdot \cdot \cdot$) of the nuclear deformation $\xi$ coincides with the one determined by Eq. (22) for the basis $|n l j j_{z}\rangle$ with $l \cdot \cdot \cdot$.

As shown in Ref. [23], in the asymptotic limit the coefficients $d_{∞, 0} = d_{∞, 0}$ also hold, similar to the Nilsson Hamiltonian.

The $U_{∞}$ transformation, and accordingly, $U_{∞} |n l j j_{z}\rangle = |n l j j_{z}\rangle$

This rule is equivalent to

$$n̄ = n - 1,$$
operator (Eqs. (13) case.) for projects out the transformation of the 

\[ \text{(18)} \]

used in the previous under consideration

\[ \text{(19)} \]

\[ \text{(20)} \]

\[ \text{(21)} \]

\[ \text{(n+2)} \] but now they act

\[ \text{which is known as the nilsson structure.} \]

\[ \text{in the spin-}

(22) oscillator frequency

\[ \text{(more precisely, it is the cornerstone term in Eq. (22) oscillator frequency} \]

\[ \text{Nilsson is isospec-}

\[ \text{the formula (12) worthy property of} \]

\[ \text{m) is its rotational}

\[ \lambda = l \pm s. \text{The proof} \]

\[ \text{description of the nu}

\[ \text{copic model, there are}

\[ \text{operators for de-}

\[ \text{same arguments [21],}

\[ \text{is well conserved, have to be chosen}

\[ \text{accordingly. (Even the conventional Nilsson scheme, which is applicable at average deformations and uses the spherical assumption (10) for these terms, is known to preserve } n \text{ within several percent accuracy up to the superdeformed region. However, at higher deformations the mixing among deformed shells becomes significant.)} \]

\[ \text{In the case of axial deformation } (\epsilon_x = \epsilon_y = \epsilon, \epsilon_z = 1/\epsilon^2) \text{ the model Hamiltonian is no longer rotationally invariant and commutes with } j_z \text{ only. An increase in } \epsilon \text{ makes a nuclear shape more prolate, and at some point the asymptotic Nilsson limit is reached where the number of longitudinal quanta } n_z \text{ is conserved. If the model Hamiltonian conserves } n \text{, as suggested above, then regardless of its exact form the spectrum is actually determined by its part that commutes with } n_z, \text{ and the terms in the Hamiltonian can be effectively written as} \]

\[ h_{\text{osc}} = -e(n_p + 1) + \frac{1}{e^2} \left( n_z + \frac{1}{2} \right), \]

\[ u_{l_x} = \xi (l_x - (l_x)_n), \]

\[ u_y = \xi (l_y - (l_y)_n), \]

\[ (23) \]

where \( \xi \) and \( \xi \) are the model parameters, \( (l_x)_n = \frac{1}{\xi} n(n+3) \) is the trace-equivalent part of \( l_x \) within the deformed shell, and \( n_p = n_z + n_t \) is the number of transverse quanta. (Note that \( l_z \) coincides with \( l_z \) in the axial case.) The eigenstates of the effective Hamiltonian, determined by Eqs. (5) and (23), are the Nilsson asymptotic states in the cylindrical basis \( |n, l_z, j_z⟩ \) with \( j_z = l_z + \frac{1}{2} \) or \( l_z - \frac{1}{2} \).

Nilsson diagrams in the deformed region clearly indicate that, even though the spin-orbit strength is large, there exist pairs of levels whose energy splitting is weak and slowly decreasing with increasing deformation. In the domain of strong deformation these pairs consist of the asymptotic orbitals \( |n, n_z, l_z, l_z + \frac{1}{2}⟩ \) and \( |n, n_z, l_z + 2, l_z + \frac{1}{2}⟩ \).

As shown in Ref. [15], this result can be understood if the pseudospin transformation in the asymptotic region is of the form

\[ U_\infty = d_\infty(d_\infty^+ d_\infty)^{-1/2}, \]

\[ (24) \]

where

\[ d_\infty = b_x \sigma_x + b_y \sigma_y. \]

\[ (25) \]

This transformation is unitary and its equivalent form

\[ U_\infty = (d_\infty^+ d_\infty)^{-1/2} d_\infty \]

\[ (26) \]

also holds, similarly to Eq. (14).

The \( U_\infty \) transformation conserves \( j_z \), the longitudinal component of the angular momentum, and acts on the cylindrical basis states according to the rule

\[ U_\infty |n, n_z, l_z, l_z \pm \frac{1}{2}⟩ \equiv |n - 1, n_z, l_z \pm 1, l_z \pm \frac{1}{2}⟩. \]

\[ (27) \]

This rule is equivalent to the relabelling procedure

\[ \bar{n} = n - 1, \quad \bar{n}_z = n_z, \quad \bar{l}_z = l_z + 2s_z, \quad \bar{s}_z = -s_z \]

\[ (28) \]
within the subspace of cylindrical states with \( n_p > l_z \). The pairs of nearly degenerate states \( |n, n_z, l_z, l_z + \frac{1}{2} \rangle \) and \( |n, n_z, l_z + 2, l_z + \frac{3}{2} \rangle \) are then relabelled as \( |\tilde{n}, \tilde{n}_z, \tilde{l}_z, l_z - \frac{1}{2} \rangle \) and \( |\tilde{n}, \tilde{n}_z, \tilde{l}_z, l_z + \frac{1}{2} \rangle \), respectively, and can be treated as pseudospin doublets. The rest of the states, with \( n_p = l_z \), span the subspace where the action of \( U_\infty \) is undefined.

Under the \( U_\infty \) transformation, the operators entering Eq. (23) acquire the following images:

\[
\begin{align*}
U_\infty n_p U_\infty^+ &= n_p + 1, \\
U_\infty n_z U_\infty^+ &= n_z, \\
U_\infty l_z \sigma_z U_\infty^+ &= -l_z \sigma_z - 1, \\
U_\infty l_z^2 U_\infty^+ &= l_z^2 + 2l_z \sigma_z + 1
\end{align*}
\]

which are in accordance with the relabelling rules. Then the transformed Hamiltonian can be written in the form

\[
U_\infty H U_\infty^+ = h_\text{osc} + \epsilon - k \left( \frac{2\xi}{\xi} \mu - 1 \right) u_{1s} + \mu u_{1s} + k_\xi (1 - \mu) + \frac{1}{2} k_\mu \xi (n + 2).
\]

Therefore, as for the spherical case, in the cylindrical limit the Nilsson-type Hamiltonian can be replaced by its “pseudo” version which is characterized by a much weaker spin–orbit splitting and slightly higher oscillator frequencies (provided the value of \( \xi/\xi \) is close to 1). As mentioned above, the action of the asymptotic pseudo Nilsson Hamiltonian is confined to the subspace of the entire space of states but this subspace is different from the normal parity subspace of the spherical limit.

3.3. Generic case

As seen from Eqs. (13), (14) and (24), (26), the structure of the pseudospin transformation operator in the spherical and cylindrical limits is very similar although the basic structural blocks (15) and (25) are somewhat different. It is natural to assume that the \( d_0 \) and \( d_\infty \) operators are limiting cases of an operator \( d \) which is the structural block of a generic pseudospin transformation

\[
U = d (d^+ d)^{-1/2} = (dd^+)^{-1/2} d.
\]

The latter transformation is unitary and valid for arbitrary deformation.

To find an explicit form of the \( d \) operator, one can assume that the pseudospin transformation is closely related to the helicity operator form. Indeed, if the helicity transformation plays the same role in the realistic nuclear structure models as the pseudospin transformation does in the oscillator shell model, there should exist a direct connection between the two transformations.

Note that the single-particle helicity transformation \( \mathcal{H} \) can also be written in the form (34) with the structural block

\[
d_{\text{het}} = p \cdot \sigma.
\]

The helicity operation is regardless of any specific is confined to a mean-field deformed oscillator, determined Eq. (35) as

\[
d_{\text{het}} = -i \sqrt{\frac{2}{\hbar \omega_0}} \sum_i \epsilon_i b_i \sigma_z,
\]

where the boson operator is the deformed shell with state requirements on the interactions with the shell be ignored since it has a structural block \( d \) can be

\[
d = \sum_i \sqrt{\epsilon_i} b_i \sigma_z.
\]

This equation, together with the pseudospin transformation, is exact and consistent with the entire structure.

(i) Correspondence with the spherical limit
(ii) Conformity between the deformed oscillator basis, the transformed regular sector of the region of unitary, the operators of the unitary basis are closely related;
(iii) Equivalence of the Hamiltonian (22).

Checking and implementation of the rest of the paper is beyond the scope of this paper. The last one needs to be discussed in the last section.
nearly degenerate states \(|\vec{n}, \vec{n}, l_z, l_z - \frac{1}{2}\) doublets. The rest is undefined.

\begin{align}
(29) \\
(30) \\
(31) \\
(32)
\end{align}

Hamiltonian

\[
\hat{H} = \frac{1}{2} \hbar \omega (n + 2).
\]

\begin{align}
(33)
\end{align}

on-type Hamiltonian, a much weaker

de the value of \(\mu\) Nilsson but this subspace

the pseudospin, similar although

\begin{align}
(34)
\end{align}

the pseudospin, if the helicity levels as the pseudospin exist a direct

written in the form

\[
d_{\text{hel}} = p \cdot \sigma.
\]

The helicity operation is a universal transformation acting on the microscopic level regardless of any specific properties of a given nucleus. However, when the discussion is confined to a mean-field description, one can model the self-consistent field with the deformed oscillator, determined by the frequencies \(\epsilon_z, \epsilon_x\), and \(\epsilon_y\), and henceforth rewrite Eq. (35) as

\[
d_{\text{hel}} = -i \frac{\hbar}{\hbar} \sum_s \sqrt{\epsilon_s} (b_s^+ - b_s^0) \sigma_s, 
\]

where the boson operators are defined by Eq. (7). Now note that \(d_{\text{hel}}\) mixes states of a deformed shell with states from the shell below as well as the shell above. To comply with requirements on the pseudospin operator, considered in Section 2, one must exclude interactions with the shell above. Also, the constant factor in the definition of \(d_{\text{hel}}\) can be ignored since it has no effect on the \(U\) operator according to Eq. (34). Finally, the structural block \(d\) can be defined as follows:

\[
d = \sum_s \sqrt{\epsilon_s} b_s \sigma_s. 
\]

This equation, together with Eq. (34), uniquely defines the generic deformation-dependent pseudospin transformation.

At this point it is expedient to formulate general requirements which guarantee internal consistency of the entire approach and its relevance for nuclear structure calculations:

(i) Correspondence with the known limiting cases;

(ii) Conformity between the Hamiltonian (5) and the transformation at any given deformation. Specifically,

- the transformation should not destroy the shell structure of the deformed oscillator basis,

- the transformed Hamiltonian should be isospectral to the original one in the regular sector of the entire space of states which is defined to be the kernel (the region of unitarity) of the transformation,

- the operators entering both the Hamiltonian and the transformation should be closely related;

(iii) Equivalence of the spherical and deformed pseudo representations for physical applications. The deformed pseudo Nilsson Hamiltonian must be the same whether it is constructed by inserting the deformation into the spherical pseudo Nilsson Hamiltonian (22) or by applying the generic transformation to the original Hamiltonian (5). This condition implies a specific form of the \(u_\text{II}\) and \(u_\text{III}\) operators.

Checking and implementing these requirements determine the structure and content of the rest of the paper. The first two requirements will be considered in this subsection. The last one needs an analytical expression for the transformed Hamiltonian and will be discussed in the last section.

It is evident that in the spherical limit the operator (37) coincides with (13) and (14). In the axial case \((\epsilon_x = \epsilon_y = \epsilon, \epsilon_z = 1/\epsilon^2)\) the \(d\) operator is reduced to \(d_{\text{axi}} = \)}
b_{3} \sigma_{3} + b_{2} \sigma_{2} + e^{-3/2} b_{5} \sigma_{5} \text{ after the elimination of a common } \sqrt{e} \text{ factor. The asymptotic prolative shape formally corresponds to the limit } e \gg 1 \text{ in which case } d_{\infty} \text{ approaches } d_{\infty}, \text{ and the results } [15], \text{ reviewed in the previous subsection, can be reproduced. Nevertheless, this limiting case should be treated with caution. Indeed, the ratio of frequencies } \omega_{2}/\omega_{5} = e^{3/2} \text{ is known to be close to } 2 \text{ in the superdeformation region, and therefore the contribution of } b_{5} \sigma_{5} \text{ is not negligible in the experimentally achievable domain.}

The shell structure of the anisotropic oscillator is preserved by the transformation (34) and (37) since the shell number gets decreased exactly by one: } \bar{n} = n - 1. \text{ However, in contrast to the spherical and cylindrical limits, the Hamiltonian eigenstates in the generic case do not coincide with the eigenstates of the harmonic oscillator. The pseudospin transformation divides the } n \text{th deformed oscillator shell in the normal representation into } 2 \text{ regions which can be named a regular sector } R_{n} \text{ and a singular one } S_{n}. \text{ Within the regular sector, which is analogous to the normal-parity subspace in the spherical limit, the transformation is unitary. The singular sector is similar to the defector subspace. It is defined as the part of the space of states which is annihilated by the } d \text{ operator,}

\[ dS_{n} = 0, \] (38)

and within this sector the transformation is undetermined. The transformation maps the region of unitarity onto an oscillator shell with one quantum less, with the latter shell in the pseudo representation becoming representative of the whole former shell in the normal representation. The eigenstates within the former and the latter shells can be determined only through the action of the original and transformed Hamiltonians, respectively. Within the region of unitarity, both the Hamiltonians are isospectral, and there exists a one-to-one mapping between the corresponding eigenstates.

A noteworthy fact, although not emphasized in Refs. [14,15], is a close relation between the structural block of the transformation and the operators entering the Hamiltonian. By comparing the relations

\[ n + \frac{3}{2} = \frac{1}{2} [d_{0}, d_{0}^{+}]_{+}, \] (39)

\[ n_{0} + 1 = \frac{1}{2} [d_{\infty}, d_{\infty}^{+}]_{+}, \] (40)

\[ l \cdot \sigma_{2} + \frac{3}{2} = \frac{1}{2} [d_{0}, d_{0}^{+}]_{+}, \] (41)

\[ l_{2} \sigma_{2} + 1 = \frac{1}{2} [d_{\infty}, d_{\infty}^{+}]_{+}, \] (42)

\[ l^{2} = (l \cdot \sigma)^{2} + l \cdot \sigma, \] (43)

\[ l_{2}^{2} = (l_{2} \sigma_{2})^{2}, \] (44)

where \([\hat{x}, \hat{y}]_{+} \{[\hat{x}, \hat{y}]_{+}\}) \text{ denotes a commutator (anticommutator), with Eqs. (10) and (23), note that in both limiting cases the harmonic-oscillator and spin-orbit splitting terms are bilinear combinations of } d \text{ and } d^{+}, \text{ and the orbit-orbit interaction is the square of the spin-orbit splitting operator plus a correction which eliminates a dependence on the spin variable. (This process poses no problem.) It is natural to suggest harmonic-oscillator to}

\[ h_{\text{osc}} = \frac{1}{2} [d, d^{+}], \] (45)

This logic suggests the

\[ u_{l_{2}} = \frac{1}{\sqrt{e_{l}}} \{[d, d^{+}] \} \] (46)

where \(\zeta\) is the mode components of the \(A_{l}\).

\[ A_{l} = \frac{1}{\sqrt{e_{l}}} \{[d, d^{+}] \}, \]

are defined through the representation, and the \(m_{l} = \sum_{n} e_{n}^{l} \)

for any real \(q\). Note also

Following this line of

\[ u_{l_{2}} = \frac{1}{\sqrt{e_{l}}} \{[d, d^{+}] \}, \]

By introducing an auxiliary

\[ A'_{l} = \sqrt{e_{l}}), \]

rewrite \(A^{2}\) in the form

\[ A^{2} = (A \cdot \sigma)^{2} + A'^{2}, \]

which is reminiscent of

It is easily seen that the spherical and cylindrical regions. However, despite Hamiltonian can statement is purely alge
d_{0} and d_{0}^{+} form a close known as the \(osp(1|2)\) the principal reason why as the original one. The it is proven that the op Hermitian conjugate and
The asymptotic \( d_{\text{stat}} \) approaches be reproduced.

\[ h_{\text{osc}} = \frac{1}{2} [d, d^+] \varepsilon. \]  

This logic suggests the following assumption for the spin-orbit interaction:

\[ u_{\text{so}} = \frac{1}{2} \xi ([d, d^+] - m_1) = \xi \Lambda \cdot \sigma \]  

where \( \xi \) is the model parameter which may depend on deformation. The Cartesian components of the \( \Lambda \) vector,

\[ \Lambda_s = \frac{1}{\sqrt{\varepsilon_s}}, \]  

are defined through the components \( \varepsilon \), of the orbital momentum of the deformed representation, and the \( m_q \) parameters are given by the formula

\[ m_q = \sum_i \varepsilon^i_q \]  

for any real \( q \). Note also that Eq. (47) uses the volume conservation condition.

Following this line of thought, choose the orbit-orbit interaction operator in the form

\[ u_{\text{orb}} = \xi (\Lambda^2 - \langle \Lambda^2 \rangle n), \]  

\[ \langle \Lambda^2 \rangle n = \frac{1}{2} m_{-1} n (n + 3). \]  

By introducing an auxiliary operator \( \Lambda' \) with components

\[ \Lambda'_s = \sqrt{\varepsilon_s} \varepsilon, \]  

rewrite \( \Lambda^2 \) in the form

\[ \Lambda^2 = (\Lambda \cdot \sigma)^2 + \Lambda' \cdot \sigma, \]  

which is reminiscent of Eqs. (43) and (44).

It is easily seen that the \( u_{\text{so}} \) and \( u_{\text{orb}} \) operators thus obtained reduce to both the spherical and cylindrical limits and are a natural extrapolation to other deformation regions. However, despite this smooth extrapolation, an analytical transform for the model Hamiltonian cannot be written in a closed form. The argument behind this statement is purely algebraic. Note that in the spherical limit bilinear combinations of \( d_0 \) and \( d^*_2 \) form a closed set under both the commutation and anticommutation relations known as the \( \mathfrak{osp}(1|2) \) superalgebra [24]. The closure of the permutation relations is the principal reason why the transformed Nilsson Hamiltonian becomes almost as simple as the original one. The same situation holds in the cylindrical limit. In Appendix B it is proven that the operator \( \sum_s f(\varepsilon_s) b_s \sigma_s \), where \( f(x) \) is an analytic function, its Hermitian conjugate and, therefore, any combination of such operators can be written in
terms of some combination of the $d$ and $d^+$ operators only. Thus, generally speaking, the above defined operators entering a generic Nilsson-type Hamiltonian, as well as their pseudospin transforms, can still be rewritten in terms of $d$ and $d^+$ only. However, in contrast with the spherical and cylindrical limits, the permutation relations no longer close for general finite deformation, and such a rewrite cannot be folded into a simple expression and thus would be of no help for practical purposes.

Nevertheless, there exists an analytical, although approximate, solution to the problem which is reasonable for higher shells, i.e. in the case of the heavy nuclei. It is based on a technique that proved accurate in the spherical representation for operators that are not reducible to combinations of $d_0$ and $d^+_0$. This technique, and the resulting approximate transforms for the Hamiltonian terms are discussed in the next section.

4. Approximate pseudospin transforms

Consider the permutation rule

$$df(d^+d) = f(dd^+)d,$$

which is a particular case of Eq. (17). By applying this rule along with the definitions (34), (37) of the transformation operator, one obtains the formula

$$Ud^+dU^+ = dd^+, \quad (53)$$

which can also be written as

$$U(h_{osc} - \Lambda \cdot \sigma - \frac{1}{2} m_1)U^+ = h_{osc} + \Lambda \cdot \sigma + \frac{1}{2} m_1. \quad (54)$$

This result and the obvious equality

$$UnU^+ = n + 1 \quad (55)$$

are apparently the only independent pseudospin transforms derivable in exact analytical form. For instance, a similar rule for the $d$ and $f(dd^+)$ operators does not exist as a consequence of the identity

$$d(dd^+) = (d^+d) + 2 \sum_s e_s^{3/2}b_s\sigma_s, \quad (56)$$

and the fact that $\sum_s e_s^{3/2}b_s\sigma_s$ is not a linear function of $d = \sum_s e_s^{1/2}b_s\sigma_s$, except for the spherical and asymptotic axial ($\epsilon \gg 1$ or $\epsilon \ll 1$) limits. (Note that Eq. (56) is just a rewrite of the commutation relation

$$\left[\sum_s e_s b_s^*, \sum_s e_s^{1/2}b_s^+ \sigma_s\right] = 2 \sum_s e_s^{3/2}b_s\sigma_s$$

with the help of the auxiliary relation $\sum_s e_s b_s^2 = d^2$.)

The above notes might seem cumbersome. Nevertheless, the Hamiltonians in the limit to expect that there exist Nilsson-type Hamiltonians associated with the simple analytical form isospectrality condition.

4.1. Approximation procedure

The procedure is to form the particle operator $F$ such that

$$dFd^+ = \frac{1}{2}(F, dd^+),$$

with $d$ and $d^+$ defined as the $d$ and $d^+$ operators in the only commutation relation $l$ h.s. of Eq. (57). The residual term $G$ is derived in the derivation. In what order to ultimately sin...
The above notes imply that the exact transform of the Hamiltonian (5) might look very cumbersome. Nevertheless, given the similarity between the original and transformed Hamiltonians in the limiting cases considered in the previous section, it seems reasonable to expect that there exists an analytical but approximate procedure for calculating the Nilsson-type Hamiltonian image for any deformation which yields a transform of the pseudo Nilsson type. This kind of procedure is proposed and discussed below. The errors associated with the use of such approximations can be roughly evaluated a priori in a simple analytical form; a rigorous a posteriori numerical estimate is provided by the isospectrality condition.

4.1. Approximation procedure

The procedure is based on the following observation: In general, for a given single-particle operator $F$ there exist several different pairs of operators $F$ and $G$ satisfying the identity

$$dFd^+ = \frac{1}{2} \{ F, dd^+ \} + G$$  

with $d$ and $d^+$ defined in Eq. (37). Different choices for $F$ and $G$ are possible because the $d$ and $d^+$ operators contain the spin variables that in turn enables applying not only commutation relations but also anticommutation and generic permutations to the L.h.s. of Eq. (57). While $F$ usually has a structure similar to $F$, the structure of the residual term $G$ is dependent upon the choice of the permutation relation that is used in the derivation. In what follows, it is assumed that the permutation relation is chosen in order to ultimately simplify the structure of $G$.

The transformed operator can therefore be written in the form

$$U F U^+ = \frac{1}{2} \left( (dd^+)^{1/2} F (dd^+)^{-1/2} + (dd^+)^{-1/2} F (dd^+)^{1/2} \right)$$

$$+ (dd^+)^{-1/2} G (dd^+)^{-1/2}.$$  

Note that the $(dd^+)^{-1/2} = (h_{osc} + A \cdot \sigma + \frac{1}{2} m_1)^{-1/2}$ and its inverse, $(dd^+)^{1/2}$, are regular operator-valued functions within a given oscillator shell. Their common argument can be represented within the $n$th shell as

$$dd^+ = \langle dd^+ \rangle_n + \sum \delta n_s + A_s \sigma_s,$$  

where

$$\langle dd^+ \rangle_n = \frac{1}{2} m_1 (n + 3)$$  

is the average value of the operator, and $\delta n_s = n_s - \langle n \rangle$ is the deviation of the number of bosons along a Cartesian axis from the mean value.

Since the pseudospin symmetry is relevant for heavy nuclei and high single-particle orbitals, it is makes sense to approximate the r.h.s. of Eq. (58) by making a formal operator-valued expansion with $(n + 3)^{-1}$ as a small parameter. The advantage of such
an expansion is that the mean value within the shell of the first-order correction often vanishes, and in some practically interesting cases the correction itself vanishes. It is important to recall, however, that average values of the single-particle angular momenta within the shell correlate with the shell number. So the above formal expansion is apparently asymptotic and should be used with appropriate caution.

The expansion technique is applied below only to those operators which are bilinear in boson operators. When formally rewritten in series of inverse powers of \( n + 3 \), such operators may contain only negative first and zeroth degree terms. Therefore, it is reasonable to neglect the impact of the terms of the positive first and higher degrees. The approximate transforms thus obtained carry no explicit dependence on the expansion parameter, i.e. they are shell-independent. As shown later, they provide the exact results in both the spherical and cylindrical limits.

The

\[
\frac{1}{2} \left( (dd^+)^{1/2} \bar{F}(dd^+)^{-1/2} + (dd^+)^{-1/2} \bar{F}(dd^+)^{1/2} \right)
\]

term in Eq. (58) can be approximated by \( \bar{F} \). This is appropriate because the two operators behave similarly under Hermitian conjugation, have the same traces in any subspace of single-particle states, and their difference can only be on the order of \( O((n+3)^{-2}) \). The latter estimate is valid since the linear term vanishes in the expansion of

\[
\frac{1}{2} \left( (dd^+)^{1/2} \bar{F}(dd^+)^{-1/2} + (dd^+)^{-1/2} \bar{F}(dd^+)^{1/2} \right).
\]

The \( G \) operator is representable in the form

\[
G = \frac{1}{2} (n+3) G_{-1} + G_0,
\]

which estimates the residual \( (dd^+)^{-1/2} G (dd^+)^{-1/2} \) term by \( G_{-1} / m_1 \). [Eqs. (59) and (60) display a particular case of representation (61).]

By combining the above considerations, the approximation

\[
U F^+ U^+ = \bar{F} + \frac{1}{m_1} G_{-1} + O \left( \frac{1}{n+3} \right)
\]

for the transform of the operator \( F \) is obtained. The \( \bar{F} \) and \( G_{-1} \) operators are defined in Eqs. (57) and (61), and the dominant part of the approximation error occurs due to the latter operator. The accuracy of the approximation is expected to increase with increasing shell number. It is also noteworthy that the procedure of averaging over the oscillator shell, which lies at the foundation of Eq. (62), is quite natural for Nilsson-type models since these models normally use the values of parameters fixed for nuclei within given shells.
4.2. Scalar operators

Approximate transforms for the $h_{\text{osc}}$, $h'_{\text{osc}} = \sum_s \varepsilon_s^{-1} \left( n_s + \frac{1}{2} \right)$, $\Lambda \cdot \sigma$, $l \cdot \sigma$, and $\Lambda' \cdot \sigma$ operators, where $l$ is the orbital momentum vector of the deformed representation whose Cartesian components are $l_s$ with $s = x, y, z$, can be obtained by applying Eq. (62). Strictly speaking, it is more than sufficient for finding the transform of the Hamiltonian defined by Eqs. (5), (6) and (46)–(52). Nevertheless, having this set of operators determined allows for a propagation of the result to other operators of similar kind.

By using the boson commutation relations (7) and the well-known multiplication rule

$$ (\sigma \cdot u)(\sigma \cdot v) = i\sigma \cdot (u \times v), $$

the validity of the identities

$$ dh_{\text{osc}}d^+ = \frac{1}{2} \left[ [h_{\text{osc}}, dd^+]_+, + \sum_s \varepsilon_s^{3/2} \left( b_s \sigma_s d^+ + db_s \sigma_s^+ \right) \right], $$

$$ dh'_{\text{osc}}d^+ = \frac{1}{2} \left[ [h'_{\text{osc}}, dd^+]_+, + \sum_s \varepsilon_s^{-1/2} \left( b_s \sigma_s d^+ + db_s^+ \sigma_s^+ \right) \right], $$

$$ d\Lambda \cdot \sigma d^+ = \frac{1}{2} \left( -[2m_1 + \Lambda \cdot \sigma, dd^+]_+ + \sum_s \varepsilon_s^{3/2} \left( b_s \sigma_s d^+ + db_s^+ \sigma_s^+ \right) \right), $$

$$ dl \cdot \sigma d^+ = \frac{1}{2} \left( [2 - l \cdot \sigma, dd^+]_+ + 2 \sum_s \varepsilon_s^{1/2} \left( b_s l_s d^+ + db_s^+ l_s \right) 
- m_{1/2} \sum_s \left( b_s \sigma_s d^+ + db_s^+ \sigma_s^+ \right) \right), $$

$$ d\Lambda' \cdot \sigma d^+ = \frac{1}{2} \left( -[\Lambda' \cdot \sigma, dd^+]_+ + 2 \sum_s \varepsilon_s^1 \left( b_s l_s d^+ + db_s^+ l_s \right) 
- \sum_s \varepsilon_s^{-1/2} \left( b_s \sigma_s d^+ + db_s^+ \sigma_s^+ \right) \right) $$

can be verified. By comparing these identities along with the expressions for the leading terms of the relevant formal expansions,

$$ \left( \sum_s \varepsilon_s^\alpha \left( b_s \sigma_s d^+ + db_s^+ \sigma_s^+ \right) \right)_{-1} = m_{\alpha + 1/2}, $$

$$ \left( \sum_s \varepsilon_s^\beta \left( b_s l_s d^+ + db_s^+ l_s \right) \right)_{-1} = 3 \sum_s \varepsilon_s^{\beta + 1/2} l_s \sigma_s - m_{\beta} \Lambda' \cdot \sigma, $$

to Eqs. (57) and (62), the approximate analytical results

$$ U_{\text{osc}}U^+ \simeq h_{\text{osc}} + \frac{m_2}{m_1}, $$

(61)

(62)

(63)

(64)

(65)

(66)

(67)

(68)

(69)

(70)

(71)
\[ U h_{\text{osc}} U^+ \simeq h_{\text{osc}}' + \frac{3}{m_1}, \]  
\[ U A \cdot \sigma U^+ \simeq -A \cdot \sigma - m_1 + \frac{m_1^2}{m_1}, \]  
\[ U 1 \cdot \sigma U^+ \simeq 1 \cdot \sigma + \frac{3}{m_1} \sum_s \epsilon_s^l \sigma_s - \frac{m_1^2}{m_1} A' \cdot \sigma + 1 - \frac{m_1^2}{m_1}, \]  
\[ U A' \cdot \sigma U^+ \simeq -2A' \cdot \sigma + \frac{3}{m_1} \sum_s \epsilon_s^{l/2} \sigma_s - \frac{6}{m_1}, \]  
for the operator transforms under discussion can be obtained.

In accordance with the Cayley–Hamilton theorem for the deformation matrix \( E = \text{diag}(\epsilon_x, \epsilon_y, \epsilon_z) \), any analytical functional of \( E \) is a linear combination of three linearly independent functions of \( E \) with coefficients which are invariants of the same matrix (see Appendix B). In particular, it is convenient to make use of the \( I^0 \) operator, whose components are defined via

\[ \rho_s = \frac{1}{2} \sum_{\sigma} (\epsilon_{s\sigma})^2 \left( \sqrt{\epsilon_s^x} + \sqrt{\epsilon_s^y} \right) \lambda_s, \]

or, equivalently,

\[ \rho_s = \frac{1}{2} \sqrt{\epsilon_s} (m_1 - \epsilon_s) \lambda_s, \]

instead of the \( l \) vector of the anisotropic orbital momentum. Indeed, by applying Eq. (B.11) from Appendix B, one can derive approximate equations

\[ U I^0 \cdot \sigma U^+ \simeq I^0 \cdot \sigma + \frac{m_1 - 1}{2m_1} A' \cdot \sigma - \frac{3}{m_1} A \cdot \sigma - 2, \]

\[ U A' \cdot \sigma U^+ \simeq A' \cdot \sigma - \frac{6}{m_1} I^0 \cdot \sigma - \frac{6}{m_1} \]

\[ j = \frac{1}{2} (\delta + \{6, d\}) \]

in such a way.

The Cartesian components of \( \{6, d\} \) and the Hermitian of the standard

given Eq. (45) and (46):

\[ d j d^+ \]
\[ n + \frac{1}{2} = \frac{1}{4} \{ d, d^+ \} + \text{h.c.}, \]  
(81)

\[ h'_{\text{onc}} = \frac{1}{2} \{ d', d'^+ \} + \text{h.c.}, \]  
(82)

\[ l^0 \cdot \sigma + \frac{1}{2} = \frac{1}{4} \{ d, d'^+ \} + \text{h.c.}, \]  
(83)

\[ \Lambda' \cdot \sigma + \frac{1}{2} m_{-1} = \frac{1}{2} \{ d', d'^+ \}. \]  
(84)

Note that since the transforms of the operators of the closed set are linear functions of the operators themselves, and the model Hamiltonian depends only on these operators, the structure of the approximate pseudospin image of the Hamiltonian occurs to be similar to the structure of the Hamiltonian itself. In other words, while the exact transformation of the Hamiltonian cannot be performed in a closed analytical form, the approximate transformation readily yields an effective Hamiltonian of the pseudo Nilsson type. An explicit form of this "pseudo" Hamiltonian will be considered in the next section.

4.3. Angular momentum

It follows from the basic definitions (7) that the \( l^0 \) operator, determined by Eq. (76), is the part of the physical orbital momentum operator \( l \) rewritten in the deformed representation which preserves the total number of quanta. As one can conclude from Eq. (83) and the above remarks regarding the \( d \) and \( d' \) operators, its occurrence in connection to the pseudospin transformation is not accidental and reflects a physically important property of this transformation. Namely, the pseudospin transformation is explicitly defined in the deformed basis but nevertheless is related to the physical angular momentum \( j = l + \sigma/2 \) rather than to the angular momentum \( j = l + \sigma/2 \) in the deformed representation. This fact can be demonstrated by pointing out that the angular momentum can be decomposed into a sum of two conjugates,

\[ j = \frac{1}{2} (\delta + \delta^+), \]  
(85)

in such a way that

\[ [\delta, d] = [\delta^+, d^+] = 0. \]  
(86)

The Cartesian components of \( \delta \) and \( \delta^+ \) are defined via the rule

\[ \delta_r = i \sum \epsilon_{rst} \sqrt{\frac{\epsilon}{\epsilon_i}} \beta_i (b_r + b_r^+) + \frac{1}{2} \sigma, \]  
(87)

(and the Hermitian conjugate to this rule) and satisfy the requirement (86) by virtue of the standard boson commutation relations (8).

Given Eq. (86), it is possible to find the approximate transform of the angular momentum vector. Indeed, the commutation rule yields the identity

\[ d j d^+ = \frac{1}{4} [j, d d^+] + \frac{1}{4} [\delta - \delta^+, d d^+]. \]  
(88)
Since the last term in the r.h.s. of this formula does not contain a contribution proportional to \( n + 3 \) (cf. Eqs. (57) and (61)), the pseudospin transform of \( f \) is written in the form

\[
U_j U^+ = j + O \left( \frac{1}{n + 3} \right)
\]

(89)
in accordance with Eq. (62).

The physical angular momentum is therefore conserved by the pseudospin transformation with better accuracy the higher the shell number. The degree of conservation of \( j \) by the transformation correlates with the degree of its conservation by the Hamiltonian – in line with the rule of conformity between the transformation and the Hamiltonian (see Subsection 3.3). In particular, the angular momentum as a whole is strictly conserved in the spherical limit, and its longitudinal component \( j_z \) under the axial deformation.

Realistic many-particle nuclear Hamiltonians are necessarily rotationally invariant. The rotational symmetry of nuclear systems is reflected by the fact that the helicity transformation (1) preserves the angular momenta of individual nucleons:

\[
\hat{H} f \hat{H} = j.
\]

(90)
while relabelling their orbital and spin momenta. The pseudospin transformation is a modification of the helicity transformation for the needs of the deformed oscillator shell model. Consequently, the rotational invariance is inherited by the pseudospin transformation, but only to the extent this can occur under the symmetry breaking induced by the deformed mean-field approximation and a decrease in shell number.

4.4. Accuracy of the approximate transformation

The approximate transforms derived thus far employed a general procedure based on a formal expansion in inverse powers of \( n + 3 \). However, for the particular case of Eqs. (71) and (73) another alternative procedure can be considered which gives some insight into the essence of the approximations as well as an a priori estimate of their accuracy.

Consider approximating the last term in the r.h.s. of Eq. (56) by a linear function of

\[
d = \sum_s e_s^{1/2} b_s \sigma_s, \quad \text{i.e.}
\]

\[
\sum_s e_s^{3/2} b_s \sigma_s \simeq \eta d,
\]

(91)
where the coefficient \( \eta \) depends only on the deformation. Under this assumption Eq. (56) can be applied iteratively to yield

\[
df(ud^+T) \simeq f(d^+d + 2\eta)d.
\]

(92)
The latter formula leads to the approximate transform

\[
U d^+ U^+ \simeq d^+d + 2\eta.
\]

(93)
To determine the optimal value of \( \eta \) as a function of deformation, minimize the norm of the difference operator

\[
D(\eta) = \sum_i \left( \epsilon_i^{3/2} - \eta \epsilon_i^{1/2} \right) \mathbf{b}_i \cdot \mathbf{\sigma}_i,
\]

with respect to \( \eta \), where the norm of an operator \( \hat{\mathbf{x}}_n \) within the shell number \( n \) is defined via its Hermitian “root mean square” value

\[
\| \hat{\mathbf{x}}_n \| = \sqrt{\langle \hat{\mathbf{x}}_n \hat{\mathbf{x}}_n \rangle_n}.
\]

For the case under discussion, one has

\[
\| D(\eta) \|_n^2 = \frac{1}{2} \left( m_3 - 2m_2 \eta + m_1 \eta^2 \right) (n + 3).
\]

The optimal value \( \eta = m_2 / m_1 \), which minimizes the r.h.s. of Eq. (96), must be substituted in the r.h.s. of Eq. (93). The resulting equation, together with Eq. (53), is equivalent to Eqs. (71) and (73).

The minimal value of the squared norm,

\[
\| D \left( \frac{m_2}{m_1} \right) \|_n^2 = \frac{1}{3} \left( m_3 - \frac{m_2^2}{m_1} \right) (n + 3),
\]

is a deformation-dependent quantitative estimate of the validity of the approximation, Eq. (91), or, which is the same, Eqs. (71) and (73). Unfortunately, similar estimates are not available for approximations (72), (78) and (79). However, they are of minor importance for the transform of the Hamiltonian (5) because of the small value of \( k\mu \), the coefficient of the \( u_{ll} \) term. By evaluating the r.h.s. of formula (97) at various deformations and fixed \( n \), it is easy to find out that the approximation under discussion is more precise for prolate-like shapes than for oblate-like ones, and the accuracy decreases with increasing deformation.

Thus, the squared norm of the difference operator provides a simple ad hoc estimate for the accuracy of the basic approximation used for transforming the model Hamiltonian. Essentially, it measures the accuracy of the approximation to the transformed Nilsson-type Hamiltonian since the exact transform cannot be written in a simple analytical form. As demonstrated below, this measure is also convenient for an analysis of the relative accuracy of different models.

It is interesting that the Nilsson-type model, developed in this paper, can be viewed as a further refinement over the triaxial model with the deformed orbital momentum that was intended to correctly reproduce the structure of the basis states in both the spherical and cylindrical limits and to extend this interpolation to arbitrary triaxial shapes [25]. The general structure of the Hamiltonian of the latter model is similar to Eq. (5). While the harmonic-oscillator term (6) is standard, the spin–orbit and orbit–orbit interactions are defined by analogy to Eq. (10) but with the spherical-representation operators \( I \) and \( n \) substituted by the deformed-representation operators \( \mathbf{l} \) and \( \mathbf{n} \), respectively:

\[
u_{ls} = \mathbf{l} \cdot \mathbf{\sigma}, \quad \nu_{ll} = \mathbf{l}^2 - \langle \mathbf{l}^2 \rangle_n.
\]
(The \( \langle \ell \rangle \) \(_{n} \) correction, absent in the original Hamiltonian, must be inserted to preserve the shell-model structure.)

Because of the analogy between the structure of the \( u_{\ell} \) and \( u_{\ell} \) terms in this model and in the spherical limit it is natural to construct the appropriate pseudospin transformation merely as the deformed analog of the transformation (13)–(15), i.e. by rewriting the structural block in the form

\[
d_{\text{def}} = \sum_{j} b_{j} \sigma_{j}.
\]

(99)

It is important to note that \( d_{\text{def}} \) commutes with the deformed angular momentum operator \( j \). As a result, the corresponding transformation operator \( U_{\text{def}} \) leaves \( j \) invariant,

\[
U_{\text{def}}^{+}U_{\text{def}} = j,
\]

(100)

in contrast to the pseudospin transformation \( U \) which approximately preserves the physical angular momentum \( j \) (see Eq. (89)). Therefore, although the eigenstates in both models coincide in the spherical and cylindrical limits, and the energy spectra in the spherical limit are the same, the pseudospin transformation-based model seems to be more adequate in general. However, within the experimentally attainable domain of deformations a serious difference in predictions of the two models is not expected.

It follows from Eqs. (19), (20), and (21) that the transforms

\[
U_{\text{def}}^{+}U_{\text{def}} = n + 1,
\]

(101)

\[
U_{\text{def}}^{+} \cdot \sigma U_{\text{def}} = -1 \cdot \sigma - 2,
\]

(102)

\[
U_{\text{def}}^{+}U_{\text{def}} = \sigma^2 + 2 \cdot \sigma + 2
\]

(103)

are exact. However, the \( h_{\text{osc}} \) term can be transformed only approximately by using techniques similar to those developed in the previous subsections. By combining those techniques with identities

\[
d_{\text{def}}h_{\text{osc}}d_{\text{def}}^{+} = \frac{1}{2} \left( [h_{\text{osc}}, d_{\text{def}}d_{\text{def}}^{+}] + \sum_{j} e_{j} \left( b_{j} \sigma_{j}d_{\text{def}}^{+} + d_{\text{def}}b_{j}^{*} \sigma_{j} \right) \right),
\]

(104)

\[
d_{\text{def}}^{+}d_{\text{def}}^{+} = n + 3 + 1 \cdot \sigma
\]

(105)

it is possible to obtain the approximate analytic expression

\[
U_{\text{def}}h_{\text{osc}}U_{\text{def}}^{+} \approx h_{\text{osc}} + \frac{1}{2} m_{1}.
\]

(106)

Therefore, the approximate image of the Hamiltonian can be written in the pseudo Nilsson form

\[
U_{\text{def}}^{+}hU_{\text{def}} \approx h_{\text{osc}} + \frac{1}{2} m_{1} - k((2\mu - 1)\nu_{LT} + \mu_{L}) + k(1 - \mu) + k \mu (n + 2),
\]

(107)

which is reminiscent of Eq. (22).
The approximate transform (106) and, consequently, (107), is alternatively derivable by means of the approximation rule
\[ \sum_{i} \varepsilon_{i} b_{i} \sigma_{i} \approx \eta' d_{\text{def}}, \] (108)
which is the analog of (91). By pursuing the analogy further, it is possible to measure the accuracy of this rule by assessing the minimal value of the squared norm of the corresponding difference operator. Similarly to Eq. (97), the numerical value of this measure is given by
\[ \| D_{\text{def}} \left( \frac{1}{n} m_{1} \right) \|_{n}^{2} = \frac{1}{3} \left( m_{2} - \frac{1}{2} m_{1}^{2} \right) (n + 3). \] (109)

The ratio of the r.h.s. of the former and the latter equations provides a quantitative shell-independent estimate for the accuracy of the approximation rule (91) (within the model developed in this paper) relative to the accuracy of the rule (108) (within the model with the deformed orbital momentum):
\[ R = \frac{\| D(m_{2}/m_{1}) \|_{n}^{2}}{\| D_{\text{def}} \left( \frac{1}{n} m_{1} \right) \|_{n}^{2}} = \frac{m_{3} - m_{2}^{2}/m_{1}}{m_{2} - \frac{1}{2} m_{1}^{2}}. \] (110)
where \( m_{k}, k = 1, 2, 3 \) are functions only on the dimensionless frequencies \( \varepsilon_{z}, \varepsilon_{x}, \varepsilon_{y} \) (see Eq. (48)).

It is convenient to parametrize the frequencies in \((\beta, \gamma)\) terms [26]:
\[ \varepsilon_{z} = \nu \left( 1 - 2 \beta \cos \gamma \right), \] (111)
\[ \varepsilon_{x} = \nu \left( 1 - 2 \beta \cos \left( \gamma - \frac{2}{3} \pi \right) \right), \] (112)
\[ \varepsilon_{y} = \nu \left( 1 - 2 \beta \cos \left( \gamma + \frac{2}{3} \pi \right) \right), \] (113)
where
\[ \nu = (1 - 3 \beta^{2} - 2 \beta^{3} \cos 3 \gamma)^{-1/3} \] (114)
is the factor providing the volume conservation. This parametrization maps the region of possible deformations (under the \( \varepsilon_{z} \leq \varepsilon_{x} \leq \varepsilon_{y} \) constraint that makes the choice unique) into a triangle in the \((\beta, \gamma)\) plane with boundaries \( 0 \leq \gamma \leq \frac{1}{3} \pi \) and \( 0 \leq \beta \cos \gamma \leq \frac{1}{2} \).

For small deformations
\[ \beta \approx \sqrt{\frac{5}{16 \pi}} \beta_{B}, \quad \gamma \approx \gamma_{B} \] (115)
where \( \beta_{B} \) and \( \gamma_{B} \) are the conventional Bohr parameters [27].

By means of the above parametrization, the ratio (110) can be rewritten in the form
\[ R = \frac{1 - \beta \cos 3 \gamma - 2 \beta^{2}}{(1 - 3 \beta^{2} - 2 \beta^{3} \cos 3 \gamma)^{1/3}}. \] (115)
The corresponding plot is displayed on Fig. 3. It clearly demonstrates that in the region \( 0 \leq \gamma \leq \frac{1}{3} \pi \) (which contains the majority of strongly deformed nuclei) the approximation (91) is preferable, falling closer to the cylindrical limit. For \( \frac{1}{3} \pi < \gamma \leq \frac{1}{2} \pi \)
4.5. Limiting cases

The approximate analytic transforms (71), (72), (73), (78), and (79) were derived exclusively by means of the techniques based on the formal expansion in inverse powers of $n + 3$. Therefore, in the limiting cases and conditions

$$h_{\text{osc}}, k_{\text{osc}} \to n + \frac{1}{2}$$

and $m_z = 3$ for any $k, l$, and (79) to (20). The calculations show Eq. (43). All the results obtained have a relatively weak dependence of degeneracy on $\gamma$ for two triads of the equations

Indicating the equation with the number of quanta and $s$

$$Un_z U^+ \simeq n_z + \frac{1}{2}$$
$$Un_p U^+ \simeq n_p + \frac{1}{2}$$
$$Un_\sigma U^+ \simeq \left(2e^2\right)$$
$$U_\sigma U^+ \simeq \frac{-l_\sigma p_\rho}{\sqrt{n_z + 1}}$$

where $l_\sigma p_\rho$ stands for $l_\sigma$ and $p_\rho$

In the cylindrical limit ($\epsilon \to 0$) the connection becomes exact and coincides with the second power of the spin–orbit splitting.

The connection among the limit ($\epsilon \to 0$) is also similar.

$$U_{-\infty} = \frac{1}{\sqrt{n_z + 1}}$$

and equations

$$U_{-\infty} n_p U_{-\infty} \simeq n_p$$
of $n + 3$. Therefore, it is expedient to analyze their functional forms in the familiar limiting cases and compare the results to the available exact expressions.

In the spherical limit, when

$$ h_{\text{osc}}, h^{'\text{osc}} \to n + \frac{1}{2}; \quad \mathbf{A}, \mathbf{A}' \to \mathbf{I}, $$

and $N_k = 3$ for any $k$, Eqs. (71) and (72) are equivalent to (19), and Eqs. (73), (78), and (79) to (20). The transform (21) of the $L^2$ operator is then derivable by means of Eq. (43). All the results are necessarily exact.

Considering the axial case ($\epsilon_1 = \epsilon_2 = \epsilon, \epsilon_3 = \epsilon^{-2}$) serves a dual purpose. First, the presence of degeneracy associated with the axial symmetry provides a strong consistency check for two triads of equations, namely, (71), (55), (72) and (73), (78), (79). Indeed, equations within these triads must become linearly dependent and mutually noncontradictory – otherwise the approach would be flawed. Second, the cylindrical and asymptotic oblate limits are easily extractable from the axial case formulae.

A direct check shows that both the triads meet the consistency test. Then the application of the definitions (6), (47) and (51) corroborates that the two triads in the axial case are equivalent to two dyads, for the longitudinal and transverse components of the number of quanta and spin–orbit splitting operators:

$$ U_{n_{1}} U^{+}_{n^{'}} \simeq n_{1} + \frac{1}{2\epsilon^{3} + 1}, $$

$$ U_{n\rho} U^{+}_{n^{'}} \simeq n_{\rho} + \frac{2\epsilon^{3}}{2\epsilon^{3} + 1}, $$

$$ U_{l_{z}\sigma_{z}} U^{+}_{l^{'}} \simeq \frac{(2\epsilon^{3} - 1)l_{z}\sigma_{z} + \epsilon^{3/2}l_{\rho}\sigma_{\rho} + 2\epsilon^{3}}{2\epsilon^{3} + 1}, $$

$$ U_{l_{\rho}\sigma_{\rho}} U^{+}_{l^{'}} \simeq \frac{(-\epsilon^{3} + 1)l_{\rho}\sigma_{\rho} + 2\epsilon^{3/2}l_{z}\sigma_{z} + 4\epsilon^{3/2}}{2\epsilon^{3} + 1}, $$

where $l_{z}\sigma_{z}$ stands for $l_{z}\sigma_{z} + l_{z}\sigma_{y}$.

In the cylindrical limit ($\epsilon \gg 1$), approximate transforms (117), (118), and (119) become exact and coincide with (30), (29), and (31), respectively. Since Eq. (32) is just the second power of Eq. (31), it follows from (119) as well. However, the image

$$ U_{l_{\rho}l_{\rho}} U^{+}_{l_{\rho}l_{\rho}} \simeq \frac{1}{1 - \epsilon^{3} l_{\rho}l_{\rho}}, $$

remains approximate because the exact image cannot be expressed through components of the spin–orbit splitting operator only.

The connection among the approximate and exact images in the asymptotic oblate limit ($\epsilon \to 0$) is similar. The transformation operator in this limit has a form

$$ U_{l_{\rho}l_{\rho}} = \frac{1}{\sqrt{\epsilon} + 1} b_{l_{\rho}l_{\rho}}, $$

and equations

$$ U_{l_{\rho}l_{\rho}} U^{+}_{l_{\rho}l_{\rho}} = n_{l_{\rho}l_{\rho}}. $$
\[ U_{-\infty} n_z U_{-\infty}^+ = n_z + 1, \quad (124) \]
\[ U_{-\infty} l_z \sigma_z U_{-\infty}^+ = l_z \sigma_z, \quad (125) \]
\[ U_{-\infty} l_z^2 U_{-\infty}^+ = l_z^2, \quad (126) \]

which follow from (117), (118), and (119), are exact. The image
\[ U_{-\infty} l_\rho \sigma_\rho U_{-\infty}^+ \simeq -l_\rho \sigma_\rho, \quad (127) \]
is again approximate; the exact image is not expressible through the transverse components of the orbital and spin momenta only but reaches the limit (127) at \( n_z \gg 1 \).

A general observation, which ensues from the analysis of this subsection and Eq. (89), is that the procedure of approximate pseudospin transformation provides exact results for the integrals of motion and reasonable approximate expressions for the rest of the operators.

5. Hamiltonian in the pseudospin representation

Since the set of required operator transforms has been obtained, it is possible to proceed with the transformation of the model Hamiltonian. This is a crucial point for the approach because all the previous results can be checked for conformity.

The Nilsson-type Hamiltonian, associated with the deformed pseudospin transformation (34) and (37), is determined by Eqs. (5), (6), and (46)–(52) with the accuracy of two deformation-dependent positive parameters, \( \xi \) and \( \zeta \), which reach the value of 1 in the spherical limit. These dependences are to be chosen in order to satisfy the third general consistency requirement from Subsection 3.3, namely, the physical equivalence of the spherical and deformed pseudo representations.

Consider the model Hamiltonian at \( \mu = 0.5 \),
\[ \hbar |_{\mu=0.5} = \hbar_{\text{osc}} - k \left( \frac{\lambda_\rho}{\lambda} \sigma_\rho + \frac{1}{3} \xi \left( \frac{2}{3} - \langle A^2 \rangle_n \right) \right). \quad (128) \]

Since this value of \( \mu \) corresponds to the “exact” pseudospin limit, the spin–orbit splitting term in the transformed Hamiltonian should vanish or, at least, acquire the minimal magnitude possible.

By applying the approximate transformation rule
\[ U_{\lambda} \sigma \lambda U_{\lambda}^+ \simeq \sigma_\lambda + \frac{2 m_{-1} A \cdot \sigma - 3 l_\rho \cdot \sigma}{m_1} + \frac{2 m_{-2}}{m_2}, \quad (129) \]
which follows from Eqs. (52), (73), and (79), as well as from Eq. (71) and Eq. (73), it is possible to obtain the following spin–orbit term in the approximate image of the Hamiltonian (128):
\[ \frac{3}{m_1} k \xi \left[ \frac{1}{3} \left( 2 m_{-1} - m_1 \xi \right) \sigma_\rho + \frac{2 m_{-2}}{m_2} \right]. \quad (130) \]

This term disappears in the spherical limit and may vanish in the cylindrical one (provided \( \xi \) and \( \zeta \) are chosen appropriately) but it cannot vanish at arbitrary deformation.

However, the ratio \( \xi/\zeta \) square brackets, and, the transformed Hamiltonian

Define the deformation
\[ \zeta = \frac{1}{3} \left( 2 m_{-1} - m_1 \xi \right), \]
and minimize the square
\[ C(\zeta) = \frac{1}{2} (\xi - \zeta)^2 \]
with respect to \( \zeta \). (The Eqs. (46), (76) and the can be shown that for \( C(\zeta) + 1 \]
and its minimum is of the form
\[ \xi_{\text{opt}} = \frac{m_{-1}}{m_1}. \]

In accordance with Eq. the relation
\[ \xi = \frac{m_{-1} m_{-2}}{m_1 + 2 m_2}. \]

Now integrate the rest
\[ U(n_\lambda) U_\lambda^+ \simeq \frac{1}{3} \left( 2 m_{-1} - m_1 \xi \right) \sigma_\rho + \frac{2 m_{-2}}{m_2} \]

The analysis of this rule
\[ \xi = \xi_{\text{opt}} = \frac{m_{-1}}{m_1}. \]

Indeed, according to the approximation to the I–the deformed representation (37), This in turn impli the conventional Nilsson general requirements fro

Finally, the transforme
\[ U_{\lambda} \sigma \lambda U_{\lambda}^+ \simeq \hbar_{\text{osc}} + \frac{m_2}{m_1} \]

\[ + k \mu \delta n_{\lambda} + \]

However, the ratio $\xi/\zeta$ may be chosen to minimize the norm of the operator in the square brackets, and, therefore, to significantly reduce its influence on the rest of the transformed Hamiltonian.

Define the deformation-dependent coefficient

$$
\xi = \frac{1}{3} \left( 2m_{-1} - m_1 \right) \frac{\zeta}{\xi}
$$

(131)

and minimize the squared norm of the difference operator

$$
C(\xi) = \mathbf{I} - \xi \mathbf{A} - \xi \mathbf{A} \cdot \mathbf{d}
$$

(132)

with respect to $\xi$. (The norm for a given operator is defined in Eq. (95).) By using Eqs. (46), (76) and the formulae $\mathbf{l}_s \mathbf{A} = 0$ and $\mathbf{I}_z^2 \mathbf{n} = \frac{1}{2} \mathbf{n}(n+3)$, $s = z, x, y$, it can be shown that for $C(\xi)$ within the shell number $n$ the squared norm equals to

$$
\|C(\xi)\|^2 \mathbf{H} = \left( \frac{1}{3} (3 + m_1 m_{-1}) - 2m_1 \xi \mathbf{c} + m_{-1} \xi \mathbf{c}^2 \right) \frac{1}{3} \mathbf{n}(n+3),
$$

(133)

and its minimum is reached at

$$
\xi_{\text{opt}} = \frac{m_1}{m_{-1}}.
$$

(134)

In accordance with Eq. (131) and identity (B.12) from Appendix B, this result yields the relation

$$
\xi = \frac{m_1 m_{-1}}{m_1 + 2m_{-1} \xi}.
$$

(135)

Now integrate the results of this section into the approximate transformation rule

$$
U(\xi_{ls} + \mu n) U^+ \approx (2\mu - 1) u_{ls} + 6\mu \frac{\xi}{m_1} \left( \xi_{opt} - \mathbf{I} \cdot \mathbf{A} - 2 \xi_{opt} - (1 - \mu) - \frac{1}{3} \mu m_{-1} \xi \mathbf{c} (n+2). \right.
$$

(136)

The analysis of this rule gives a unique prescription for $\xi$, namely

$$
\xi = \frac{m_{-1}}{m_1}.
$$

Indeed, according to this choice and Eq. (132), the $u_{ls}$ operator becomes the best approximation to the $\mathbf{I} \cdot \mathbf{A}$ operator given the constraints of conservation of quanta in the deformed representation and conformity with the pseudospin transformation (34), (37). This in turn implies that the Hamiltonian thus obtained combines the advantages of the conventional Nilsson scheme (physical vs deformed orbital momentum) with the general requirements from Subsection 3.3.

Finally, the transformed Hamiltonian acquires the following form:

$$
U h U^+ \approx h_{\text{oc}} + \frac{m_2}{m_1} \mathbf{n} - k((2\mu - 1) u_{ls} + \mu n) + k \mu \delta u_{ls} + 2k(1 - \mu) + \frac{1}{3} k \mu m_{-1} \xi (n+2),
$$

(138)
where

\[ u_{ls} = \frac{m_1}{m_{-1}} A \cdot \sigma, \]  
(139)

\[ u_{ll} = \frac{\xi}{m_1} \left( A^2 - (A^2)n \right), \]  
(140)

\[ \delta u_{ls} = \frac{\xi}{m_1} \left( u_{ls} - l^0 \cdot \sigma \right), \]  
(141)

\[ \xi = \frac{m_1^2}{m_1 + 2m_{-2}}. \]  
(142)

Given this formula, one can check the equivalence requirement between the spherical and deformed representations and numerically test the isospectral character of the original and transformed Hamiltonians in the regular sector.

The equivalence requirement states that the operations of altering the deformation of and transforming the Hamiltonian should be essentially interchangeable. In other words, the image (18) of the Hamiltonian at a given deformation can also be obtained by putting the deformation into the pseudo Nilsson Hamiltonian of the spherical limit (22). This requirement ensures that the conventional way of doing shell-model calculations in the pseudospin-adapted basis [9–11], which utilizes a diagonalization of the many-particle Hamiltonian in the spherical pseudo representation (and thus generates the deformation dynamically), is physically equivalent to the diagonalization of the Hamiltonian already transformed in the framework of the shell-model basis with a corresponding deformation.

Since the general structure of the model Hamiltonian (5) remains the same at any deformations, and the only varying component is the content of the \( h_{osc} \), \( u_{ls} \), and \( u_{ll} \) terms as a function of oscillator frequencies, the operation of deforming converts the transformed spherical Hamiltonian (22) into the "pseudo" form

\[ U_\delta h U^+ = h_{osc} + \frac{m_2}{m_1} - k \left( 2\mu - 1 \right) u_{ls} + \mu u_{ll} + 2k(1 - \mu) + k\mu(n + 2). \]  
(143)

(Note that the deformation-dependent term \( m_2/m_1 - 1 \), which only shifts the energy spectrum as a whole, was added to the r.h.s. of this equation for conformity with rule (71).)

Ideally, the r.h.s. of Eqs. (138) and (143) should coincide. In fact, they are essentially the same. A slight distinction exists because of the presence of \( \delta u_{ls} \) in the former equation and the difference in the coefficient at \( k\mu(n + 2) \) (which is \( \frac{1}{2} m_{-1} \xi \) in (138) vs 1 in (143)). However, an estimate by means of the operator norms demonstrates that the distinction is not significant, at least within the experimentally attainable domain of deformation. The relevant data are presented graphically in Figs. 4 and 5. The first of the plots roughly evaluates the magnitude of spin–orbit term in (138) relative to the spin–orbit splitting magnitude in (143) by means of the ratio \( \|(2\mu - 1)u_{ls} + \mu \delta u_{ls}\|/\|(2\mu - 1)u_{ls}\| \) for neutrons (\( \mu = 0.4 \)). The estimated ratio is about 1.1 for normal rotational bands (lantanides and actinides) and reaches 1.4 in the hyperdeformation region, and decreases with increasing \( \gamma \), the nonaxiality angle. In the
between the spherical and character of the original

considering the deformation exchangeable. In other
situation can also be ob-

tainable of the spherical
of doing shell-model
izes a diagonalization
ation (and thus gen-

diagonalization of
ell-model basis with a

pends the same at any
if the $h_{\text{osc}}, u_{ij}$ and $u_{ij}$
forming converts the

$+ k\mu(n+2)$. \hspace{1cm} (143)

only shifts the energy
r conformity with rule

In fact, they are es-
resence of $\delta h_{M}$ in the
) (which is $\frac{1}{2} m_{-1} \xi$ in
parator norms demon-
experimentally attainment
igraphically in Figs. 4
of spin-orbit term in
by means of the ratio
The estimated ratio is
and reaches 1.4 in the
axiality angle. In the

region of asymptotically high deformation the ratio reaches 1 again which means that
the $\delta h_{M}$ term vanishes. The deformation dependence of the coefficient $\frac{1}{2} m_{-1} \xi$ is plotted
in Fig. 5. This coefficient is very close to 1 (within 10\%) up to the hyperdeformation
region, and stays within 30\% for all shapes except those in the physically unattainable
region of asymptotically strong oblate deformation.

Therefore, the a priori criteria demonstrate excellent fulfillment of the equivalence
requirement for attainable deformations and very reasonable fulfillment over the whole
of the $(\beta, \gamma)$ plane. This means that within the accuracy of the approximate pseudospin
transformation, the pseudospin dynamical symmetry is a perfectly valid concept which
can be used reliably in calculations of any deformed heavy nuclei in a spherical, as well as a deformed, shell-model basis.

The final test for the entire approach is a numerical calculation. It serves as a critical and objective measure of all the approximations involved and also of the equivalence requirement. The reason for this statement is that in the absence of approximations the spectra of the original and transformed Hamiltonians would be identical in the regular sector of the single-particle space of states. The test calculation has been done with the deformation parameter $\beta$ (as defined in Subsection 4.4) spanning the interval up to 0.4 and different values of the nonaxiality angle $\gamma$. This deformation domain extends out to the hyperdeformation area and completely covers the region of experimental interest. The values used for the parameters, $k = 0.0637$ and $\mu = 0.42$, are characteristic of lantanides. Since the deformed representation and $\hbar = 4$ which are related.

The calculated spectra of the levels of the origin in the pseudo constant and belong to a single $j$ angular momentum $j$. This separation of the space of pseudospin transforms into a regular sector, within which the transformed and $^{\text{ps}}$ coincide which is a desirable feature of the pseudo representation.

3-4%, usually much less than 100%. This is better accuracy than for the ad hoc estimations made on a practical standpoint, this correlation between strong and weak approximation displays h
Fig. 6. Neutron levels of the original Hamiltonian. Axial case. The basis is confined to the deformed oscillator shell $n = 5$. Model parameters: $k = 0.0637, \mu = 0.42$. Energies in units of $\hbar \omega$. Positive values of $\beta$ correspond to prolate deformation, negative values to oblate deformation. (Use of negative $\beta$ is based upon the physical equivalence of $(-\beta, \gamma)$ and $(\beta, 1/2 \pi - \gamma)$ parameter sets.)

of lantanides. Since the model Hamiltonian conserves the number of quanta in the deformed representation, the calculation was confined to the neutron shells with $n = 5$ and $\tilde{n} = 4$ which are relevant for the rare-earth region of the periodical system.

The calculated spectra are plotted in Figs. 6–11. First of all, observe that some of the levels of the original Hamiltonian are not reproduced by the Hamiltonians acting in the pseudo representation. In the spherical limit they are known as defecter levels and belong to a single $j$ subshell of the given oscillator shell with the maximal possible angular momentum $j$. This is no longer true in the presence of deformation; however, the separation of the space of states into the regular and singular sectors with respect to the pseudospin transformation depends smoothly on the nuclear shape parameters. As for the regular sector, within this vast deformation domain the corresponding energy levels of the transformed and "pseudo" Hamiltonians are very close and in most cases practically coincide which is a direct validation of the equivalence requirement. The spectra in the pseudo representation fairly closely (with the difference in energy of no more than 3–4%, usually much less) follow the "parent" levels of the original Hamiltonian. Note that for the prolate-like shapes ($\gamma \leq \frac{1}{2} \pi$) the energy levels tend to be reproduced with better accuracy than for the oblate-like shapes ($\gamma \geq \frac{1}{6} \pi$). Such an asymmetry is in line with the ad hoc estimates in Subsection 4.4 and, in particular, with Fig. 3. From the practical standpoint, this situation is the most favorable since it reflects the observed correlation between strong deformation and prolate nuclear shapes. Consequently, the approximation displays high accuracy exactly where it is required by nature.
Fig. 7. Neutron levels of the transformed and "pseudo" Hamiltonians. Axial case. The basis is confined to the deformed oscillator shell \( n = 4 \). Model parameters: \( k = 0.0637, \mu = 0.42 \). Energies in units of \( \hbar \omega \). Positive values of \( \beta \) correspond to prolate deformation, negative values to oblate deformation. Continuous lines correspond to the approximately transformed Hamiltonian, dotted lines to the "pseudo" version.

Fig. 8. Neutron levels of the original Hamiltonian. Intermediate triaxiality. The basis is confined to the deformed oscillator shell \( n = 5 \). Model parameters: \( k = 0.0637, \mu = 0.42 \). Energies in units of \( \hbar \omega \). Positive values of \( \beta \) correspond to \( \gamma = \frac{1}{12} \pi \), negative values to \( \gamma = \frac{1}{3} \pi \).

Fig. 9. Neutron levels of the transformed Hamiltonian. Axial case. The basis is confined to the deformed oscillator shell \( n = 4 \). Model parameters: \( k = 0.0637, \mu = 0.42 \). Energies in units of \( \hbar \omega \). Positive values of \( \beta \) correspond to prolate deformation, negative values to oblate deformation. Continuous lines correspond to the approximately transformed Hamiltonian, dotted lines to the "pseudo" version.

Fig. 10. Neutron levels of the original Hamiltonian. Intermediate triaxiality. The basis is confined to the deformed oscillator shell \( n = 5 \). Model parameters: \( k = 0.0637, \mu = 0.42 \). Energies in units of \( \hbar \omega \). Positive values of \( \beta \) correspond to \( \gamma = \frac{1}{12} \pi \), negative values to \( \gamma = \frac{1}{3} \pi \).

6. Conclusion

An explicit deformation-dependent Hamiltonian has been suggested and studied in the context of nuclear structure.
Fig. 9. Neutron levels of the transformed and "pseudo" Hamiltonians. Intermediate triaxiality. The basis is confined to the deformed oscillator shell $n = 4$. Model parameters: $k = 0.0637, \mu = 0.42$. Energies in units of $\hbar \omega$. Positive values of $\beta$ correspond to $\gamma = \frac{1}{6} \pi$, negative values to $\gamma = \frac{1}{4} \pi$.

Fig. 10. Neutron levels of the original Hamiltonian. Maximal triaxiality. The basis is confined to the deformed oscillator shell $n = 5$. Model parameters: $k = 0.0637, \mu = 0.42$. Energies in units of $\hbar \omega$.

6. Conclusion

An explicit deformation-dependent form of the pseudospin transformation has been suggested and studied in the framework of the single-particle harmonic-oscillator shell
model. By construction, the transformation operator is a special quanta-decreasing projection of the momentum helicity transformation which in turn is known to accomplish the relevant relabelling of the spin and orbital momenta in the scope of more realistic mean-field and many-particle approaches to nuclear structure [17]. The connection between the two transformations is displayed in detail. In the limiting cases of both the spherical and strongly prolate nuclear shapes the deformation-dependent pseudospin transformation reduces to already familiar functional forms [14,15].

The results of transforming the single-nucleon space of states and the Hamiltonian are found to corroborate one another and to qualitatively describe the concept of dynamical pseudospin symmetry at any reasonable deformation. The concept offers an effective description of the many-nucleon systems since the basis of single-particle states involves only a subset of the entire set of the relevant states. However, this subset is what primarily contributes to the dynamical generation of deformation for the nucleus as a whole. In the normal representation it coincides with the region of unitarity of the pseudospin transformation and by this reason is determined in a consistent manner. The basic idea of the dynamical pseudospin symmetry concept is that in the pseudospin representation the strength of the spin-orbit splitting term of the single-particle potential is drastically reduced regardless of the degree of deformation and nonaxiality of the nuclear shape. While in the axial case the validity of the pseudospin symmetry was already demonstrated by the existence of nearly degenerate nucleon energy levels [5], the present study proves this validity directly for arbitrary nonaxiality even though the near degeneracy is no longer observable in the nucleon spectra.

The modified Nilsson Hamiltonian employed in this study is constructed out of the same structural blocks as in the traditional description of a close coupled approximation. The structural features of the conventional model [21,22] with that having a developed oscillation behavior in the transverse coordinates, and can be related to deformations including the axial component. Since the pseudospin transformation has been recently applied to operate, it results in approximately accurate [23] The reproducing all the known results exist only for a procedure is based upon of the approximations. However, the transformation dependence on the shell model. By construction, the transformation operator is a special quanta-decreasing projection of the momentum helicity transformation which in turn is known to accomplish the relevant relabelling of the spin and orbital momenta in the scope of more realistic mean-field and many-particle approaches to nuclear structure [17]. The connection between the two transformations is displayed in detail. In the limiting cases of both the spherical and strongly prolate nuclear shapes the deformation-dependent pseudospin transformation reduces to already familiar functional forms [14,15].

The results of transforming the single-nucleon space of states and the Hamiltonian are found to corroborate one another and to qualitatively describe the concept of dynamical pseudospin symmetry at any reasonable deformation. The concept offers an effective description of the many-nucleon systems since the basis of single-particle states involves only a subset of the entire set of the relevant states. However, this subset is what primarily contributes to the dynamical generation of deformation for the nucleus as a whole. In the normal representation it coincides with the region of unitarity of the pseudospin transformation and by this reason is determined in a consistent manner. The basic idea of the dynamical pseudospin symmetry concept is that in the pseudospin representation the strength of the spin-orbit splitting term of the single-particle potential is drastically reduced regardless of the degree of deformation and nonaxiality of the nuclear shape. While in the axial case the validity of the pseudospin symmetry was already demonstrated by the existence of nearly degenerate nucleon energy levels [5], the present study proves this validity directly for arbitrary nonaxiality even though the near degeneracy is no longer observable in the nucleon spectra.

The modified Nilsson Hamiltonian employed in this study is constructed out of the same structural blocks as in the traditional description of a close coupled approximation. The structural features of the conventional model [21,22] with that having a developed oscillation behavior in the transverse coordinates, and can be related to deformations including the axial component. Since the pseudospin transformation has been recently applied to operate, it results in approximately accurate [23] The reproducing all the known results exist only for a procedure is based upon of the approximations. However, the transformation dependence on the shell model. By construction, the transformation operator is a special quanta-decreasing projection of the momentum helicity transformation which in turn is known to accomplish the relevant relabelling of the spin and orbital momenta in the scope of more realistic mean-field and many-particle approaches to nuclear structure [17]. The connection between the two transformations is displayed in detail. In the limiting cases of both the spherical and strongly prolate nuclear shapes the deformation-dependent pseudospin transformation reduces to already familiar functional forms [14,15].

The results of transforming the single-nucleon space of states and the Hamiltonian are found to corroborate one another and to qualitatively describe the concept of dynamical pseudospin symmetry at any reasonable deformation. The concept offers an effective description of the many-nucleon systems since the basis of single-particle states involves only a subset of the entire set of the relevant states. However, this subset is what primarily contributes to the dynamical generation of deformation for the nucleus as a whole. In the normal representation it coincides with the region of unitarity of the pseudospin transformation and by this reason is determined in a consistent manner. The basic idea of the dynamical pseudospin symmetry concept is that in the pseudospin representation the strength of the spin-orbit splitting term of the single-particle potential is drastically reduced regardless of the degree of deformation and nonaxiality of the nuclear shape. While in the axial case the validity of the pseudospin symmetry was already demonstrated by the existence of nearly degenerate nucleon energy levels [5], the present study proves this validity directly for arbitrary nonaxiality even though the near degeneracy is no longer observable in the nucleon spectra.

The modified Nilsson Hamiltonian employed in this study is constructed out of the
same structural blocks as the pseudospin transformation itself. This is a natural generalization of a close connection between the two which is realized in both the known limiting cases. The structure of the spin–orbit and orbit–orbit terms combines important features of the conventional triaxial Nilsson model (with the physical orbital momentum) [21,22] with that of a model that uses a deformed representation of the orbital momentum [25]. Namely, the orbital momentum-dependent terms are optimally fitted to the conventional model while the Hamiltonian exactly preserves the number of quanta in the deformed oscillator representation. For this reason, the current model displays adequate behavior in the well-established regions of medium to very strong prolate deformations, and can be recommended for mean-field calculations over a broader range of deformations including oblate and triaxial shapes.

Since the pseudospin transformation of the model Hamiltonian cannot be accomplished in exact analytical form for arbitrary deformation, a procedure of approximate transformation has been developed. This procedure generalizes the technique which was recently applied to operators in the spherical representation and which appeared to be quite accurate [23]. The approximate transforms obtained in this paper happened to reproduce all of the known exact results in both limiting cases. Incidentally, those exact results exist only for operators of conserved quantities. Since the approximation procedure is based upon an operator-valued power expansion, the accuracy of the approximation could be increased by utilizing higher orders in the expansion. In this case, however, the transforms will acquire a complicated functional form with an explicit dependence on the shell number.

Outside the regions of weak and strong prolate deformations, the goodness of the approximation has been confirmed by both a priori and a posteriori tests. The a priori estimates utilize an analytical evaluation of some operator norms within the deformed oscillator shell that is conceptually close to methods used in statistical spectroscopy [30]. The a posteriori check is a numerical test based on isospectrality of the original Hamiltonian and its exact transform within the region of the unitarity of the pseudospin transformation. The tests demonstrate that the approximation procedure yields reliable results for all deformations and is most accurate for prolate-like shapes – which are observed in the majority of strongly deformed nuclei.

Apart from validating the approximation procedure, both tests prove the physical equivalence between the spherical and deformed pseudospin representations at all experimentally attainable deformations. Thus, at the mean-field level the deformed "pseudo" Hamiltonian, which can be obtained by inserting the proper deformation dependence into the familiar pseudo Nilsson Hamiltonian of the spherical limit without any change in coefficients, is a very good approximation to the exactly transformed Hamiltonian (which cannot be written down in a closed analytical form). The equivalence condition is especially important for the many-particle pseudospin-adapted nuclear algebraic models like the pseudo SU(3) model and its pseudo symplectic extension which traditionally employ the spherical oscillator shell-model basis. This way of doing the calculation is therefore guaranteed to produce practically the same results as the calculation in the deformed pseudo shell-model basis provided the Hamiltonian is adequately transformed.
to the pseudospin representation in either case.

Acknowledgements

This work was supported in part by grants PHY-9312628 and PHY-9317877 from the U.S. National Science Foundation.

Appendix A. Helicity transformation for oscillator wavefunctions

The action of the helicity transformation on a function $\phi(r)$ of the coordinate and spin variables is conveniently representable by means of an auxiliary Fourier transformation:

$$\mathcal{H}(\phi(r, \sigma)) = \mathcal{F}^{-1}(e_k \cdot \sigma \mathcal{F}(\phi(r, \sigma)))$$

(A.1)

where

$$\mathcal{F}(\phi(r, \sigma)) = \frac{1}{(2\pi)^{3/2}} \int dr e^{ik \cdot r} \phi(r, \sigma),$$

(A.2)

$$\mathcal{F}^{-1}(\phi'(k, \sigma')) = \frac{1}{(2\pi)^{3/2}} \int dr e^{-ik \cdot r} \phi'(k, \sigma')$$

(A.3)

denote the direct and inverse Fourier transforms, respectively, and $e_k = k/k$ is the unit vector in the $k$ direction.

By utilizing the spherical wave expansion of a vector plane wave

$$e^{ik \cdot r} = 4\pi \sum_{l \geq 0} i^l j_l(kr) Y_l(e_k) \cdot Y_l(e_k),$$

(A.4)

the normalization condition for spherical harmonics, and the formula

$$e_k \cdot \sigma e^i (Y_l(e_k) \otimes \chi)_{ji} = i^l (Y_l(e_k) \otimes \chi)_{ji},$$

(A.5)

(which was obtained in Ref. [12] for the coordinate space), it is straightforward to derive the transform (3) for the oscillator function (2) in the spherical representation. The transformed radial function $R_{\tilde{a} \tilde{l} \tilde{b}}(r)$ is then calculable with the help of the Hankel integral transformation:

$$R_{\tilde{a} \tilde{l} \tilde{b}}(r) = \sqrt{\frac{2}{\pi}} \int_0^\infty dk k^2 j_l(kr) P_{\tilde{a} \tilde{b}}(k),$$

(A.6)

$$P_{\tilde{a} \tilde{b}}(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty dr r^2 j_l(kr) R_{\tilde{a} \tilde{b}}(r),$$

(A.7)

where the change $l \rightarrow \tilde{l}$ in the index of the spherical Bessel function occurs as a result of the spin-angular transformation (A.5). (The definition of $\tilde{l}$ and $\tilde{r}$ is given in Eq. (4); it implies that the number of Fourier transformations.

It is convenient to define

$$U_{\tilde{a} \tilde{b}}(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty U_{a b}(x') e^{-x'} dx'$$

Then, from the symmetries of the Fourier transform it follows

$$R_{\tilde{a} \tilde{l} \tilde{b}}(r) = \frac{1}{r_0^{3/2}} U_{a b}(e^{-r_0})$$

where $r_0 = \sqrt{\hbar/(m \omega)}$ by using the explicit form

$$L^{l+\ell/2}_\nu(x^2) = \sum_{\nu=0}^\infty \frac{1}{n^2} \int_0^\infty$$

and the integral

$$\int_0^{\infty} dr r^{\nu-1} j_l(xr) e^{-r}$$

which is a particular case

$$R_{\tilde{a} \tilde{l} \tilde{b}}(r) = \frac{1}{r_0^{3/2}} U_{a b}(e^{-r_0})$$

Here, by definition,

$$U_{a b}(x) = (-1)^{r^2/2}$$

for any nonnegative integer matter for the transformed $U_{a b}(x) = U_{a b}(x)$. The example of behavior of the oscillator radial function $R_{\tilde{a} \tilde{l} \tilde{b}}(r)$ is
it implies that the number of radial nodes $\nu = \frac{1}{2}(n - l)$ is conserved under the direct Fourier transformation.

It is convenient to introduce a family of dimensionless oscillator radial functions

$$U_\nu(x) = \sqrt{\frac{\nu!}{2\pi \Gamma(\nu + l + \frac{1}{2})}} x^{l+1/2} \psi_\nu(x^2) e^{-x^2/2}. \quad (A.8)$$

Then, from the symmetry of the spherical oscillator Hamiltonian with respect to the Fourier transform it follows that

$$R_{nl}(r) = \frac{-1}{r_0^{n+3/2}} U_{(n-l)/2} \left( \frac{r}{r_0} \right), \quad (A.9)$$

$$P_{nl}(k) = (-1)^{(n-l)/2} \frac{2^n}{r_0^{n+3/2}} U_{(n-l)/2} (kr_0), \quad (A.10)$$

where $r_0 = \sqrt{\hbar/(m\omega)}$ is the oscillator radius. The inverse transform can be performed by using the explicit form of the Laguerre polynomials

$$L_{\nu}^{l+1/2}(x^2) = \sum_{m=0}^{\nu} \frac{(-1)^m}{m!} \binom{\nu + l + \frac{1}{2}}{\nu - m} x^{2m}, \quad (A.11)$$

and the integral

$$\int_0^\infty dt t^{\nu-1} J_l(x t) e^{-t^2/2} = 2^{\nu-l-2} \frac{\Gamma(l + \nu + 2)}{\Gamma(l + 2)} x^{l+1/2} \times F_1 \left( \frac{1}{2} (l - \nu + 3), l + \frac{1}{2}, \frac{1}{2} x^2 \right) e^{-x^2/2} \quad (A.12)$$

(which is a particular case of Eq. (11.4.28) in Ref. [28]), and results in the formula

$$R_{nl}(r) = \frac{1}{r_0^{n+3/2}} U_{(n-l)/2} \left( \frac{r}{r_0} \right). \quad (A.13)$$

Here, by definition,

$$U_{\nu\ell q}(x) = (-1)^\nu 2^{\nu+1/2} \sqrt{\frac{\nu!}{2\pi}} \frac{\Gamma(\nu + \ell + q + \frac{1}{2})}{\Gamma(\ell + \frac{1}{2})} \frac{1}{\ell!} x^{\ell+1/2} \times \sum_{m=0}^{\nu} \frac{(-1)^m}{m!(\nu - m)!} \frac{\Gamma(\ell + m + \frac{1}{2}(q + 3))}{\Gamma(\ell + m + q + \frac{1}{2})} \times F_1 \left( -m - \frac{1}{2} q, \ell + \frac{1}{2}, \frac{1}{2} x^2 \right) \quad (A.14)$$

for any nonnegative integer $\nu$, $\ell$, and $q = -1, 0, \text{or} 1$. (Only the values $q = 1$ or $-1$ matter for the transformed functions; $q = 0$ makes sense just for an error check since $U_{\nu\ell 0}(x) \equiv U_{\nu\ell}(x)$.)

The example of behavior of the transformed radial functions compared to the closest oscillator radial function is given in Fig. 1 (Section 1) for $\nu = 1$, $\ell = 3$. As can be
seen, the bulk behavior of the three functions is very similar, while \( U_{\ell q} \) is the dominant component in the oscillator function expansion of both \( U_{\ell q} \) and \( U_{\ell-q} \). Moreover, it will be shown below that both the latter functions in the bulk are to good accuracy obtainable from the former by a mere scaling transformation and subsequent normalization; and this scaling rule is valid for more realistic mean fields as well.

The expansion coefficients \( c_{\ell'q'p'} \), occurring in the series

\[
U_{\ell q}(x) = \sum_{q' \geq 0} c_{\ell'q'p'} U_{p'}(x),
\]

(A.15)

are directly calculable either in coordinate or in momentum space. The latter way is definitely simpler since in this case the orbital momentum label is the only parameter to be changed. Then the calculation is reducible to expanding the \( x^{q/2}L_{\ell+1/2}^{p+1/2}(x) \) function in terms of \( L_{\ell+1/2}^{p+1/2} \) polynomials, and results in

\[
c_{\ell'q'p'} = \frac{\sqrt{\Gamma(\nu + \ell + q + \frac{3}{2})\Gamma(\nu' + \ell + \frac{3}{2})}}{\Gamma(\nu + \frac{3}{2})(\nu' + \frac{3}{2})} \left( \ell + \frac{1}{2}(q + 1) \right)! \times \sum_{k=0}^{\min(q, q')} \left( \ell + \frac{1}{2}(q + 3) + k - 1 \right) \left( \frac{1}{2}q - \nu + k \right) \left( -\frac{1}{2}q - \nu + k \right).
\]

(A.16)

The diagram of these coefficients versus the shell number is given in Fig. 2 for the same case as in Fig. 1. The total contribution from higher shells is fairly small for both values of \( q \); however, for \( q = -1 \) the expansion converges more slowly because of the larger tail of the corresponding radial function.

It is noteworthy that there exists another example of a wavefunction whose helicity transform is easily calculable analytically. Indeed, by using Eq. (A.5), the Hankel transformation procedure, and the normalization integral

\[
\int_0^\infty dr r^2 j_l(kr) j_l(\kappa r) = \frac{\pi}{2k\kappa} \delta(k - \kappa),
\]

(A.17)

one can prove the following formula:

\[
\mathcal{H}^l_j j_l(\kappa r) (Y_l(e_r) \otimes \chi)_{jk} = \hat{\mathcal{H}}^l_j j_l(\kappa r) (Y_l(e_r) \otimes \chi)_{jk}.
\]

(A.18)

The importance of this equation for realistic mean-field nuclear models is underscored by the fact that within the bulk the leading term of a bound-state wavefunction with the angular momentum \( j \), the orbital momentum \( l \) and the energy \( \epsilon = -\hbar^2 \kappa^2 / 2m \) is proportional to \( j_l(\kappa r) (Y_l(e_r) \otimes \chi)_{jk} \). Therefore, after the helicity transformation the binding energy is conserved, the orbital momentum changes from \( l \) to \( \hat{l} \) according to rule (4), and the central potential remains rather flat in the nuclear bulk. This model-independent result reemphasizes the understanding of the helicity transformation as a microscopic precursor of the pseudoscalar transformation.

Appendix B. Identities

The dimensionality of linearly independent factors beyond the choice of transforms are derived in.

To explicitly determine the deformation matrix

\[
\mathbf{\epsilon} = \text{diag}(\epsilon_x, \epsilon_y, \epsilon_z).
\]

Generally speaking, \( \mathbf{\epsilon} \) is in its principal frame un both diagonal and unimorphs.

The quintessential of the Cayley–Hamilton theorem example, Ref. (291). For

\[
\det(\mathbf{\rho} - \mathbf{\epsilon}) = \rho^3 - \cdots
\]

where \( I \) denotes the unit form

\[
\epsilon^3 - m_1 \epsilon^2 + m_{-1} \epsilon - \cdots
\]

which makes explicit use of \( \mathbf{\epsilon} \). By induction, it follows that \( f(\mathbf{\epsilon}) \) is expressible as a polynomial involving all \( m \)-th powers of the eigenvalues of \( \mathbf{\epsilon} \). This is one of the most convenient forms.

Furthermore, any operator

\[
\text{Tr}(f(\mathbf{\epsilon}) \mathbf{\xi}) = \sum_i f(\epsilon_i) \mathbf{\xi}_i
\]

For the spherical oscillator, behaves like \( j_l(\sqrt{2n+3}) \). The forming function \( R_{n\ell}(r) \) is equivalent to the scaling

\[
R_{n\ell}(r) \propto R_{\ell}(\sqrt{2n+3})
\]

where \( n = \bar{n} + q \) in conjunction with Fig. 1.
For the spherical oscillator eigenstates $\kappa = \sqrt{2n + 3}/r_0$, and the radial function $R_{nl}(r)$ behaves like $j_l(\sqrt{2n + 3}r/r_0)$ within the bulk. According to the rule (A.18), the transformed function $R_{nl}(r)$ is then proportional to $j_l(\sqrt{2n + 3}r/r_0)$ which in turn is equivalent to the scaling relation

$$R_{nl}(r) \propto R_{nl} \left( \sqrt{\frac{2n + 3}{2n + 3}}r \right),$$

where $n = \tilde{n} + q$ in correspondence with (4). This scaling rule is mentioned above in conjunction with Fig. 1.

Appendix B. Identities for the deformation matrix

The dimensionality of the physical space places very strict constraints on the number of linearly independent functions of the frequencies $\epsilon_x$, $\epsilon_y$ and $\epsilon_z$. This is also a principal fact beyond the choice of independent operator set for which the approximate pseudospin transforms are derived in Section 4.

To explicitly determine the dimensionality constraints, it is convenient to introduce the deformation matrix

$$\epsilon = \text{diag}(\epsilon_x, \epsilon_y, \epsilon_z).$$

(A.17)

Generally speaking, $\epsilon$ is a second-rank symmetric tensor. Since the discussion is held in its principal frame under the condition of volume conservation, it is assumed to be both diagonal and unimodular ($\epsilon_x, \epsilon_y, \epsilon_z = 1$).

The quintessential of the dimensionality restrictions for matrices is expressed in the Cayley–Hamilton theorem: A square matrix obeys its characteristic equation (see, for example, Ref. [29]). For the deformation matrix the characteristic determinant is

$$\det(\rho \mathbf{I} - \epsilon) = \rho^3 - m_1 \rho^2 + m_{-1} \rho - 1,$$

(A.18)

where $\mathbf{I}$ denotes the unit matrix. Consequently, the matrix equation is written in the form

$$\epsilon^3 - m_1 \epsilon^2 + m_{-1} \epsilon - \mathbf{I} = 0,$$

where $\mathbf{I}$ denotes the unit matrix. Consequently, the matrix equation is written in the form

$$\epsilon^3 - m_1 \epsilon^2 + m_{-1} \epsilon - \mathbf{I} = 0,$$

(A.19)

which makes explicit use of the unimodularity and the definition (48) of the invariants of $\epsilon$. By induction, it follows from this equation that an arbitrary analytical function $f(\epsilon)$ is expressible as a linear combination of three basis matrices with the coefficients which are invariants of $\epsilon$. The choice of the three basis matrices is by no means unique; one of the most convenient options is $\epsilon, \mathbf{I}$ and $\epsilon^{-1}$.

Furthermore, any operator of the form

$$\text{Tr}(f(\epsilon) \hat{x}) = \sum_i f(\epsilon_i) \hat{x}_i$$
can then be rewritten as a linear combination of only three independent operators. In particular, the operator $\sum_s f(\varepsilon_s) b_s \sigma_s$ can be rewritten as a linear function of the operators $d$, $\sum_s b_s \sigma_s$ and $d^\dagger$ (see Eqs. (37) and (80)). This in turn leads to the proof of the fact that the same operator, $\sum_s f(\varepsilon_s) b_s \sigma_s$, is expressible in terms of $d$ and $d^\dagger$ only. Indeed, the repetitive use of the commutation relation

$$\left[ \sum_s \varepsilon_s^k b_s \sigma_s, \hat{h}_{\text{loc}} \right] = \sum_s \varepsilon_s^{k+1} b_s \sigma_s, \quad (B.4)$$

allows one to construct the operator $\sum_s \varepsilon_s^k b_s \sigma_s$ for virtually any $k$ starting from the $d$ operator. By combining this relation with the Hadamard operator identity

$$e^{-\frac{1}{2}\hat{\mathcal{H}}_0} = \sum_{k \geq 0} \frac{1}{k!} \left[ \ldots \left[ \ldots \left[ \left[ \hat{\gamma}, \hat{\mathcal{H}}_0 \right], \hat{\mathcal{H}}_0 \right], \ldots, \hat{\mathcal{H}}_0 \right] \right], \quad (B.5)$$

the following formula can be obtained:

$$e^{\phi_{\text{loc}}} de^{-\rho_{\text{loc}}} = \sum_s \sqrt{\varepsilon_s} e^{-\rho_s} b_s \sigma_s. \quad (B.6)$$

The integration of the latter operator identity over the parameter $\rho$ yields the relations

$$\int_{\rho=0}^\infty dp e^{\phi_{\text{loc}}} de^{-\rho_{\text{loc}}} = d', \quad (B.7)$$

$$\int_{\rho=0}^\infty \frac{dp}{\sqrt{\pi \rho}} e^{\phi_{\text{loc}}} de^{-\rho_{\text{loc}}} = \sum_s b_s \sigma_s, \quad (B.8)$$

which prove the desired result since $h_{\text{loc}}$ is also expressible via $d$ and $d^\dagger$ (see Eq. (45)).

Finally, several identities are listed below for the functions and invariants of the deformation matrix. They follow from Eq. (B.3) and the unimodularity condition and are utilized to derive various equations in this paper (predominantly in Section 4):

$$\varepsilon^2 = 3m_1 \varepsilon - 3m \varepsilon - 1 + \varepsilon^{-1}, \quad (B.9)$$

$$\varepsilon = m_{1/2} \varepsilon^{1/2} - m_{-1/2} \varepsilon^{-1/2}, \quad (B.10)$$

$$\varepsilon^{1/2} (\varepsilon - m_1) = m_{1/2} \varepsilon^{1/2} - (m_{1/2} m_{-1/2} - 1) \varepsilon + m_{1/2} \varepsilon^{-1/2}, \quad (B.11)$$

$$m_3^2 = m_2 + m_{-2}, \quad (B.12)$$

$$4m_3 = (m_2 - m_{-2}) m_1 + 3. \quad (B.13)$$

References

(B.4) 

Finally any $k$ starting from the $d$-operator identity

(B.5) 

(B.6) 

(B.7) 

(B.8) 

and $d^+$ (see Eq. (45)). Expressions and invariants of the anisomodularity condition and dominantly in Section 4) :

(B.9) 

(B.10) 

(B.11) 

(B.12) 

(B.13) 

**NUCLEAR PHYSICS A**

*Journal devoted to the experimental and theoretical study of the fundamental constituents of matter and their interactions*

Instructions to Authors — Short Version

(A more detailed version of these instructions is published in the preliminary pages of each volume of the journal)

Submission of papers

Manuscripts should be sent to:

**Before acceptance**

Nuclear Physics A, Editorial Office
Street Address: Sara Burgerhartstraat 25, 1055 KV Amsterdam, The Netherlands
Postal Address: P.O. Box 103, 1000 AC Amsterdam, The Netherlands
Tel: +31 20 485 23 55 Fax: +31 20 485 23 70
e-mail: npa@ijn.nucphys.nl

Original material. Submission of a paper implies that the material has not been published before and that it is not being considered for publication elsewhere. **Referees.** All submitted papers are subject to a refereeing process. Electronic submission of LaTeX files by e-mail is preferred. Hard copy contributions should be sent in triplicate.

Types of paper

Concisely written research papers are welcome. Letter-type contributions and unnecessarily long papers cannot be accepted.

Manuscript preparation

**Language.** Manuscripts should be written in good English. **Structure.** Please adhere to the following order: Title, Authors, Affiliations, Abstract, PACS codes, Keywords, Main text, Acknowledgements, Appendix, References, Vitae, Figure legends, Tables. **Corresponding author.** Please indicate the corresponding author: full postal address, e-mail address, telephone and fax numbers on the title page. **Abstract.** All papers should have an abstract of no more than 150 words. PACS classification codes/Keywords. Please supply us with one or more relevant PACS classification codes and 1–6 keywords of your own choice for indexing purposes. **References.** References to other work should be consecutively numbered in the text using square brackets and listed by number at the end of the article. **Illustrations.** Illustrations should also be submitted in triplicate. One set must be in publishable condition. Figures should be clearly numbered. **Colour.** If judged essential by the Editor figures may be published in colour. The Publisher and the author will each bear part of the extra costs involved. Further information is available from the Publisher.

Copyright transfer

You will be asked to transfer the copyright to the Publisher. This will ensure the widest possible dissemination of scientific information.

Electronic publishing

The Publisher welcomes the receipt of your accepted manuscript as an electronic file (LaTeX). For further information, please refer to the more detailed Instructions to Authors or contact the Publisher at the address below.

Author benefits

No page charge. Publishing in **Nuclear Physics A** is free. **Free offprints.** The corresponding author will receive 50 offprints of the article free of charge. An offprint order form will be supplied for ordering any additional paid offprints. **Discount.** Contributors to Elsevier Science journals are entitled to a 30% discount on all Elsevier books. A coupon will be sent to you. **Contents Alert.** Nuclear Physics A is included in Elsevier's pre-publication service Contents Alert. **Nuclear Physics Electronic.** Nuclear Physics A is included in Nuclear Physics Electronic.

For further information

**After acceptance**

Elsevier Science BV, Nuclear Physics A
Issue Management Physics and Materials Science
P.O. Box 2759, 1000 CT Amsterdam, The Netherlands
Tel: +31 20 485-2573 Fax: +31 20 485-2431
e-mail: nphnuclear@elsevier.nl

North-Holland, an imprint of Elsevier Science