ON THE SURVIVAL OF THE SD-PAIR SHELL MODEL UNDER PSEUDO-SPIN TRANSFORMATION

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The pseudo-spin concept and the SD-pair shell model provide for tractable truncations of large model spaces. We tested the compatibility of these concepts, and hence the applicability of the SD-pair shell model to its pseudo SD-pair analog, for the \((1\,\hbar/2\,f_7/2\,f_5/2\,3p_{3/2}\,3p_{1/2})^8\) system. The validity of the SD-pair concept is found to survive under pseudo-spin transformation.

Keywords: SD-pair shell model; pseudo-spin transformation.

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The solvability of a model for describing a physical system is closely linked to understanding and exploiting the system’s symmetries. In particular, the eigenstates of an exactly solvable model can be identified by the quantum numbers of a group-subgroup chain that characterize the symmetries of the system. The irreducible representation (irrep) of the highest unbroken symmetry in the group-subgroup chain provides a band label for a classification of the states.\(^1\) A quasi-exactly solvable model emerges when a large-dimensional Hamiltonian matrix can be brought into near-block diagonal form with at least one of the blocks being solvable.\(^2\) A special case is realized when the inter-block couplings do not destroy the intra-block structure. When a separation of this type is realized, the concept of a band label survives, but with the band being a coherent admixture of representations (a “mixed-irrep” rather than a “single-irrep” band).\(^3,4\) The latter can be thought of
as a separation of the dynamics into “collective” and “intrinsic” modes with similar structures adiabatically mixed.

The pseudo-LS coupling scheme is known to work well for heavy nuclei.\textsuperscript{5,6} Recently, this scheme was applied to the lighter nuclei of the upper \textit{fp}-shell.\textsuperscript{7,8} The success of the theory lies in the near degeneracy of single-particle levels that form pseudo-spin doublets; that is, the levels with radial, angular momentum, and total angular momentum quantum numbers \((n_r, l, j = l + \frac{1}{2})\) and \((n_r - 1, l + 2, j = l + \frac{3}{2})\) are approximately degenerate. This near degeneracy can be traced to a specific combination of the scalar and vector potentials of the relativistic nuclear mean field (RMF).\textsuperscript{9–11} In a spherical RMF theory, the relevant quantum numbers are related to the lower two components of the Dirac wave function with pseudo-orbital \(\tilde{l} = l + 1\) and the pseudo-spin \(\tilde{s} = \frac{1}{2}\). The usual quantum numbers of the upper two components, namely, the orbital angular momentum \(l\) and spin \(s\), are related to the lower two via a helicity transformation \(\sigma \cdot \hat{p}\), which is the microscopic form of the pseudo-spin transformation.\textsuperscript{12} An extension of the RMF theory that includes Fock terms also supports the pseudo-spin symmetry, even though in this case its origin is not the same as in the RMF theory without Fock terms.\textsuperscript{13}

The pseudo-SU(3) model is a natural many-body shell model extension of the pseudo-spin concept.\textsuperscript{14} In the model, the approximate symmetry is enhanced so long as the defector level \((j = \eta + \frac{3}{2}\) for oscillator shell \(\eta\)) is fully occupied so that the dynamics is driven by the particles that occupy the remaining levels \((j = \eta - \frac{1}{2}, j = \eta - \frac{3}{2}, \ldots, \frac{1}{2})\) of the normal-parity space. This situation applies if the Fermi level is above the defector level and below the intruder level \((j = \eta + \frac{3}{2}, \text{ coming from the shell above, } \eta + 1)\). If the Fermi level is above the intruder level, the participation of active particles in the intruder level is expected to be adiabatic.\textsuperscript{8,15} In this subspace of the truncated oscillator shell, the unitary pseudo-spin transformation effectively acts as \(N\sigma \cdot b\), where \(b\) is the oscillator annihilation operator.\textsuperscript{16} Consequently, various physical operators, except for the total angular momentum \(J\), are realized differently in the pseudo-spin transformed space. For example, as will be used later, the quadrupole operator transforms approximately as\textsuperscript{17,18}

\[
Q^{\text{pseudo}} \rightarrow \kappa_1 Q - \kappa_2 \sum_\alpha [\tilde{l}_\alpha \otimes \tilde{s}_\alpha]^{(2)} + \cdots, \tag{1}
\]

where \(\kappa_1 \approx \frac{2\eta^2 + 15\eta + 24}{2(\eta + 1)^2}\) and \(\kappa_2 \approx \frac{3(\eta + 2)}{2(\eta + 3)^2}\).

The \(SD\)-pair shell model (SDPM) employs a highly truncated shell model space. The validity of this model is tractable because the building blocks of the model are directly constructed from collective nucleon pairs.\textsuperscript{19}

\[
S^l = \sum_j y(jj0)(a_j^\dagger \times a_j^\dagger)_{j0}, \tag{2}
\]

\[
D^l_\mu = \sum_{jj'} y(jj'2)(a_j^\dagger \times a_{j'}^\dagger)_{\mu}, \tag{3}
\]
where $y(jj')$ are structure coefficients, and $a^\dagger$ and $a$ the usual fermion creation and annihilation operators that obey the anticommutation relations, $\{a^{jm}, a^{jm'}_{\dagger}\} = \delta_{jj'} \delta_{mm'}$. The single-particle quantum numbers $j$ and $m$ label the total angular momentum and its $z$-projection. The structure coefficients $y(jj')$ are determined self-consistently by the single-particle levels. In the limit of degenerate single-particle levels, the structure coefficients are given by $y(jj0) = \frac{1}{\sqrt{2j+1}}$ and $y(jj'2) = -\frac{1}{2} \langle j||2||j'\rangle$.

To study the effect of the pseudo-spin transformation on the SDPM, we considered a Hamiltonian that captures the essence of low-lying nuclear structure,

$$H = H_0 - \alpha G S^\dagger S - (1 - \alpha) \frac{1}{2} \chi Q \cdot Q,$$

where $H_0 = \sum_{jm} \epsilon_j a^{jm}_{\dagger} a^{jm}$. For simplicity, calculations were performed with the $\epsilon_j$ taken to be degenerate and the interaction strengths set to $G = 0.14$ MeV and $\chi = 0.008$ MeV. The parameter $\alpha$ was varied between 0 (quadrupole limit) and 1 (pairing limit). The case we studied had $n = 8$ identical particles (or $N = 4$ pairs) in the $1h_9/2f_{7/2}f_{5/2}p_{1/2}$ (denoted by $h_9/2fp$) valence space. (The $1_{11/2}$ level that is part of a full $hfp$ shell was taken to be a defector; that is, the $1_{11/2}$ level was considered to be part of the core.) Under the pseudo-spin transformation, this transforms into a $1g_{9/2}1g_{7/2}d_{5/2}2d_{3/2}3s_{1/2}$ valence space (or full $gds$ shell) with the modified strength of the quadrupole interaction $\approx \kappa^2 \chi = 0.011$ MeV as prescribed by Eq. (1). The strength of the pairing interaction, however, remained $G = 0.14$ MeV because the pairing interaction is invariant under pseudo-spin transformation. Thus, the transformed Hamiltonian is

$$\tilde{H} \approx \tilde{H}_0 - \alpha G \tilde{S}^\dagger \tilde{S} - (1 - \alpha) \frac{1}{2} \kappa^2 \tilde{Q} \cdot \tilde{Q}.$$

Results for $H$ are shown in Fig. 1(b) and $\tilde{H}$ in Fig. 1(c). Generally, the spectrum in the transformed space follows that in the original space with slightly lower values at higher energy towards the quadrupole limit. In the pairing limit, the spectra of both spaces agree extremely well, within 0.007 MeV or a relative error of about 0.1%. By comparison, in the full shell-model space this agreement is realized exactly because the pairing interaction $S^\dagger S$ is invariant under pseudo-spin transformation. At the other end of the spectrum, i.e. the quadrupole limit ($\alpha = 0$), the agreement is within 0.1 MeV error for the yrast states (ground band) and is still within a relative error of about 7% for others. Between these two limits, the spectra of both spaces agree rather well.

We also note that the pairing and quadrupole limits of the SDPM are approximately solvable. In the pairing ($\alpha = 1$) limit, we obtain a superconducting spectrum that follows the formula,

$$\langle S^\dagger S \rangle = \frac{1}{4} (n - v) (2\Omega - v - n + 2),$$
Fig. 1. (color online). Energy levels (in MeV) of an $n = 8$-particle ($N = 4$ pair) system: (a) in the SU(3) limit; (b) as they vary with the parameter strength $\alpha$ in the $h_{9/2}f_{7/2}p_{3/2}p_{1/2}$-shell; (c) the pseudo-spin-transformed counterpart of (b), namely as they vary with the parameter strength $\alpha$ in the $g_{9/2}d_{5/2}d_{3/2}s_{1/2}$-shell; and (d) in the quasi-spin SU(2) limit. The label of the line in column (b) and (c) is same as those in column (a) with the same color and line style.
approximately with the yrast states tracking the seniority quantum number \( v \) (here, \( \Omega = \frac{1}{2} \sum_j 2j + 1 = 15 \); that is, \( J_1 = L_1 = v \), and so on, where the seniority \( v \) belongs to the highest weight irrep \((\kappa, \kappa, \kappa)\) with \( \kappa = 5 \) of \( \text{Sp}(3) \subset \text{Sp}(\Omega) \). As can be seen through a comparison with Fig. 1(d), the calculated energy levels are in good agreement with relative errors in energy of less than about 3%. Despite this remarkable agreement in the energy for the pairing limit, we should be aware of differences between the SDPM and the full shell-model spaces. A very important consideration is that unlike the shell model where the 0\( ^3 \) state is pure two-body interaction in the full shell-model space is a consequence of the severe truncation. The spectrum of the Hamiltonian, \( H_{\text{SU}(3)} = -3\xi c_2 + \eta K^2 + \frac{3}{2} \xi L^2 \),

\begin{equation}
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\end{equation}

where, just as above, \( c_2 \) is the second-order Casimir invariant of \( \text{SU}(3) \), \( K^2 \) the \( \gamma \)-band splitting operator, and \( L^2 \) the usual angular momentum operator with their respective eigenvalues in the \( \text{SU}(3) \) bases \( |(\lambda \mu)KLM)\):

\begin{equation}
\langle c_2 \rangle = \frac{2}{3}(\lambda^2 + \mu^2 + \lambda \mu + 3\lambda + 3\mu) , \quad \langle K^2 \rangle = K^2 , \quad \langle L^2 \rangle = L(L + 1) .
\end{equation}

Unlike \( c_2 \) and \( L^2 \) (which are two-body operator), \( K^2 \) is a special \( 2 + 3 + 4 \)-body operator, specifically,

\[ K^2 = aL^2 + bL \cdot (Q \times L) + c(L \times Q) \cdot (Q \times L) . \]

The fact that we obtain 3- and 4-body effective interactions in the \( \text{SU}(3) \) model space from a pure two-body interaction in the full shell-model space is a consequence of the severe truncation. The spectrum of the Hamiltonian, \( H_{\text{SU}(3)} \) of Eq. (7), is depicted in Fig. 1(a) with the parameters fit to the results of \( H_{\text{SDPM}}(\alpha = 0) \). The \( \text{SU}(3) \) irreps are (80), (42), (04), and (20). The \( \gamma \)-band parameter strength is found to be \( \eta = -0.163 \) MeV; and, interestingly, the other parameter strengths \( \xi = \zeta = 4.53\chi = 0.0362 \) MeV show that the relation \( Q \cdot Q = 6c_2 - 3L^2 \) approximately holds in the truncated space. Generally, except for the 0\( _1 \) level, relative errors in energy are less than about 10%. The level ordering within each band is the same, except that the 0\( _4 \) level is significantly higher than the 2\( _3 \) level even though they belong to the same band. Detail analyses of the Hamiltonian in Eq. (4) in connection with the shell model will be presented in a separate paper.24

The reduced \( E2 \) transition strength is

\begin{equation}
B(E2; J_i \rightarrow J_f) = \frac{1}{2J_1 + 1} |\langle J_f | e_{\text{eff}}^{p} Q^{p} + e_{\text{eff}}^{n} Q^{n} | J_i \rangle |^2 .
\end{equation}

In a theory that incorporates a full, infinite-dimensional Hilbert space, the effective charges are bare charges, i.e. \( e_{\text{eff}}^{p} = e \) and \( e_{\text{eff}}^{n} = 0 \), where \( e \) is the proton charge.
A theory like the symplectic model accounts for such correlations to higher shells and therefore we use bare charges.\textsuperscript{25,26} In a shell-model theory restricted to a major oscillator shell, we normally take $e_{\text{eff}}^p = 1.5\epsilon$ and $e_{\text{eff}}^n = 0.5\epsilon$. In the SDPM the effective charges are expected to be even larger. As can be seen from Fig. 2, the $B(E2)$ value from the SDPM calculations in the pseudo-spin-transformed space follows that of its original space according to the factor in Eq. (1). The discrepancies in the $B(E2)$ values are more pronounced than those found in the energies. The difference in both can be attributed to the correction term $(\tilde{l}_\alpha \times \tilde{s}_\alpha)^{(2)}$ in Eq. (1). Nonetheless, as can be seen in Table 1, the ratio of $B(E2)s$ to the $B(E2; 2_1 \rightarrow 0_1)$ in the pseudo-spin-transformed space is almost identical to that in the original space. As a caveat, we also find a phase transition at $\alpha \approx 0.5 - 0.6$ in this study which is indicated by a rise in the $B(E2; 0_2 \rightarrow 2_1)$ and $B(E2; 2_2 \rightarrow 2_1)$ (Fig. 2).

In conclusion, the pseudo-spin symmetry is preserved in the $SD$-pair truncation scheme. The symmetry in the truncated space follows that in the full shell model space in nearly every detail. The contribution from the quadrupole interaction in the $SD$ space follows the pseudo-spin factor $\kappa_1$ without re-fitting, while the contribution from the pairing interaction remains invariant. Furthermore, the $B(E2)$ strength in
Table 1. $B(E2; J_i \to J_f)/B(E2; 2^+ \to 0^+)$ ratios for $n = 8$ particles ($N = 4$ pairs) in the $(h_{9/2}fp)$ and for some specific values of $\alpha$. The SU(3) and SU(2) quasi-spin limits are also given.

<table>
<thead>
<tr>
<th>$J_i \to J_f$</th>
<th>SU(3) limit</th>
<th>$\alpha = 0.0$</th>
<th>$\alpha = 0.5$</th>
<th>$\alpha = 1.0$</th>
<th>SU(2) quasi-spin limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4^1 \to 2^1$</td>
<td>$1.266$</td>
<td>$1.257$</td>
<td>$1.290$</td>
<td>$1.302$</td>
<td>$1.304$</td>
</tr>
<tr>
<td>$6^1 \to 4^1$</td>
<td>$1.073$</td>
<td>$1.050$</td>
<td>$1.101$</td>
<td>$1.139$</td>
<td>$1.143$</td>
</tr>
<tr>
<td>$8^1 \to 6^1$</td>
<td>$0.636$</td>
<td>$0.610$</td>
<td>$0.646$</td>
<td>$0.670$</td>
<td>$0.675$</td>
</tr>
<tr>
<td>$0^2 \to 2^1$</td>
<td>$0.000$</td>
<td>$0.00686$</td>
<td>$0.01029$</td>
<td>$0.346$</td>
<td>$1.570$</td>
</tr>
<tr>
<td>$2^2 \to 2^1$</td>
<td>$0.000$</td>
<td>$0.00940$</td>
<td>$0.00819$</td>
<td>$0.336$</td>
<td>$1.070$</td>
</tr>
</tbody>
</table>

the pseudo-spin transformed space also scales approximately according to $\kappa_1$. Even though in this article we only reported the case of $n = 8$ particles ($N = 4$ pairs), we did a similar study on $(g_{7/2}ds)^6$, $(h_{9/2}fp)^6$, and $(g_{7/2}ds)^8$ [equivalent to $(\tilde{f}p)^6$, $(\tilde{g}ds)^6$, and $(\tilde{f}p)^8$] systems and they give the same general conclusion.

In the context of quasi-exactly solvable models, we have shown that the quasi-dynamical symmetry plays a role in the quadrupole limit of the SDPM and there is a strong coupling between the truncated collective $SD$-pair subspace to the remaining subspace. The strong coupling is indicated by a scaling of the rotational structure $[\kappa_1^2 \chi$ in the shell model space versus $4.53 \chi$ in the SDPM space for the case of $(h_{9/2}fp)^6$], the scaling of effective charge in $B(E2)$ values, and the pseudo-spin symmetry. It remains to be shown how the $su(3)$ algebra works in the SDPM space; that is, to verify whether the quadrupole limit possesses quasi-dynamical symmetry. Nevertheless, the success of the IBM and FDSM in this limit underscores this view of quasi-dynamical symmetry. In the pairing limit, however, it is apparent that the collective $SD$-pair subspace couples weakly to the remaining subspace as the pairing gap in the SDPM is approximately the same as in the case of the shell model.

While this is the first study to incorporate non-degenerate single-particle-energy levels, we expect the pseudo-spin to remain an approximate symmetry, due to the adiabatic nature of the rotational structure and of the pair structure, even when the spin-orbit splitting in the shell model space is strong.$^{3,27}$

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