Many-particle states of the anisotropic oscillator

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Received 14 January 1976

The invariance under canonical transformations of the fundamental Cartesian coordinates and conjugate momenta commutators is used to show that the state labeling schemes of Elliott and Moszkowski are applicable to the case of many particles in an anisotropic harmonic oscillator potential. The relationship of the states obtained to the group theoretically equivalent isotropic harmonic oscillator states is investigated. The results reveal explicitly the collective nature of the particle-hole structures induced by the deformation.

NUCLEAR STRUCTURE Theory, many particle states, anisotropic (deformed) harmonic oscillator, relation to isotropic harmonic oscillator states.

1. INTRODUCTION

The three-dimensional isotropic harmonic oscillator is known to serve as a mathematically convenient choice of the average potential used in an attempt to describe the bound states of a many nucleon system. As early as 1949, Such and Hill\(^\text{(1)}\) pointed out the invariance of the corresponding particle Hamiltonian to unitary transformations in three dimensions, U(3), or equivalently to unitary unimodular transformations in three dimensions, SU(3). This remained, however, for Elliott\(^\text{(2)}\) in 1958 to exploit this invariance in firmly establishing (at least for the 3s-1d shell) the physical significance of the three-dimensional isotropic harmonic oscillator states in many-particle shell model calculations. More recently Moszkowski and his associates (see Ref. 3 from which a suitable bibliography may be obtained) have further exploited this U(3) invariance to obtain a canonical labeling for the many-particle isotropic harmonic oscillator states by relating the boson calculus techniques familiar from the works of Schwinger\(^\text{(3)}\) and Bargmann\(^\text{(4)}\) to the abstract Weyl basis vectors.\(^\text{(5)}\) In a series of two papers\(^\text{(6)}\) we established the connection between these two labeling schemes thus allowing one to utilize the physical significance of the state labels in the U(3) x U(3) scheme of Elliott while maintaining the mathematical elegance of the state labels in the U(3) x U(3) x U(3) scheme of Moszkowski. Regardless of the state labeling scheme involved, however, the three-dimensional isotropic harmonic oscillator is just too simple to provide an adequate description of the detailed properties of nucleons. For this reason nuclear structure physicists have attempted to take into account residual nucleon interactions within the framework of this simple model. Quadrupole-quadrupole and pairing interactions have been shown to play an important (often dominant) role in obtaining results in agreement with experiment.\(^\text{(7,8)}\) Since the quadrupole-quadrupole interaction is related to deformation of the average nuclear potential, the results indicate the desirability of incorporating nuclear deformation into many-particle theories from the beginning. Nilsson\(^\text{(9)}\) initiated such a study by giving a description of a single particle moving in an axially symmetric harmonic oscillator potential created by a deformed even-even core of nucleons which itself was allowed to rotate adiabatically. Newton\(^\text{(10)}\) extended Nilsson's work to include axially symmetric deformations of the nuclear potential. In this article we initiate an investigation into the consequences of generalizing their results to the case of many particles moving in an anisotropic harmonic oscillator potential.

In order to establish both notation and definitions we begin by presenting in Sec. II a brief review of the relevant mathematical results of Refs. 3, 7, and 8. Then in Sec. III we shall exploit the invariance of the fundamental commutators of U(3) to canonical transformations in order to introduce deformation into the theory. In Sec. IV we will study the structure of the deformed potential many-particle states by expanding them in terms of the undeformed potential many-particle states. In Sec. V we will discuss the physical significance of the results obtained.

901
result to label each state vector by two Gell-Mann* patterns \([I(\alpha), I(\beta)]\). The \(H_{12}\) and \(H^{*}\) are integers which satisfy the betweenness condition\(2,10\)

\[H_{12} \leq H_{1} < H_{2} \leq H_{12} + \delta, \quad 1 \leq \alpha < \beta < 3 \quad (2.9a)\]

and

\[H^{*} > H_{12} + \delta > H_{1} + \delta > H^{*}, \quad 1 \leq \alpha < \beta < 3, \quad (2.9b)\]

In terms of the canonical reductions \(U(3) \supset U(2)\) \(\supset U(1)\) and \(U(4) \supset U(3) \supset U(2) \supset U(1)\) the \(H_{12}\) and \(H^{*}\) specify, respectively, the length of the core in the Young patterns\(11\) for the irreducible representation \(\mathbf{H}\) of \(U(3)\) and \(U(2)\) to which the state belongs. Several features of these states are of importance in what follows. First of all, since they are labelled by a distinct set of \(\mathbf{H}\) labels for the subgroups in the canonical reductions of \(U(3)\) and \(U(4)\) they are necessarily orthogonal. The second point has to do with their weights defined by

\[C_{\text{red}}(H_{12}) = W_{\alpha}(H_{12}^{*})^{\text{th}}, \quad (2.10a)\]

and

\[C_{\text{red}}(H^{*}) = W_{\text{tot}}(H_{12}^{*})^{\text{th}}, \quad (2.10b)\]

where the weights \(W_{\alpha}\) and \(W_{\text{tot}}\) are given by

\[W_{\alpha} = \sum_{\beta} H_{12}^{*} = \sum_{\beta} H_{12}^{*}, \quad (2.11a)\]

and

\[W_{\text{tot}} = \sum_{\beta} H^{*} = \sum_{\beta} H^{*}. \quad (2.11b)\]

Physically \(H_{12}(H^{*})\) corresponds to a distribution of \(W_{\alpha}\) phonons of excitation in the Cartesian direction \(\alpha\) and \(W_{\text{tot}}\) phonons of excitation on particle \(s\). Thus, a definite shell structure determined by the \(W_{\alpha}\) is associated with each state. These states do not, however, possess definite particle permutation symmetry. To resolve this question it is necessary to reduce \(\mathbf{H}\) with respect to the symmetric group in \(A\) dimensions or \(\mathbf{S}\). Because of (2.7-9) the solution to that particular problem is independent of the results we shall present for \(U(3)\). Finally, because the \(A_{12}\) operator acting on \(\mathbf{H}\) can only form a basis for completely symmetric \(\mathbf{H}\) of \(U(3)\), the reduction \(U(3) \supset U(2) \supset U(1)\) admits only these three \(\mathbf{H}(\mathbf{S})\) and \(U(1)\) for which \(H_{12}^{*} = H_{12}\), where the \(H_{12}\) form sparticle of \(H_{1}\), the total number of oscillator quanta. This fact is important in determining the type of particle-hole excitation generated by our introduction of deformation into the theory.

b. \(U(3)\) basis

In addition to not possessing definite particle permutation symmetry the \(H_{12}(H^{*})\) are not eigen-
states or the physical orbital angular momentum operator, \( L^z \). However, states of definite angular momentum can be obtained by projection, \[
(\Omega, K, L, M) \rightarrow \int \delta(L, M|\Omega, K, L) G(\Omega) \delta.
\] (2.12)

Here, as in Eq. 8, we have for notation simplicity introduced \((\Omega, K, L)\) and suppressed the \( \mathcal{B}^t \) labels which remain unchanged as a result of Eq. (2.7). It should be pointed out, however, that although the \((\Omega, K, L)\) labels are unique, it is only the \( \mathcal{B}^t \) labels \( M_L \) which remain valid state labels. In Eq. (2.13), \( \delta(L, M|\Omega, K, L) \) is a \( \mathcal{B}(3) \) rotation matrix and \( R(\Omega) \) is an \( \mathcal{B}(3) \) rotation operator with infinitesimal generators
\[
L_0^z = \frac{1}{2} \sum_{\Omega, K, L} \Theta(\Omega, K, L) G(\Omega),
\] (2.13)

which satisfy the usual \( \mathcal{B}(3) \) commutation relations
\[
[L_0^z, L_1^z] = L_2^z, \quad (\text{ab cyclic}),
\] (2.14)

The integration in Eq. (2.12) is over the \( \mathcal{B}(3) \) group manifold and \( K \) labels the multiple occurrences of a given \( \mathcal{B}(3) \) IR in a given \( \mathcal{B}(3) \) IR. Any of the \((\Omega, K, L)\) such that \(|\Omega| < L\) may be used, but if \( G \) is an arbitrary Gell-Mann state, in many cases the \( |\Omega, K, L\rangle \) will turn out to be identical zero.

This has origins in the fact that while the \( R(\Omega) \) and \( H^\Omega \) uniquely specify the weights \( \omega_\Gamma \) and \( \omega_\Lambda \), the converse is not in general true. States of particular interest here are extremal states\(^1\) in which the \( \omega_\Gamma \) assume either their maximum or minimum values allowed by Eqs. (2.13). These states are uniquely specified by their weights. One is guaranteed a set of linearly independent, although not necessarily orthonormal \((\Omega, K, L)\) basis states if \( G \) is an extremal state and the values of \( \omega_\Gamma \) are properly restricted. In particular, if \( G \) is in the highest weight \((\Omega, K, L)\) state \( \omega_\Gamma = \omega_{\lim} \), then a physically significant role\(^2\) for determining \( K \) and the corresponding \( L \) values is given in terms of \( \Lambda = \omega_\Lambda - \omega_\Gamma \) and \( \mu = \omega_\mu - \omega_\Gamma \) by
\[
K + \Lambda = -2, \quad \Lambda = 0, \quad \Lambda = 0, \quad \mu = 0.
\]
(2.15)

On the other hand, if \( G \) is the lowest weight \((\Omega, K, L)\) state \( \omega_{\lim} = (\Omega, K, L) \), then we have the complementary role
\[
\Lambda = \mu = \mu = 0, \quad \Lambda = 0, \quad \mu = 0.
\]
(2.16)

For each \( L \), the projection \( L_0^z \) can assume the usual values \( -L^M + L \).

A linear relationship between the \( \mathcal{B}(3) \otimes \mathcal{B}(3) \) states of Eq. (2.18) and those of the \( \mathcal{B}(3) \otimes \mathcal{B}(3) \otimes \mathcal{U}(1) \) scheme is given in terms of transformation coefficients \( \langle G' | G, K, L, M \rangle \) defined by
\[
| G, K, L, M \rangle = \sum_{G'} \langle G' | G, K, L, M \rangle | G' \rangle.
\] (2.17)

These coefficients are evaluated in Ref. 8 and their symmetry properties given in Ref. 20. They determine properly orthonormalized \( \mathcal{B}(3) \otimes \mathcal{B}(3) \) projected states via a Schmidt orthogonalization based on the relation
\[
\langle G' | K', L', M' \rangle | G, K, L, M \rangle \cap_{G'} \frac{1}{8} \sum_{G'} \langle G' | G, K, L, M \rangle | G' \rangle.
\] (2.16)

III. ANISOTROPIC HARMONIC OSCILLATOR

A necessary and sufficient condition for a transformation between generalised coordinates and their conjugate momenta to be canonical is that the fundamental commutators (henceforth we take \( \hbar = c = 1 \))
\[
\{\mathcal{Q}_i, \mathcal{P}_j \} = i \hbar \delta_{ij}, \quad \{\mathcal{Q}_i, \mathcal{Q}_j \} = \{\mathcal{P}_i, \mathcal{P}_j \} = 0 \quad \text{(3.1)}
\]
be invariant under the transformation. The transformation from the coordinate-momentum representation to the creation-annihilation representation for the anisotropic harmonic oscillator is an example in point masses
\[
\mathcal{Q}_i = \mathcal{P}_i = 0.\quad \mathcal{Q}_i = 0, \quad \mathcal{Q}_i = 0, \quad \mathcal{P}_i = 0, \quad \mathcal{P}_i = 0,
\] (3.2)

where
\[
\mathcal{Q}_i = (1/\sqrt{2}) (\mathcal{Q}_i + \mathcal{Q}_j - \mathcal{Q}_k), \quad \mathcal{Q}_i = (1/\sqrt{2}) (\mathcal{Q}_i - \mathcal{Q}_j + \mathcal{Q}_k), \quad \mathcal{P}_i = (1/\sqrt{2}) (\mathcal{P}_i + \mathcal{P}_j - \mathcal{P}_k).
\] (3.3)

and forms (3.1)
\[
\{\mathcal{Q}_i, \mathcal{Q}_j \} = \{\mathcal{Q}_i, \mathcal{Q}_j \} = \{\mathcal{P}_i, \mathcal{P}_j \} = 0. \quad \text{(3.4)}
\]

A more general transformation then that given by (3.3) is however possible. We define
\[
\mathcal{Q}_i = (1/\sqrt{2}) \mathcal{Q}_i, \quad \mathcal{P}_i = (1/\sqrt{2}) \mathcal{P}_i, \quad \mathcal{Q}_j = (1/\sqrt{2}) \mathcal{Q}_j, \quad \mathcal{P}_j = (1/\sqrt{2}) \mathcal{P}_j
\] (3.5)

By direct substitution Eq. (3.4) again holds so that for
\[
\mathcal{Q}_i = \mathcal{Q}_i, \quad \mathcal{P}_i = \mathcal{P}_i
\] (3.6)

we have that
\[
\{\mathcal{Q}_i, \mathcal{Q}_j \} = \{\mathcal{Q}_i, \mathcal{Q}_j \} = \{\mathcal{P}_i, \mathcal{P}_j \} = 0. \quad \text{(3.7)}
\]

In direct analogy with Eqs. (3.4), if we choose \( \omega = \omega_{\lim} \). This transformation...
reduces the coordinate-momentum representation for the anisotropic harmonic oscillator analog of Eq. (2.6), namely,
\[ H = \sum_{\alpha=1}^{3} \sum_{\sigma=1}^{4} \mathcal{a}_{\alpha}^{\dagger} \mathcal{a}_{\alpha} + \mathcal{a}_{4}^{\dagger} \mathcal{a}_{4}. \] (3.9)

This essential transformation simply corresponds to an anisotropic scale change for the Cartesian coordinates as may be seen by substituting
\[ \delta_{\alpha} \rightarrow \delta_{\alpha}^{'} = \delta_{\alpha} P_{\alpha}^{'} \mathcal{N}_{\alpha}. \] (3.10)

The $\delta_{\sigma}$'s and $\delta_{\sigma}^{'}$'s are then the same functions of the $\sigma$'s and $\sigma^{'}$'s as the $\delta_{\alpha}$'s and $\delta_{\alpha}^{'}$'s are of the $\delta_{\sigma}$'s and $\delta_{\sigma}^{'}$'s. The consequences of this correspondence are basic to the development of our many-particle theory. Explicitly it implies that: Any result for the isotropic harmonic oscillator which depends only upon the commutation relations of the creation and annihilation operators is also valid for the anisotropic harmonic oscillator. The correspondence will be reflected in our adoption of the convention that the lower case form for a quantity introduced as referring to the isotropic harmonic oscillator refers to the anisotropic harmonic oscillator analog of that quantity. For example, the $\alpha_{4}^{\dagger}$ and $\alpha_{4}^{\dagger}$ introduced in Eq. (3.8) refer to the anisotropic harmonic analog of the $\mathcal{A}_{4}^{\dagger}$ and $\mathcal{A}_{4}^{\dagger}$ introduced in Eq. (2.2), or equivalently in Eq. (3.2). Employing this convention, one finds that the operators $\alpha_{4}^{\dagger}$, $\alpha_{3}^{\dagger}$, and $\alpha_{4}^{\dagger}$ defined by $\alpha_{4}^{\dagger} = \sum_{\alpha=1}^{3} \delta_{\alpha}^{'} P_{\alpha}^{'} \mathcal{N}_{\alpha}$, $\alpha_{3}^{\dagger} = \sum_{\alpha=1}^{3} \delta_{\alpha}^{'} P_{\alpha}^{'} \mathcal{N}_{\alpha}$, and $\alpha_{4}^{\dagger} = \sum_{\alpha=1}^{3} \delta_{\alpha}^{'} P_{\alpha}^{'} \mathcal{N}_{\alpha}$, satisfy precisely the same commutation relations as the $\mathcal{A}_{4}^{\dagger}$, $\mathcal{A}_{3}^{\dagger}$, and $\mathcal{A}_{4}^{\dagger}$, respectively. Therefore these operators may also be interpreted as the infinitesimal generators of the unitary groups in the reduction U(4)$\rightarrow$U(3)$\times$U(1). Furthermore, a Hilbert space representation of the U(4)$\rightarrow$U(3)$\times$U(1) eigenstates of the reduced three-dimensional anisotropic harmonic oscillator Hamiltonian, Eq. (3.8), may be given in terms of the $\mathcal{A}_{4}^{\dagger}$ operating on the deformed vacuum ket $|0\rangle$ defined by
\[ \alpha_{\alpha}^{\dagger} |0\rangle = 0; \quad 1 \leq \alpha \leq 3, \quad 1 \leq \sigma \leq 4. \] (3.10)

Such states will be labeled by $|b_{4}\rangle$, $|b_{4}\rangle$, etc. in the cross notation of Sec. 2 B by $|b_{4}\rangle$, etc. in the usual notation of Sec. 2 B by $|b_{4}\rangle$. Notice that although the $|b_{4}\rangle$, $|b_{4}\rangle$ may be obtained from the $|b_{4}\rangle$, $|b_{4}\rangle$ by simply replacing $\alpha_{4}$ by $a_{4}$, $\mathcal{A}_{4}^{\dagger}$ by $\mathcal{A}_{4}^{\dagger}$ they are not equal, and in particular, the $|0\rangle$ defined by Eq. (3.10) is not equal to the $|0\rangle$ defined by Eq. (2.8). The relationship between the $|b_{4}\rangle$, $|b_{4}\rangle$ and the $|b_{4}\rangle$, $|b_{4}\rangle$ will be investigated in the next section. It reveals explicitly the collective nature of the so-called particle-hole structure induced by the deformation.

The $|b_{4}\rangle$, $|b_{4}\rangle$ again represent a distribution of $\delta_{\alpha}$ and $\delta_{\sigma}$ phonons of excitation on particle $\delta$. It is no longer true, however, that a definite shell structure may always be associated with each. The reason is that phonons oscillating in different Cartesian directions may carry different energy quanta. For instance, if $\delta_{\alpha}$ and $\delta_{\sigma}$ then two phonons of excitation could be associated with the energies $2\delta$, $3\delta$, or $4\delta$. For small deformation, a shell structure does of course persist. This is simply a manifestation of the more general result
\[ \lim_{\delta \rightarrow 0} |b_{4}\rangle, |b_{4}\rangle = |b_{4}\rangle, |b_{4}\rangle. \] (3.11)

Regardless of the deformation, however, the weights of the $|b_{4}\rangle$, $|b_{4}\rangle$ are significant in that they play precisely the same role as the weights of the $|b_{4}\rangle$, $|b_{4}\rangle$ in the physical reduction of $U(4)$ with respect to $U(3)$. A U(3)$\times$U(1) basis

Outside of the definition of Eq. (2.10) the results given in Sec. 2 B are purely group theoretical, depending only upon the labeling scheme involved. Therefore, if we define in analogy with Eq. (2.10)
\[ (g_{A} \otimes H_{B}) |\phi\rangle = \sum_{g_{A} \otimes H_{B}} \langle g_{A} \otimes H_{B} |\phi\rangle \] (3.12)

the results of that section are applicable to these deformed R(3)$\times$U(1) projected states. In Eq. (3.12) the $R(3)$ is the R(3) rotation operator with infinitesimal generators
\[ I_{3} = \sum_{\alpha=1}^{3} \mathcal{N}_{\alpha}^{*} \mathcal{N}_{\alpha} \] (3.10)

which also satisfy the R(3) commutation relations
\[ [I_{3}, I_{3}] = 2 I_{3}; \quad [I_{3}, I_{3}] = 0; \quad \text{cyclic}. \] (3.14)

Notice that the $I_{3}$ are not the components of the physical orbital angular momentum, $\mathbf{L}$, of Eq. (3.13). Their relationship will be discussed in Sec. IV. The states of Eq. (3.12) are however well defined, and those of Eq. (3.12) possess simple mathematical properties. Specifically, they are eigenstates of $I_{3}$ defined by
\[ I_{3} = \sum_{\alpha=1}^{3} \mathcal{A}_{\alpha}^{\dagger} \mathcal{A}_{\alpha} = \sum_{\alpha=1}^{3} \mathcal{A}_{\alpha}^{\dagger} \mathcal{A}_{\alpha} \] (3.15)

and $I_{3}$. The label $\mathbf{I}$ may be given the same interpretation as the $\mathbf{S}$ of Sec. 2 B and is to be given together with the corresponding I values by either Eq. (2.15) or (2.16).

Although the $|g_{A} \otimes H_{B}\rangle$ introduced in Eq. (3.12) are obtained from the $|g_{A}\rangle$, they do not diagonalize the physical Hamiltonian given by Eq. (3.8). This may most easily be seen by recognizing that al-
from \(|\varphi_{\pm}\rangle = 0\) as for the isotropic harmonic oscillator, \(|\varphi_{\pm}\rangle = 0\). That is, the \(|g; b, m\rangle\) eigenstates neither the physical Hamiltonian \(H\) nor the physical angular momentum operator \(L^2\) and \(S_z\). For this reason they possess no obvious physical significance; they are merely the group theoretical generalization of the \(|G; K\rangle\).

\[ |\varphi_{G; K}\rangle = (\langle G | K \rangle) \]

(3.10)

Both sets of states, the \(|g; b, m\rangle\) and the \(|G; K\rangle\), are complete, however, and therefore either forms an adequate basis with which to study nuclear structure. Which is more useful remains to be determined. Nuclear structure physicists have preferentially chosen the \(|G; K\rangle\) because of the physical significance of the individual states\(^{21}\). The \(|g; b, m\rangle\) should not however be excluded; they may well be the more suitable choice. This is particularly true in light of the fact that many nuclei are known to have physical properties which can only be adequately understood in terms of intrinsic state deformations.\(^{22}\)

IV. DEFORMED BASKS EXPANDS IN TERMS OF NONDEFORMED BASKS

Any attempts to investigate the collective and rotational structure of the deformed basis states necessitate a consideration of the expansions of these states in terms of their nondeformed counterparts. These expansions are carried out below.

A. \(\sum_{K=M}^{M}(2M+1)\langle K| g (b, m) \rangle \langle g (b, m) | K\rangle\)

For all, consider the special case of a single particle moving in a one-dimensional harmonic oscillator potential. Let \(\|\varphi\rangle\) and \(\|\psi\rangle\) denote, respectively, the deformed \((\alpha = 1)\) and nondeformed \((\alpha = 1)\) eigenstates of the corresponding particle Hamiltonian. Using the properties of Hermite polynomials it is a straightforward matter\(^{23}\) to show that

\[ \langle \varphi | = \sum_{n=0}^{\infty} (\langle n | \langle n + 1 | \psi \rangle ) \]

(4.1)

where

\[ (2n+1) \langle n+1 | (2n+1) \langle n+2 | \psi \rangle )^{\frac{1}{2}} \]

\[ x_i = \sqrt{\gamma_i} \langle \langle n \rangle | (x_i) \rangle \].

(4.2)

Here \(F\) is the second Appell function as given for example by Taimina\(^{24}\) and \(\gamma = \cosh(b/2)\) or \(-\cosh(b/2)\) where \(b\) is related to the deformation parameter, \(\epsilon\), by \(\epsilon = \gamma \sqrt{\alpha}\) and \(\alpha\) and \(\beta\) are integers which may independently be taken equal to zero or one. The canonical transformation to the form \(\alpha^2 = \alpha^2 - \alpha \epsilon = \alpha \epsilon + \alpha \epsilon^2 \epsilon \) with \(\alpha = \beta^2\) following from Eqs. (3.4) and (3.7). Hence we have a quasi-bose expansion analogous to the quasiparticle (fermionic) expansion of the RSC theory of superconductivity.\(^{25}\) For the special case of the deformed vacuum Eq. (4.1) reduces to

\[ \langle \varphi | = \sum_{n=0}^{\infty} (\langle n \rangle | \langle n + 1 \rangle ) \]

(2n+1) \langle n+1 | (2n+1) \langle n+2 | \psi \rangle )^{\frac{1}{2}} \]

(4.3)

For a first order theory the result is

\[ \langle \varphi | = \langle 1 \rangle | \langle b \rangle \rangle \]

(4.4)

where \(F\) is a homogeneous polynomial in the \(\alpha^2\). That is, an arbitrary term in the polynomial expansion of \(F\) has the form

\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \alpha^n \beta^m \]

(4.5)

where \(k\) is the total number of oscillator quanta. The deformed vacuum expands as a product generalization of Eq. (4.2) while the substitution \(\alpha^2 = \alpha^2 \epsilon - \alpha^2 \epsilon \) may be used to expand such factors in \(F\)\(^{11}\). One then uses the commutation properties of creation and annihilation operators to collect together terms of different order in the deformation parameters. In this way as appropriate generalization of Eq. (4.1) to the mass-case is obtained. The results of carrying out such a term-wise expansion to first order is

\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \alpha^n \beta^m \]

(4.6)

Since for the same state labels, i.e., \(|\alpha\rangle = |\alpha\rangle\), \(\langle \varphi |\) in the same function of the \(\alpha^2\) as \(\langle \varphi |\) is of the \(\alpha^2\) it follows that to lowest order

\[ \langle \varphi | = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \alpha^n \beta^m \]

(4.7)

(4.8)
with $g = G$. For the special case of axial symmetry with $b_1 = b_2 = 0$ and $b_3 = 2b_1$ this reduces to

$$|\phi| = [1 - (b_1/Dp)]|G|,$$  

(4.3)

where

$$P_x = 2b_2 - D_{11} - D_{22}$$  

(4.8)

is the eighth component of the second rank spherical tensor $P$ formed from the Cartesian tensor

$$L_{00} = \sum_{m=0}^{\infty} |P_{m0}|^2.$$  

(4.9)

The choice of the $b_n$ is consistent with the constant volume condition, $c_4V = 1$, derived from the incompressibility of nuclear matter.69

S. LEHANN (3) (1983) behind

To expand the $|g; k(m)|$ in terms of the $|G; KLM>$ it is necessary to expand the deformed rotation operator $\hat{R}(\theta)$ in terms of the nondeformed rotation operator $R(\theta)$. First of all we find that

$$L_0 = \cos[2(\theta_3 - b_3)/2]L_0 + \sin[2(\theta_3 - b_3)/2]K_2,$$  

(4.10)

where

$$L_0 = \sum_{k=1}^{\infty} \sin[(2k+1)\theta_3]/D_{22},$$  

(4.11)

with the $D_{ij}$ given by (4.6). If we now define

$$K_0 = -\phi \cos[2(\theta_3 - b_3)/2],$$  

(4.12)

then we have the theorem of Hantsch (57) that we find that

$$l_{\phi} = a^{-1} l_{\phi} a, a = \exp [i \cdot \cdot \cdot i_0 [K_0, L_0],$$

(4.13)

where $a = \exp [i \cdot \cdot \cdot i_0 [K_0, L_0], \cdot \cdot \cdot i_0]$ is the n-fold m-commutator of $a$ and $n$ is defined by

$$\{v, a, \cdot \cdot \cdot \} = \{\cdot \cdot \cdot [u, v], \cdot \cdot \cdot [u, v], \cdot \cdot \cdot [u, v] \}. \quad (1.19)$$

We also obtain the mixed basis result

$$|G; KLM> = \int d\theta \sum_{m=1}^{\infty} |\phi \cdot \cdot \cdot |G; KLM> \cdot \cdot \cdot |\phi \cdot \cdot \cdot$$

(4.23)
In deriving these expressions we have used in addition to Eqs. (4.7) and (4.21) the rotational properties of $P$ and the R(3) Coburn-Gordan series for the rotation matrixes. It should be pointed out that in Eqs. (4.23) and (4.24) the $(k,l,m)$ are mixed basis states in that they are physical angular momentum states projected from the deformed basis.

The expansions of Eqs. (4.7), (4.22), (4.23), and (4.24) involve only the second rank spherical tensor $P$ corresponding to $L = 2$. From Eq. (4.8) together with the fact that

$$D_{L} \cdot D_{-m} \cdot A_{0 \alpha} = \sum_{L \alpha \beta} (A_{L \beta 0} - A_{L 0 \beta})^{L}$$

(4.25)

the components of $P$ are seen to correspond to linear combinations of $0, 2$, and $3$ SU(3) tensors.

The necessary mathematical tools to examine their tensorial properties are available in the literature. Here it is sufficient to note that the coupling among the deformed states induced by the deformation is to these $3 \otimes 3 \otimes 3$ excitations $(V \otimes V \otimes V)$ associated with a quadrupole interaction. In particular, the particle-hole structure induced by the deformation into the many-particle system involves the promotion of a single nucleon across two major oscillator shells and never involves a pair promotion across a single shell.

V. DISCUSSION AND SUMMARY

We have exploited the invariance under canonical transformation of the commutators of the fundamental Curvilinear coordinates and their conjugate momenta to obtain the anisotropic analog of the isotropic harmonic oscillator results. In effect, this theory represents a many-particle generalization of the mathematical input into the theoretical work of Nilsson and Newton.

The physical consequences of this generalization are best understood in terms of the particle-hole structure induced by the deformation into the spherical oscillator states. In first order the deformation corresponds to promoting a single nucleon across two major shells. Pairwise promotions across a single shell are not included. This implies, for example, that the four particle-two hole $(4p-2h)$ renormalization effects introduced by Roy and Brown to obtain realistic two-body matrix elements for the $2s$ and $2p$ shell are not included. On the other hand, $3p-1h$ contributions to the renormalization can in principle be reproduced by introducing such deformation into the many-particle wave functions. The question which then arises is simply: Can the use of the anisotropic oscillator as a better first approximation to the many-particle wave functions of the $sd$ shell nuclei? Preliminary results simulating the deformations via a quadrupole-quadrupole renormalization (equivalent to $P_{2}$ within a single shell) are encouraging. 20

The deformation also allows a sequence of angular momentum states intermediate to those found in the Elliott and the rigid rotor models. This may be most easily seen by comparing the angular momenta obtained by projection from an extremal deformed basis state and the corresponding non-deformed basis state. For example, consider the effect of an axially symmetric deformation on those states of the completely symmetric $(\lambda, 0)$ IR of SU(3). In the limit of zero deformation we have from rule Eq. (2.18) that $K = 0$ and $L = \lambda$, $\lambda - 2$, $\ldots$, $\lambda - 2$, $\ldots$, $\lambda - 2$, $L = 0$. Since the $K$ label is given the interpretation of the projection of $L$ on the symmetry axis in the intrinsic system, from Eq. (4.7) it is clear that for an axially symmetric deformation $K$ remains unchanged. To determine what new $L$ values are introduced we must consider the coupling of the $(0, 0)$ and $(0, 2)$ IR of SU(3) to the $(\lambda, 0)$ IR of SU(3). The $(0, 2)$ IR couples to the $(\lambda + 2, 0)$, $(\lambda + 1, 0)$, and $(\lambda - 2, 2)$ IR of SU(3). Similarly, the $(0, 1)$ couples to the $(\lambda, 2)$, $(\lambda - 1, 0)$, and $(\lambda - 2, 0)$. For $K = 0$, the $(\lambda + 2, 0)$ IR adds something new, namely an $\lambda \leq 2$ terms to the angular momentum sequence. The $2\lambda$ involved either do not have $\lambda = 0$ or, as can be seen from Eq. (4.23), only yield a nonvanishing result for already existing $L$ values. Therefore, the deformation may be interpreted as having the effect of extending the $K = 0$ "rotational band" in the direction associated with that of a rigid rotor. For other IR of SU(3) similar results hold except several values of $K$ will in general be involved. Only for nonaxially symmetric deformations, however, will new $K$ values be involved.

In conclusion then, we may say that we have developed a many-particle theory for the anisotropic oscillator which shares the mathematically elegant properties of both the Elliott and Moshinsky models. By expanding the deformed states in terms of their nondeformed counterparts, we have been able to show quantitatively the relationship of the deformation to a quadrupole interaction as well.
as the fact that associated with the deformed states are rotational sequences intermediate to those of the Elliott and rigid rotor models. Calculations simulating the deformation via a quadrupole-quadrupole renormalization in the spherical basis support the physical validity of our theory.

9) P. Elliott, Group Theory and The Nuclear Shell Model (Laxmi American School of Physics, University of Mexico, Mexico City, 1963).
25) M. Lane, Nuclear Theory (Benjamin, New York, 1965).