Macroscopic limit of the microscopic SU(3)⊗SO(3) integrity basis interaction

Yosef Leibflether and J. P. Draayer
Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana 70899-4001

(Received 15 October 1985)

It is shown that a fourth degree $SU(3)$⊗$SO(3)$ integrity basis interaction, encountered in both fermion and boson theories of nuclear structure, maps onto an axially symmetric rotor Hamiltonian for a special linear combination of the third (LQQ) and fourth (LQQQ) order scalar operators.

A long-time goal of nuclear physics has been to provide a macroscopic shell-model interpretation of nuclear rotational phenomena that are as simply and ably described within the framework of the Bohr-Mottelson picture of collective motion. 1 Elliott made a giant step forward in that direction when he proposed the now acclaimed $SU(3)$ model.2 The latter is microscopic in the sense that the angular momentum $(L\mu,\mu=0,\pm 1)$ and quadrupole $(Q_{\mu},\mu=0,\pm 1,\pm 2)$ operators that generate $SU(3)$ are given in terms of individual nucleon coordinate and momentum variables. Here that microscopic-macroscopic link is reinforced by a direct demonstration of the equivalence of the rotational model Hamiltonian and an $SU(3)$⊗$SO(3)$ integrity basis interaction.3 It is shown that for a special linear combination of the third (LQQ) and fourth (LQQQQ) order $SU(3)$ noninvariant $SO(3)$ scalar, the integrity basis (IR) interaction reduces to the symmetric rotor (SR) Hamiltonian. In a subsequent paper the connection of the integrity basis form to a truly microscopic interaction will be presented. The integrity basis technology therefore yields at a fundamental (operator) level the sought-after microscopic interpretation of nuclear rotational phenomena.

The $SU(3)$⊗$SO(3)$ integrity basis consists of six operators: $[C_1\hat{R}x_1, C_2\hat{R}x_2, x_1x_2, x_1x_3, x_2x_3, x_1x_2x_3]$. The operators $C_1$ and $C_2$ are the second and third order invariants of $SU(3)$; $L^2$ is the invariant of $SO(3)$, and $x_1, x_2, x_3$ are independent third, fourth, and sixth order $SU(3)$ noninvariant $SO(3)$ scalar operators.4 Here an invariant means a group Casimir invariant operator, and a scalar operator is one that transforms like a multiple of the identity under $SO(3)$. So invariants are scalars but the opposite is not necessarily so. The integrity basis principle ensures that all $SU(3)$ scalars in $SU(3)$ can be written as polynomial functions of these six basic ones. It follows that within a single representation of $SU(3)$, $H_{\text{SU(3)}} = aL^2 + a_1x_1 + a_2x_2 + a_3x_3$. This is the form of the most general interaction of degree four.5 Here $x_1, x_2, x_3$ are $L=Q$ and $X_{\pm 1}, X=(\pm 1, \pm 1)$, where $X_{\pm 1}$ is $X_{\pm 1}(L)\bigcirc(Q,L)$, and the $\bigcirc$ denotes the direct product of $L$ and $Q$. The simple Hamiltonians have been used to reproduce spectra and electromagnetic transition rates data of the $132$-shell nuclei $^{238}U$ as well as that of several rare earth and actinide species.6–8 Whereas this was done within the context of a fermion dynamics, it has also now been applied to Gd and Er isotopes with the $SU(3)$⊗$SO(3)$ algebra that of the rotational limit of the interacting boson model.9 In all cases, fermion and or boson, (1) gives a good representation of the data.

Consider the operator

\[ Y = X_1 + X_3. \]

It will be shown that for a special value of the parameter $\lambda$, $Y = aL^2 + \beta L^2$. That is, there is a value of $\lambda$ for which the off-diagonal matrix elements of $Y$ vanish and the diagonal matrix elements reduce to a linear function of $L$ and $L^2$. It follows that for $\lambda = \lambda_0$ the integrity basis Hamiltonian can be mapped onto the Hamiltonian of a symmetric rotor,

\[ H_{\text{SU(3)}} - H_{\text{SU(3)}} = \frac{1}{2}L^2 + \frac{1}{2}\lambda_0(L+1)(L+2). \]

(4)

In (4), $J$ and $I$ are the moments of inertia about axes perpendicular and parallel, respectively, to the principal symmetry axis of the rotor. Thus this correspondence is strictly true only in a $2\lambda > \lambda_0 > 1$ limit, it has been found to be ready so for representations that enter far near and actinide applications using the $SU(3)$ shell model. It is true to a lesser degree for Elliott model $\alpha$-shell applications since the $SU(3)$ representations here correspond to intrinsic states of smaller deformation. As the results depend only on the $SU(3)$⊗$SO(3)$ structure, they extend to the rotational limit of the interacting boson model $[U(1)\otimes SU(3)\otimes SO(3)]$ and the compact group substructure of the microscopic collective model $[Sp(3, \mathbb{R})\otimes SU(3)\otimes SO(3)]$.9,10

To illustrate the fact that there is a value of $\lambda$ for which $Y$ is nearly diagonal, we introduce a standard statistical spectroscopy measure,$^1$ $\Delta(W) = \frac{1}{2} \sum_{i \neq j} |d_{ij}(L)(2L+1) - (\pi)^{1/2}$. Here $d_{ij}(L)$ measures the contribution from the off-diagonal matrix elements to the spread in the eigenvalues of the $L$th submatrix and $\pi L^2 + a_1x_1 + a_2x_2 + a_3x_3$ is the variance of the $L$th submatrix which includes, in addition to $aL^2$, a contribution $c(L)$ from the displacement of diagonal elements about the centroid value $c(L)$. The term $(\pi L^2 + a_1x_1 + a_2x_2 + a_3x_3)$ is a contribution to the variance of the $Y$ in the full $(\lambda_0)$ space from the displacement of the $L$-subspace centroids about the average value $c$. The factor $\Delta(W)$ is just $(\pi^2/2) - 1$ times the multiplicity of $L$ in $(\lambda_0)$, hence $\Delta(W) = 0$ for $\Delta = 0$. The eigenvalues of $Y_{\text{SU(3)}}$ are diagonal and $\Delta = 0$ when the diagonal values are all eigenvalues.

In Fig. 1, $\Delta$ as defined by (5) is plotted as a function of $X$.
FIG. 1. A plot of the measure $\Delta$ vs $l$ for the $(\lambda_{k}) = (30, 8)$ representation of SU(3). The minimum occurs for $x = 2.69 \times 10^{-2}$. For larger $\lambda$ the resonant state is narrower and the minimum deeper. For $x = 0$ the $l \Delta$ is a measure of the off-diagonality of the $J_{l}(x)$ operator.

for the $(\lambda_{k}) = (30, 8)$ representation of SU(3). The results shown for $l = \lambda + \mu$, the maximum allowed value for $L$. The (30,8) representation is the leading one in a pseudo-SU(3) description of $^{161}\text{Er}$. The sharp resonant behavior shown is typical of all cases studied. For smaller $\lambda$ the resonant region is broader and less deep while for $\lambda$ larger it is narrower and the minimum is even smaller. $x_{\text{min}}$ value, which is $2.69 \times 10^{-2}$ for the (30,8) representation, was found to be insensitive to the choice of $N$. The point labeled $\Omega$ is for $y = 0$ when $Y$ reduces to the $\Omega$ operator of Moshtinsky. Note that $\Omega$ yields high off-diagonal matrices as does $J_{l}$ which is obtained for $x = \infty$. A surprising feature of the $\Delta$ vs $x$ curve is the sharpness of the dip and the smallness of $\Delta$ for $x = x_{\text{min}}$. Attempts to refine the $\Delta$ measure by restricting the sum in (5) to lower $L$ states ($N < \lambda + \mu$) led to no significant changes.

In addition to demonstrating the diagonality of $H_{\Omega}$ for $x = x_{\text{min}}$, it is necessary to show that each $J_{l}^{\Omega}$ is a linear function of $L^{2}$ and $K^{2}$. This is shown in Fig. 2 for the $(\lambda_{k}) = (30, 8)$ representation. The $J_{l}$ bands are all very nearly linear with slopes that depend only weakly upon the value of $K$. Note that there is a pronounced odd-even effect; for a given $K = 0$ the even-$L$ values lie above the odd-$L$ values. In Table I the values of $a$ are given for each $K$ band together with deviations from the average value which is $a = 7.83$. The $K$-band intercepts are also given in Table I and yield for $\beta$ an average value of 109.4 with uncertainty $\pm 3.8$. For larger $\lambda$ the uncertainty in $\beta$ is less and the linearity of each band greater, and, of course, the opposite is so for $\lambda$ smaller.

In general, one expects $x_{\text{min}}$ and the $a$ and $\beta$ values to be a function of the SU(3) representation labels $\lambda$ and $\mu$. In fact, using simple size arguments $L(L+1) - L(Q) - (2\lambda + \mu + 1)$ it is easy to show that $x_{\text{min}} = - (2\lambda + \mu + 1)^{2}$ for the off-diagonal cancellation to occur and to be $K$ and $L$ independent. Analytic forms for $a$ and $\beta$ are not so easily derived. A discussion of this matter will be part of a longer paper on the same subject. For the present it should be clear that gives $(\lambda_{k}),$ which dictate the $x_{\text{min}}, a, \beta$ values, and the required inertial parameters $l$ and $L_{0}$, one can determine $a$, $b$, and $c$ of (5).

\[ a = \frac{1}{2} - \alpha b = \left[ 1 + \frac{1}{2} \sqrt{1 - 4 \alpha} \right] \left( 1 - \frac{\beta}{2} \right) \left( 1 - \frac{\alpha}{2} \right). \]  

The parameter $d$ multiplying $L^{2}$ in $H_{\Omega}$ would, of course, be determined if the centrifugal stretching or anisotropic correction were included in $H_{\Omega}$. With the above values one has that $H_{\Omega} = H_{\text{ns}}$. Work leading to the results presented above was stimulated by the realization that the coefficient $a^{2}$ of $L^{2}$ in $H_{\Omega}$ as

<table>
<thead>
<tr>
<th>Band</th>
<th>$a$</th>
<th>$\beta$</th>
<th>Intercept</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-8.22</td>
<td>109.4</td>
<td>0.05</td>
</tr>
<tr>
<td>2$^a$</td>
<td>-6.56</td>
<td>109.4</td>
<td>0.03</td>
</tr>
<tr>
<td>4$^a$</td>
<td>-5.50</td>
<td>109.4</td>
<td>0.02</td>
</tr>
<tr>
<td>6$^a$</td>
<td>-4.61</td>
<td>109.4</td>
<td>0.16</td>
</tr>
<tr>
<td>8$^a$</td>
<td>-3.90</td>
<td>109.4</td>
<td>0.07</td>
</tr>
<tr>
<td>10$^a$</td>
<td>-3.18</td>
<td>109.4</td>
<td>0.20</td>
</tr>
<tr>
<td>12$^a$</td>
<td>-2.59</td>
<td>109.4</td>
<td>0.14</td>
</tr>
</tbody>
</table>

TABLE 1. Slopes $a$ for $K$ bands of the $(\lambda_{k}) = (30, 8)$ representation of SU(3). Uncertainties are also given and values for the intercepts. From the latter the value for $\beta$ was determined to be 109.4 ± 3.8. The average value for $a$ is -7.83.
determined by a best nonlinear least-squares fit to the ground and gamma band energies of $^{168}$Er is considerably greater than the 1/21 value of the Hartree theory, $a = 26.4$ keV versus 17/2 = 11.3 keV. The main difference it now uses to be the $-\alpha$ term in the expression for $a$ in (6). Because of $-\alpha$ is negative and $\lambda (J=0)$ is positive, the $-\alpha$ term contributes an additional 13.2 keV which gives a value of 26.4 keV for the parameter $a$. The best-fit value for $a$ is $2.59 \times 10^{-3}$ as compared to $2.69 \times 10^{-3}$ for $\lambda (J=0)$.

One additional fact bears mentioning. Rauh was apparently the first to study the SU(3)⊗SU(3) structure in depth. His objective was to find a canonical resolution of the state-labeling problem; that is, to construct an operator with simple eigenvalues that would distinguish between multiple occurrences of a given $\lambda$. A representation ($\lambda \rho$)

This work was supported in part by the National Science Foundation.