The influence of the intruder level on vibrational modes in atomic nuclei is examined within the framework of the nucleon-pair shell model truncated to a $SD$-subspace. The vibrational character of the spectra is found to depend upon the size of the active model space and not on the parity of the populated levels.

Keywords: $SD$ pair shell model; intruder level; vibrational mode.

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We know that truly large-scale nuclear shell model calculation remain out of reach with currently available computational facilities, even though the shell-model codes have been improved tremendously in recent years.\(^1\) This is because the time needed for large-scale calculations grows in a "non-polynomial" way with increasing dimensionality of the model space. This problem is often referred to as a NP (non-polynomial) problem in mathematics. One way to overcome this NP challenge in nuclear physics is to introduce a truncation scheme for normal parity configurations and to reduce the intruder or unique parity level — which has the opposite parity to the normal parity levels — to low- or even zero-seniority configurations. This has been done, for example, in the Fermion Dynamical Symmetry Model (FDSM) investigations.\(^2\)

Against this background, it is important to recognize that the role of the intruder levels has been receiving increased attention. Its significance for correctly reproducing the available data and its role in determining nuclear deformation have raised questions about how to properly incorporate intruder levels into current models. For example, in Ref. 3 the authors claim that both the normal and unique parity states contribute significantly to the overall collectivity of a nuclear system. Studies in the single-shell asymptotic model and the universal Woods–Saxon model imply that the valence nucleons in the intruder level contribute significantly to
measurable quantities like \( B(E2) \) values.\(^4\) Some mean-field theories claim that the particles in the intruder level play a dominant role in determining deformation.\(^5\) In the \( SD\)-pair shell model (SDPSM)\(^6,7\), studies of the so-called \( O(6)\)-limit nuclei \(^{132,134}\)Ba show that the omission of non-\( S \) pairs in the intruder level may destroy the SO(6) dynamical symmetry.\(^8,9\) As stated in Ref. 10, spectra with the character of the IBM\(^{11}\) U(5) limit can also be reproduced within the SDPSM. Motivated by these observations, we now use the SDPSM to study whether a model-dependent truncation scheme with only normal parity configurations taken into consideration is sufficient to model vibrational phenomena, or if it is essential to take intruder levels into account explicitly.

The Hamiltonian is given by

\[
H = H_\pi + H_\nu - \kappa Q^2(\pi) \cdot Q^2(\nu),
\]

\[
H_\sigma = H_\sigma(0) - G_\sigma S^\dagger(\sigma)S(\sigma),
\]

\[
H_\sigma(0) = \sum_{\sigma a} \varepsilon_{\sigma a} n_{\sigma a}, \quad \sigma = \pi, \nu.
\]

\[
S^\dagger = \sum_{a} ^\dagger \frac{1}{2} (C^\dagger_{\alpha} \times C^\dagger_{\beta})_{\alpha\beta}, \quad ^\dagger = \sqrt{2a + 1}.
\]

\[
Q^2_{\mu}(\sigma) = \sqrt{16\pi/5} \sum_{i=1}^{n} r_i^2 Y_{2\mu}(\theta_i, \phi_i),
\]

where \( \varepsilon_{j\sigma} \) is the single particle energy of orbit \( j \), and \( G_\sigma \) and \( \kappa \) are the pairing and quadrupole-quadrupole interaction strengths, respectively. The \( E2 \) transition operator is

\[
T(E2) = e_\pi Q^2_\pi + e_\nu Q^2_\nu,
\]

where \( e_\pi \) and \( e_\nu \) are the effective neutron and proton charges, respectively.

Collective pairs \( A^\dagger_{\mu r} \) with angular momentum \( r = 0, 2 \) and projection \( \mu \) are built from non-collective pairs \( (C^\dagger_{j1} \times C^\dagger_{j2})_{\mu} \) in the single-particle levels \( j_1 \) and \( j_2 \) as,

\[
A^\dagger_{\mu r} = \sum_{j_1, j_2} y(j_1, j_2 r) (C^\dagger_{j1} \times C^\dagger_{j2})_{\mu},
\]

\[
y(j_1, j_2 r) = -\theta(j_1, j_2 r) y(j_2, j_1 r), \quad \theta(j_1, j_2 r) = (-)^{j_1 + j_2 + r},
\]

where \( y(j_1, j_2 r) \) are the structure coefficients. Specifically,

\[
S^\dagger = \sum_{j} \frac{\hat{j} \cdot \bar{u}_j}{u_j} (C^\dagger_{j} \times C^\dagger_{\dot{j}})^0,
\]

\[
D^\dagger = \frac{1}{2} [Q^2, S^\dagger],
\]

where \( \hat{j} = \sqrt{2j + 1} \). The \( v_j \) and \( u_j \) factors are the occupied and unoccupied amplitudes obtained from solving the BCS equation.
The analytical expression for the overlap between two \( N \)-pair states is

\[
\langle s_1 s_2 \cdots s_N; J'_1 \cdots J'_{N-1} J_N | r_1 r_2 \cdots r_N; J_1 \cdots J_N \rangle
\]

\[
= (\hat{J}_{N-1}/\hat{J}_N) (-)^{J_N + s_N - J'_{N-1}} \sum_{k=N}^{1} \sum_{L_{k-1} \cdots L_{N-1}} H_N(s_N) \cdots H_{k+1}(s_N)
\]

\[
\times [\psi_k \delta_{L_{k-1}, J_{k-1}} \langle s_1 \cdots s_{N-1}; J'_1 \cdots J'_{N-1} | r_1 \cdots r_{k-1}, r_{k+1} \cdots r_N \rangle
\]

\[
J_1 \cdots J_{k-1} L_k \cdots L_{N-1}) + \sum_{i=k-1}^{1} \sum_{r_{L_k \cdots L_{k-2}}} \psi_i
\]

\[
\times \langle s_1 \cdots s_{N-1}; J'_1 \cdots J'_{N-1} | r_1 \cdots r'_{i-1} \cdots r_{k-1}, r_{k+1} \cdots r_N \rangle
\]

\[
J_1 \cdots J_{i-1} L_i \cdots L_{N-1} \rangle, \tag{5}
\]

where \( \hat{J} = \sqrt{2J+1} \), \( H_k(s_N) \) is essentially a Racah coefficient. \( \psi_k \) is a constant depending on the structure of the pairs \( A^{r_k \dagger} \) and \( A^{s_N \dagger} \). \( r'_i \) represents a new collective pair \( A^{i' \dagger} \) with a new distribution function \( \psi'(a_k, a'_i) \) depending on the structure of the pair \( A^{r_k \dagger} \), \( A^{i' \dagger} \) and \( A^{s_N \dagger} \). The intermediate quantum numbers \( L_i \cdots L_{k-2} L_{k-1} \), \( L_{i'}(i' = i, \ldots, k-2, k-1) \) are the angular momentums of the first \( i' \) pairs in the bra vector on the right hand side of Eq. (5). Since the right hand side of Eq. (5) is a linear combination of overlaps for \( N - 1 \) pairs, all overlaps can be calculated recursively starting from the simplest two-particle configuration. The details of the model can be found in Refs. 6 and 7.

To explore the influence of the intruder level on the vibrational spectrum, the identical nucleon system with \( N = 4 \) in the \( 50-82 \) shell was considered first. In this case, the Hamiltonian and the \( E2 \) transition operator reduce to

\[
H = H(0) - GS^\dagger S, \tag{6}
\]

\[
T(E2) = eQ^2.
\]

For simplicity, we set the normal parity levels to be degenerate and vary the single-particle energy \( \varepsilon \) of the intruder level from 0 MeV to 1.0 MeV. The pairing interaction strength \( G \) was taken to be 0.1 MeV.

The distribution coefficient of the BCS-determined \( S \) pair configuration versus the position of the intruder level relative to the normal parity levels is shown in Fig. 1. We see that as the position of the single-particle energy of the intruder level increases, the distribution coefficient of the intruder level decreases. The higher the position of the intruder level relative to the normal parity levels, the smaller its distribution coefficient, and accordingly, the lesser intruder level contributes to the overall structure of the low-lying states.

The energy ratios \( E_{J^+_i}/E_{J^+_2} \) for some important states are listed in Table 1. For the degenerate case (\( \varepsilon = 0 \) MeV), the energy ratios are 1.835, 2.556, and 3.189 for the \( 4^+_1 \), \( 6^+_1 \), and \( 8^+_1 \) states, respectively. These are all smaller than the corresponding
Fig. 1. Distribution coefficients for an identical $N = 4$ particle system in the 50-82 shell.

<table>
<thead>
<tr>
<th>$\varepsilon$ (MeV)</th>
<th>$4^+_1$</th>
<th>$2^+_2$</th>
<th>$0^+_2$</th>
<th>$6^+_1$</th>
<th>$8^+_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.835</td>
<td>1.840</td>
<td>1.848</td>
<td>2.556</td>
<td>3.189</td>
</tr>
<tr>
<td>0.1</td>
<td>1.830</td>
<td>1.839</td>
<td>1.840</td>
<td>2.547</td>
<td>3.181</td>
</tr>
<tr>
<td>0.5</td>
<td>1.794</td>
<td>1.835</td>
<td>1.833</td>
<td>2.474</td>
<td>3.085</td>
</tr>
<tr>
<td>1.0</td>
<td>1.754</td>
<td>1.826</td>
<td>1.854</td>
<td>2.384</td>
<td>2.948</td>
</tr>
<tr>
<td>10.0</td>
<td>1.691</td>
<td>1.777</td>
<td>1.757</td>
<td>2.234</td>
<td>2.680</td>
</tr>
</tbody>
</table>

$U(5)$ limit in IBM 2.0 2.0 2.0 3.0 4.0

vibrational limits of 2.0, 3.0, and 4.0. The energy ratios for the $2^+_2$ and $0^+_2$ states are 1.840 and 1.848, respectively, that is, the $4^+_1$, $2^+_2$ and $0^+_2$ states are almost degenerate, which is a typical feature of the $U(5)$ limit in the IBM. Table 1 also shows that the difference between the SDPSM results and the vibrational limit values in the IBM increases with $J$; that is, the larger the $J$ value, the larger the difference. For example, the difference between the two models is 0.444 for $E_{6^+_1}/E_{4^+_1}$, while it is 0.811 for $E_{8^+_1}/E_{6^+_1}$. In addition, the difference between the SDPSM results and those of the vibrational limit in the IBM increases as the intruder level moves up in energy; that is, the higher the intruder level, the lesser it contributes and the greater the deviation from the vibrational limit.

In addition to the energy ratios, some important relative $B(E2)$ ratios are listed in Table 2, from which one can see that except for $B(E2; 2^+_2 \rightarrow 2^+_1)/B(E2; 2^+_1 \rightarrow 0^+_1)$, the $B(E2)$ ratios for $4^+_1 \rightarrow 2^+_1$, $6^+_1 \rightarrow 4^+_1$ and $0^+_2 \rightarrow 2^+_1$ decrease with an increase in the single-particle energy of the intruder.
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Table 2. The relative \( B(E2) \) ratios for identical system \( N = 4 \). The position of the intruder level relative to other normal parity levels is labeled by \( \varepsilon \).

<table>
<thead>
<tr>
<th>( \varepsilon ) (MeV)</th>
<th>( \frac{B(E2; 4^+_1 \rightarrow 2^+_1)}{B(E2; 2^+_2 \rightarrow 0^+_1)} )</th>
<th>( \frac{B(E2; 6^+_1 \rightarrow 4^+_1)}{B(E2; 2^+_1 \rightarrow 0^+_1)} )</th>
<th>( \frac{B(E2; 2^+_2 \rightarrow 2^+_1)}{B(E2; 2^+_1 \rightarrow 0^+_1)} )</th>
<th>( \frac{B(E2; 0^+_2 \rightarrow 2^+_1)}{B(E2; 2^+_1 \rightarrow 0^+_1)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.316</td>
<td>1.195</td>
<td>1.405</td>
<td>1.109</td>
</tr>
<tr>
<td>0.1</td>
<td>1.302</td>
<td>1.150</td>
<td>1.425</td>
<td>1.068</td>
</tr>
<tr>
<td>0.5</td>
<td>1.246</td>
<td>1.044</td>
<td>1.489</td>
<td>0.942</td>
</tr>
<tr>
<td>1.0</td>
<td>1.196</td>
<td>0.955</td>
<td>1.537</td>
<td>0.834</td>
</tr>
<tr>
<td>10.0</td>
<td>1.178</td>
<td>0.941</td>
<td>1.586</td>
<td>0.792</td>
</tr>
</tbody>
</table>

U(5) limit in IBM | 1.5 | 1.5 | 1.5 | 1.5 |

It is known that for the vibrational limit the relative \( B(E2) \) ratios of these four transitions are all 1.5 for \( N = 4 \)\(^{11} \) but Table 2 shows that except for \( \frac{B(E2; 2^+_2 \rightarrow 2^+_1)}{B(E2; 2^+_1 \rightarrow 0^+_1)} \) with \( \varepsilon > 0.5 \) MeV, the other three transition strengths are all smaller than those of the vibrational limit. The higher the position of the intruder level, the smaller the active model space, and the greater the deviation from the vibrational limit, or vice versa, the larger the active model space, the closer the results lie to the vibrational limit.

Due to parity differences, nucleons in the intruder level cannot couple directly to those in the normal parity levels — even if they had the same total angular momentum, which they do not — to form positive-parity pairs. This means nucleons in the intruder level and those in the normal parity sector cannot bind to one another as strongly as nucleons in the same sector can. However, the above results suggest that this has no effect on the vibrational limit of the SDPSM, which appears to be sensitive only to the magnitude of the active model space and not to details regarding the mixes between different parity levels. The results also demonstrate that \( S \)-pair configurations play an important role in the vibrational limit since there are no admixtures involving different parity levels.

It is well-known that for a fermion system, Pauli-blocking effects play a very important role. It is also understood that for a fixed number of particles this effect is expected to become less important with increasing size of the model space. In fact, if the dimension of the model space tends to infinity, the role of a nucleon pair is expected to be similar to that of a boson. Therefore, with the Hamiltonian parameters fixed at values appropriate for the 50-82 shell, a toy case with \( j = 1/2, 3/2, 5/2, 7/2, 9/2, 11/2, 13/2, 15/2, 17/2 \) and 19/2 (the \( N = 9 \) shell) with \( N = 4 \) nucleon pairs was studied. Some energy and relative \( B(E2) \) ratios are given in the panel on the left of Fig. 2. The results shows that for this toy model, the vibrational spectrum can be well reproduced. From these results, we can also see that the typical relative \( B(E2) \) ratios for the vibrational limit realized; that is, 1.45, 1.51 and 1.39 for \( \frac{B(E2; 4^+_1 \rightarrow 2^+_1)}{B(E2; 2^+_2 \rightarrow 0^+_1)} \), \( B(E2; 2^+_2 \rightarrow 2^+_1)/B(E2; 2^+_1 \rightarrow 0^+_1) \) and \( B(E2; 0^+_2 \rightarrow 2^+_1)/B(E2; 2^+_1 \rightarrow 0^+_1) \), respectively.

In the panel on the right of Fig. 2, we also give the spectrum and the relative \( B(E2) \) ratios for the toy model with \( j = 19/2 \) being treated as if it were an intruder.
level. The Hamiltonian and the parameters used in this case are the same as before. The results show that the results are the same as those achieved with the \( j = 19/2 \) level treated as a normal parity level. This is further confirmation that the vibrational limit depends on the magnitude of the model space, with the parity difference among the levels playing no distinguishing role in the theory. In short, the normal and unique parity states contribute in the same way to vibrational spectra. One might be tempted to argue that this result is obvious because the fundamental operators of the theory do not link the two spaces; but this is not necessarily the case since neighboring shells can contribute to the formation of \( D^1 \) only, if they are of the same parity.

To further explore the effect of intruder level on vibrational spectrum, we also study the neutron-proton \( N_\pi = N_\nu = 2 \) in the 50-82 shell. For simplicity, single-particle energies of the intruder state and normal parity states were set to be degenerate, and \( G_\pi = G_\nu \cong G = 0.6 \) MeV was used. By fitting \( E_4^+ / E_2^+ = 2 \), \( G/\kappa = 60r_0^4 \) was adopted. The results are shown in Fig. 3. It can be seen that even though the intruder level is degenerate with the normal parity levels, the \( U_\pi(5) \otimes U_\nu(5) \) limit spectrum can still be realized in the SDPSM for the coupled system.

In summary, this analysis shows that the presence of an intruder level neither diminishes or enhances the vibrational character of a nucleus; rather, it contributes “on par” with the normal parity level to the vibrational character of the spectrum. The vibration character is sensitive to the magnitude of the active model space, not the mixture of normal and intruder level contributions. For a proton and neutron coupled system with intruder levels, the vibrational spectrum can still be realized.

Fig. 2. Energy and \( B(E2) \) ratios for the toy model. The left panel is for the case without an intruder level, while the one on the right is for the case with \( j = 19/2 \) treated as the intruder level.
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Fig. 3. Relative energies and $B(E2)$ ratios for the system with $N_p = N_n = 2$ in the 50-82 shell. The effective charges were set to be $e_x = 3e_p = 1.5e$.

if the quadrupole-quadrupole interaction is negligible. Therefore, intruder levels should be taken into consideration in any truncation scheme since they could have notable contributions to the model space.

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References