Exact Solutions for Some Nuclear Many-Body Problems

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Exact solutions for eigenvalues and eigenfunctions of some nuclear many-body systems are found by using an infinite-dimensional, Lie-algebraic approach based on the corresponding Bethe ansatz. Applications of the theory, including solutions of some nuclear pairing problems and \( U(5) \rightarrow SO(6) \) transitional Hamiltonians of the interacting boson model, are given. The relationship between this new method and other Bethe ansatz solutions in completely integrable systems is discussed. © 1999 Academic Press

1. INTRODUCTION

Exactly and quasi-exactly solvable models are of interest in both physics and mathematics. A large number of quantum integrable and exactly solvable models have been found by using the inverse scattering method [1–4]. The main idea of this method rests on the use of a special associative algebra known as a Yang-Baxter construction [5, 6]. In such cases the spectrum of the Hamiltonian and the corresponding eigenfunctions can be computed in a purely algebraic way. An intermediate class of Hamiltonians also exist which allows for an algebraic construction of part of their spectra. Spectral problems of this kind are called quasi-exactly solvable [7, 8]. Up to now, most efforts have focused on quantum spin systems with nearest neighbour interactions [9], vertex models in statistical mechanics [10], and one-dimensional Schrödinger equations [11].

There are also some many-body problems in nuclear physics that are exactly solvable, for example, exact solutions exist for the pure pairing problem (\( SU(2) \) quasi-spin) and a quadrupole-quadrupole interaction (\( SU(3) \) shell model). The three limits of the interacting boson model are other examples. To obtain exact solutions for theories that reach beyond exactly solvable models requires matrix diagonalizations. In some cases there are methods that can be used to give approximate results. For example, for the case of generalized pairing the BCS and Hartree–Fock–Bogolyubov (HFB) approximations are often used, sometimes in conjunction with correction terms evaluated within the random-phase approximation.
(RPA). But in this case there are some potentially serious pitfalls. First of all, not only is the number of nucleons in a nucleus of interest often small, the number of valence particles \( n \approx 10 \) which dominates the behavior of its low-lying states is usually too few to justify underlying assumptions of the approximation, specifically, \( \delta n/n \) is not negligible. As a result, particle-number-nonconservation effects can enter and this can lead to other serious difficulties, like spurious states, nonorthogonal solutions, etc. Furthermore, an essential feature of pairing correlations are even-odd differences which are driven primarily by Pauli blocking. It is difficult to treat these differences in approximate ways because different quasi-particle bases must be introduced for different blocked levels. Another problem with an approximate treatment of pairing in nuclei is related to the fact that both the BCS and the HFB approximations break down for a very important class of physical situations. The usual remedy in terms of particle number projection techniques complicates the algorithms considerably without yielding a better description of the higher-excited part of the spectrum. It is for these reasons that particle-number-conserving methods, even if only approximate, are important for probing the true nature of pairing effects in nuclei. The first attempt to find exact solutions of the non-degenerate nuclear pairing problem was made by Richardson who considered the equal pairing strength approximation which has an orbital independent solution \([12, 14]\).

Similarly, in the case of the interacting boson model for nuclei, it is well known that the Hamiltonian has exact solutions in three limiting cases, which are usually called the \( U(5), SU(3), \) and \( SO(6) \) limits of the theory \([15]\). These three limiting cases have, respectively, the anharmonic vibrator, an axial rotor, and the \( \gamma \)-unstable rotor as their geometrical analogs. In these cases one can obtain analytic results for energy spectra and wave functions with the help of group representation theory. The next most complicated case arises when the Hamiltonian is written in terms of invariant operators of two of the three limiting cases, that is, in cases classified by the sides of the so-called Casten triangle \([16]\). These cases normally require diagonalizing the Hamiltonian using numerical methods.

In this paper we identify boson and fermion Hamiltonians with a separable interaction matrix that can be solved exactly through the use of an infinite-dimensional algebra. The algebraic Bethe ansatz wave function plays a key role in the method \([17]\). In Section 2, as a heuristic example, we consider the diagonalization of a boson Hamiltonian with only one-body interactions using the infinite-dimensional Heisenberg algebra. This illustrates how the Bethe ansatz can be used to determine eigenvalues in the context of an infinite-dimensional, Lie-algebraic approach. The method can also be applied to many-fermion systems. Indeed, in Section 3, we discuss the affine Lie algebra \( sl(2) \) and exact solutions of some many-body Hamiltonians with special emphasis on the nuclear pairing problem. In Section 4, we will study the affine algebra \( sl(1, 1) \) and other exactly solvable Hamiltonians. Applications to orbital-dependent pairing interactions and solutions of \( U(5) \leftrightarrow SO(6) \) transitional Hamiltonian in the interacting boson model for nuclei are also presented.
2. INFINITE-DIMENSIONAL HEISENBERG ALGEBRA AND
AN EXACTLY SOLVABLE EXAMPLE

The Heisenberg algebra \( \mathcal{H}_p \) is generated by boson creation, anihilation, and particle number operators \( \{ a_j, a_j^+, n_j \} \), with \( j = 1, 2, ..., p \), which satisfy the commutation relations
\[
[a_i, a_j^+] = \delta_{ij}, \quad [n_i, a_j^+] = \delta_{ij} a_j^+, \quad [n_i, a_j] = -\delta_{ij} a_i.
\] (2.1)

Let \( \{ \epsilon_j \} \) and \( \{ c_j \} \) be two set of independent parameters, where the \( \{ \epsilon_j \} \) are real and the \( \{ c_j \} \) are complex. One can construct the \( j \)-independent operators
\[
N_n = \sum_j \epsilon_j n_j, \quad A_n^+ = \sum_j \epsilon_j c_j a_j^+, \quad A_n^- = \sum_j \epsilon_j c_j^* a_j, \quad n = 0, 1, 2, ...
\] (2.2)

It can be verified that operators (2.2) satisfy the commutation relations
\[
[N_n, A_m^+] = \pm A_m^+ A_n^-, \quad [A_n^+, A_m^-] = L_{m+n},
\] (2.3)

where
\[
L_n = \sum_j |c_j|^2 \epsilon_j^2.
\] (2.4)

The algebra generated by (2.2) is called a positive half-infinite affine Heisenberg algebra.

Using these operators, one can show that the Hamiltonian
\[
\hat{H} = \sum_j \epsilon_j n_j + \sum_j c_j c_j^* a_j
\] (2.5)
is easily diagonalizable. The paramters \( \epsilon_j + |c_j|^2 \) can be identified as single-particle energies. Using operators (2.2), one can rewrite (2.5) as
\[
\hat{H} = N_1 + A_0^+ A_0^-.
\] (2.6)

In this case, one can see that the interaction matrix in the second term of (2.5) is assumed separable, namely, \( c_{ij} = c_i c_j^* \).

In order to diagonalize (2.6), we use the following algebraic Bethe ansatz: Firstly, there exists a unique vacuum state \( |0\rangle \) obeying
\[
N_n |0\rangle = 0, \quad A_n^- |0\rangle = 0 \quad \forall n.
\] (2.7)

Then, because the total number of particles is a conserved quantity, we can write \( k \)-particle eigenvectors of (2.6) as a linear hull of vectors,
\[
|k\rangle = A^+(x_1) A^+(x_2) \cdots A^+(x_k) |0\rangle,
\] (2.8)
where \( x_i \) with \( i = 1, 2, \ldots, k \) are a set of \( c \)-numbers that are to be determined. The operator \( A^+(x_i) \) can then be expanded in terms of \( \{ A_n^+ \} \) for \( x_i \approx 0 \) as
\[
A^+(x_i) = \sum_n x_i^n b_n A_n^+ ,
\]
(2.9)
where the \( b_n \) are \( c \)-number coefficients and \( b_n A_n^+ \) are nothing but Fourier-Laurent coefficients in the expansion of \( A^+(x_i) \), namely
\[
b_n A_n^+ = \frac{1}{2\pi i} \int_0^\infty dx_i x_i^n A^+(x_i).
\]
(2.10)
Hence, eigen-energies and the set of unknown \( c \)-numbers \( b_n \) and \( \{ x_i \} \) are determined by the eigenvalue equation
\[
\hat{H} | k \rangle = E_k | k \rangle ,
\]
(2.11)
Using ansatz (2.8)–(2.9) and commutation relations (2.2), one can determine that \( b_n \) is an \( n \)-independent factor and the \( c \)-numbers \( x_i \) around \( x_i \approx 0 \) should satisfy the following set of equations, called the Bethe ansatz equations,
\[
\frac{1}{x_i} = \sum_j \frac{|c_j|^2}{1 - \epsilon_j x_i} , \quad i = 1, 2, \ldots, k ,
\]
(2.12)
while the eigen-energy is given by
\[
E_k = \sum_{i=1}^k \frac{1}{x_i} .
\]
(2.13)
The operator \( A^+(x_i) \) can now be written explicitly as
\[
A^+(x_i) = \sum_j \frac{c_j}{1 - \epsilon_j x_i} a_j^+ .
\]
(2.14)
Though relations (2.12)–(2.14) were obtained for \( x_i \approx 0 \), by analytic continuation they are valid in the entire complex plane. Therefore, the eigenvalues and eigenvectors are simultaneously determined by (2.8) and (2.12)–(2.14).

It should be noted that the \( c \)-numbers \( x_i \) in the Bethe ansatz wave function (2.8) have \( S_k \) symmetry. Any permutation among different roots \( x_i \) for \( i = 1, 2, \ldots, k \) in (2.8), (2.12), and (2.13) gives the same excitation wave functions and eigen-energies. Therefore, in general, there will be \( k! \) different solutions with the same eigenenergies, the only difference being that the \( c \)-numbers \( x_i \) are interchanged. These solutions should be regarded as repeated roots with only one of them a solution to the problem. It is useful to organize the roots \( \{ x_i \} \) such that \( |x_1| < |x_2| < \cdots < |x_k| \). Furthermore, if two roots \( x_i \) and \( x_{i+1} \) are conjugates of one another, we
always write \( x_i = \text{Re}(x_i) - i \text{Im}(x_i) \) and \( x_{i+1} = \text{Re}(x_i) + i \text{Im}(x_i) \). The solutions are then all different sets of such roots \( \{x_i\} \).

Now consider some of the properties of a new algebra generated by the operator (2.14) obtained from the infinite-dimensional algebra approach, and operators defined by

\[
A^-(x) = \sum_j \frac{c_j^+}{1 - \epsilon_j x} a_j, \quad (2.15)
\]

\[
N(x) = \sum_j \frac{1}{1 - \epsilon_j x} \tilde{a}_j. \quad (2.16)
\]

In order to keep the hermiticity of the algebra, one can assume that the spectral parameter \( x \) is real. It can be shown that the new algebra \( \{A^\pm, N(x)\} \) satisfies

\[
[N(x), A^\pm(y)] = \frac{1}{x - y} (xA^\pm(x) - yA^\pm(y)), \quad (2.17)
\]

\[
[A^-(x), A^+(y)] = \frac{1}{x - y} (xL(x) - yL(y)),
\]

where

\[
L(x) = \sum_j \frac{|c_j^2|}{1 - \epsilon_j x}. \quad (2.18)
\]

Let \( \mathcal{N}(x) = xN(x), \mathcal{A}^\pm(x) = xA^\pm(x), \) and \( \mathcal{L}(x) = xL(x), \) and make a variable substitution with \( \lambda = \frac{1}{2} \). One can prove that the algebra \( \{\mathcal{A}^\pm(\lambda), \mathcal{A}^\pm(\mu)\} \) satisfies

\[
[\mathcal{A}^-(\lambda), \mathcal{A}^\pm(\mu)] = \pm \frac{1}{\mu - \lambda} (\mathcal{A}^\pm(\lambda) - \mathcal{A}^\pm(\mu)), \quad (2.19)
\]

\[
[\mathcal{A}^-(\lambda), \mathcal{A}^+(\mu)] = \frac{1}{\mu - \lambda} (\mathcal{L}(\lambda) - \mathcal{L}(\mu)).
\]

In completely integrable systems, an algebra satisfying relations similar to (2.19), especially for the SU(2) case, is known under the name of Gaudin [18], which can be obtained from the Yangian \( \mathcal{Y} \) in some limiting cases. Therefore, we call the algebra generated by \( \{\mathcal{A}(x), \mathcal{A}^\pm(x)\} \) the Heisenberg–Gaudin algebra \( \mathcal{G}(\mathcal{H}) \). One can check that the Hamiltonian

\[
\hat{H}(\lambda) = \bar{\alpha} \mathcal{V}(\lambda) + \mathcal{A}^+(\lambda) \mathcal{A}^-(\lambda) + \mathcal{A}^-(\lambda) \mathcal{A}^+(\lambda), \quad (2.20)
\]

where \( \bar{\alpha} \) is a real parameter, satisfies

\[
[\hat{H}(\lambda), \hat{H}(\mu)] = 0. \quad (2.21)
\]
Using commutation relations (2.19), one can prove that eigenvalues of (2.20) are given by

\[ E_k(\lambda) = -\frac{\partial}{\partial \lambda} \mathcal{P}(\lambda) + \sum_i \frac{x}{\lambda - x_i} \]  

under the Bethe ansatz

\[ |k\rangle = \mathcal{P}^+(x_1) \mathcal{P}^+(x_2) \cdots \mathcal{P}^+(x_k) |\rangle, \]  

where the \( c \)-numbers \( x_i \) satisfy

\[ x + 2 \mathcal{P}(\lambda) = 2 \mathcal{P}(x_i), \quad i = 1, 2, \ldots, k. \]  

It can easily be verified that the results consist of two parts. The first part (2.22) is proportional to the Bethe wave function (2.23), while the second part will contain \( n \) terms not having the initial Bethe form, which are the so-called unwanted terms. The condition of cancellation of these terms imposes some special conditions (2.24) on the \( c \)-numbers \( x_i \), which are commonly known as Bethe ansatz equations.

From these examples, one can see that building blocks \( A(\lambda) \) of the Bethe ansatz derived from the infinite-dimensional algebraic approach are equivalent, after a variable substitution, to generators of a Gaudin algebra. The infinite-dimensional algebraic elements are simply the Fourier–Laurent expansion coefficients of \( A(x) \) in this case. It should be clear that the explicit forms of the building blocks \( A(x) \) for the Bethe ansatz wave function can be obtained by using the infinite-dimensional algebra method. This approach will be effective and convenient if the Hamiltonians can be expressed in terms of the generators of the infinite-dimensional algebra.

### 3. \( su(2) \) AND SOME EXACTLY SOLVABLE NUCLEAR PAIRING HAMILTONIANS

The infinite-dimensional algebraic method introduced in Section 2 can be extended to other affine Lie algebraic cases. In this section, we consider some nuclear pairing Hamiltonians. The affine Lie algebra \( su(2) \) will prove to be useful.

The general pairing Hamiltonian for spherical nuclei can be written as

\[ \hat{H} = \sum_{jm} e_j a^\dagger_{jm} a_{jm} - \sum_{j'} c_{j' j} S^+(j) S^-(j'), \]  

where the \( e_j \) are single-particle energies and \( S^+(j) \) and \( S^0(j) \) are the pairing operators for a single-\( j \) shell defined by
\[ S^+(j) = \sum_{m > 0} (-)^{j+m} a_{jm}^\dagger a_{jm}^\dagger, \]

\[ S^-(j) = \sum_{m > 0} (-)^{j-m} a_{jm} a_{jm}, \]

\[ S_0^0(j) = \sum_{m > 0} (a_{jm}^\dagger a_{jm} + a_{jm}^\dagger a_{jm} - 1) = \frac{1}{2}(\bar{N}_j - \Omega_j). \] (3.2)

In (3.2), \( \Omega_j \equiv j + 1/2 \) is the maximum number of pairs in the \( j \)th shell, \( \bar{N}_j \) is the particle number operator for the \( j \)th shell, and \( c_{jj}^\dagger \) is the strength of the pairing interaction between the \( j \) and \( j' \) shells.

In general, for \( k \) pairs, Hamiltonian (3.1) can be diagonalized in bases states that are products of the single-\( j \) shell pairing wave functions,

\[ |k\rangle = \sum_{k_1, k_2, \ldots, k_p} B_{k_1, k_2, \ldots, k_p} \ S_0^{k_1}(j_1) \ S_0^{k_2}(j_2) \cdots \ S_0^{k_p}(j_p) \ |0\rangle, \] (3.3)

where the summation is restricted by

\[ \sum_{i=1}^p k_i = k. \] (3.4)

\( p \) is the total number of orbitals, the \( B_{k_1, k_2, \ldots, k_p} \) are expansion coefficients that need to be determined, and \( |0\rangle \) is the pairing vacuum state which satisfies the condition

\[ S^-(j) \ |0\rangle = 0 \quad \text{for all} \ j. \] (3.5)

Though simple to formulate, this problem is algebraically intractable because there are no analytic expressions or recursion relations for determining the \( B_{k_1, k_2, \ldots, k_p} \) coefficients. Furthermore, the number of single-\( j \) shell basis vectors included in (3.3) has to be fixed on a case-by-case basis because of the Pauli Principle.

As an approximation to the general theory, we assume a separable strength pairing (SSP) interaction, \( c_{jj}^\dagger = c_j^\dagger c_j^\dagger \). Though strong, this assumption is physically motivated because it links the pair-pair interaction strength to the individual pair formation probability. In this case, \( |c_j|^2 \sum_{j'} |c_{jj'}|^2 \) gives the percentage of single-\( j \) shell pairing in the Hamiltonian. Furthermore, it is expected to be better than the equal strength pairing (ESP) approximation which sets \( c_{jj}^\dagger = |G| \) for all orbitals. The ESP approximation has been used in many applications because it is much simpler than the general case.

To diagonalize the SSP Hamiltonian, we can introduce the following two-parameter algebra generated by \{ \( S_0^\mu; \mu = 0, +, -; m = 0, 1, 2, \ldots \} \) with
\begin{align*}
S_{mn}^+ &= \sum_j c_j^m |e_j|^{2n} c_j S^+(j), \\
S_{mn}^- &= \sum_j c_j^m |e_j|^{2n} c_j^* S^+(j), \\
S_{mn}^0 &= \sum_j c_j^m |e_j|^{2n} S^0(j).
\end{align*}

(3.6)

It is easy to show that these generators satisfy the commutation relations

\begin{align*}
[S_{mn}^+, S_{m'n'}^-] &= 2S_{m+m'+n+n'+1}^0, \\
[S_{mn}^0, S_{m'n'}^-] &= \pm S_{m+m'+n+n'}^\pm.
\end{align*}

(3.7)

Therefore, the \( S_{mn}^\pm \) form an infinite-dimensional algebra, which is the two-parameter affine Lie algebra \( su(2) \) without central extension. The Hamiltonian (3.1) with \( c_{jj'} = c_j c_{j'}^* \) can be written in terms of the \( S_{mn}^\pm \) operators as

\[ H = \sum_j c_j \Omega_j - 2S_{10}^0 - S_{00}^+ S_{00}^- \]

(3.8)

In the following, we assume that the parameters \( c_j \) and \( c_j \) are all different for different \( j \) values, that is, we only consider the nondegenerate case. The situation is different for the degenerate and other simpler cases and these will be discussed later. For the nondegenerate case, the unique lowest-weight state is simply the product of the single-\( j \) shell pairing vacua with arbitrary seniority quantum numbers. Therefore, it suffices to consider the total seniority zero case. The lowest-weight state satisfies

\[ S_{mn}^- |0\rangle = 0 \quad \forall m, n. \]  

(3.9)

From the example shown in the previous section, it seems that a two-parameter Bethe ansatz wave function might be needed in this case. However, a careful analysis shows that the following one-parameter Bethe-ansatz suffices to diagonalized Hamiltonian (3.8),

\[ |k\rangle = \mathcal{N} S^+(x_1) S^+(x_2) \cdots S^+(x_k) |0\rangle, \]

(3.10)

where \( \mathcal{N} \) is a normalization constant, \( S^+(x_i) \) is a functional operator of \( S^+_j \) to be determined, and \( x_i \) with \( i = 1, 2, ..., p \) are unknown \( c \)-numbers.

To determine the \( c \)-number variables \( \{ x_i, i = 1, 2, ..., k \} \), we first expand (3.10) in terms of the \( x_i \) around \( x_i \approx 0 \),

\[ |k\rangle = \sum_n x_1^n x_2^n \cdots x_k^n a_{n_1} a_{n_2} \cdots a_{n_k} S^+_{n_10} S^+_{n_20} \cdots S^+_{n_k0} |0\rangle, \]

(3.11)
where $a_n S^+_{n0}$ are Fourier–Laurent coefficients in the expansion of $S^+(x_i)$, namely,

$$a_n S^+_{n0} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dx_j x^n_j S^+(x_j).$$  \hspace{1cm} (3.12)

However, when one applies $\hat{H}$ on (3.11), the result will not only consist of vectors spanned by a linear combination of products of $S^+_{n0}$, but also consist of those of $S^+_{n1}$. It can also be proved that the results will no longer be algebraically closed, if the whole family of the generators $S^+_{mn}$ given by (3.6) is used instead of only $\{ S^+_{n0} \}$ assumed in (3.11). Though in both cases the basis vectors spanned by $S^+_{n0}$ and $S^+_{n1}$ are within the same Hilbert sub-space, of which both can be expanded in terms of single-$j$ pairing operators $S^+(j)$, the parameterization of $S^+_{n0}$ and $S^+_{n1}$ is different, namely

$$S^+_{n0} = \sum_j \varepsilon_j^c c_j S^+(j), \hspace{1cm} (3.13a)$$

while

$$S^+_{n1} = \sum_j \varepsilon_j^a |c_j|^2 c_j S^+(j). \hspace{1cm} (3.13b)$$

However, this difficulty can be overcome if the parameters $|c_j|^2$ can be expressed in terms of a simple analytical function of $\varepsilon_j$. Therefore, there is a freedom to choose a simpler relation between $|c_j|^2$ and $\varepsilon_j$. We found that the following auxiliary relations are not only possible, but also convenient,

$$\sum_{j=1}^p \frac{b_j}{1 - \varepsilon_j z_i} = |c_j|^2, \hspace{1cm} j = 1, 2, ..., p. \hspace{1cm} (3.14)$$

where $\{b_i, z_i; i = 1, 2, ..., p\}$ are $c$-numbers that need to be determined. Because we have assumed the $\varepsilon_j$ and $c_j$ are all different for different $j$ values, (3.14) is always valid. Under the conditions given by (3.14), it can be shown that the result obtained by acting $\hat{H}$ on (3.11) can be expressed uniformly by a linear combination of products of $S^+_{n0}$ only. Using (3.11), (3.14), and commutation relations (3.7), one can prove that $a_n$ is also an $N$-independent factor, and $x_i$, $b_j$, and $z_j$ must satisfy the relations

$$E_k = \sum_{i=1}^k \frac{2}{x_i}, \hspace{1cm} (3.15a)$$

$$\frac{2}{x_i} = \sum_j \Omega_j^c |c_j|^2 + 2 \sum_{j \neq i} \left( \sum_m \frac{b_m}{z_j - z_m} + 1 \right) x_j / (x_j - x_i) \hspace{1cm} (3.15b)$$
for $i = 1, 2, ..., k$, where $E_k$ is the $k$-pair excitation energy, under the conditions

$$\sum_{i=1}^{k} b_i \sum_{r > q} \frac{z_i}{(x_r - z_i)(x_q - z_i)} = \sum_{r > q} \frac{1}{(1 - \epsilon_r x_r)(1 - \epsilon_q x_q)}$$

(3.16)

for $j = 1, 2, ..., p$. Though these relations were derived for $x_i \approx 0$, they are valid in the entire complex plane. Therefore, the coefficients $x_i$ ($i = 1, 2, ..., k$), $z_j$, $b_j$ ($j = 1, 2, ..., p$), and eigenvalues of the pairing energy are simultaneously determined by the system of equations (3.14), (3.15), and (3.16).

Let

$$U_{ji} = \frac{b_i}{1 - \epsilon_j z_i}, \quad W_j = \sum_{r > q} \frac{z_i}{(x_r - z_i)(x_q - z_i)}, \quad V_j = \sum_{r > q} \frac{1}{(1 - \epsilon_r x_r)(1 - \epsilon_q x_q)}$$

(3.17)

Then the auxiliary conditions (3.14) are just unity conditions for the transformation matrix $U$,

$$\sum_{i} \frac{U_{ji}}{|c_j|^2} = 1 \quad \text{for} \quad j = 1, 2, ..., p,$$

(3.18)

and (3.16) is a “linear transformation” from vector $W$ to $V$,

$$\sum_{i} U_{ji} W_i = V_j \quad \text{for} \quad j = 1, 2, ..., p.$$  

(3.19)

It can easily be seen that Eqs. (3.14)-(3.16) are necessary and sufficient conditions for solving the eigenvalue problem. However, it should be stated that the choice of conditions (3.14) and (3.16) is not unique. Different choices will lead to different versions of the Bethe equations (3.14)-(3.16). Of course, the final results for the eigenvalues must be the same and the corresponding eigenvectors should be equivalent up to a normalization factor. Finally, the operator $S^+(x_i)$ can now be written explicitly as

$$S^+(x_i) = \sum_{j} \frac{c_j}{1 - \epsilon_j x_j} S_j^+.$$  

(3.20)

So far we have assumed that the $c_j$ and $c_j'$ are all different for different $j$ values. When the $c_j$ are all the same or the $c_j$ are all the same the situations are greatly simplified. In the following, we will discuss some limiting cases of (3.8).

1) Equal Strength Pairing (ESP)

In this case the parameters $c_j'$ in (3.1) can be written as

$$c_j' = |G| e^{i \phi_j - i \phi_j'} \quad \text{for all} \ j, j'.$$

(3.21)
where \( \delta_j \) is an orbital dependent phase. One can introduce the following generators of an affine \( SU(2) \) algebra,

\[
S^{\mu}_m = \sum_j \epsilon_j^m e^{i\delta_j S_0(j)},
\]

(3.22)

where \( \mu = 0, +, - \), \( \epsilon_j = \epsilon_j^0 |G| \). It is easy to show that these generators satisfy the commutation relations

\[
[S^+_m, S^-_n] = 2S^0_{m+n},
\]

(3.23)

\[
[S^0_m, S^\pm_n] = \pm S^\pm_{m+n}.
\]

The ESP Hamiltonian can be written in terms of the \( S^\mu_m \) operators as

\[
\hat{h} = \sum_j \epsilon_j \Omega_j + 2S^0_1 - S^+_0 S^-_0,
\]

(3.24)

where

\[
\hat{h} = \hat{H} |G|.
\]

(3.25)

The unique lowest-weight state of this algebra is simply the product of the single-\( j \) shell pairing vacua with arbitrary seniority quantum numbers. Therefore, it again suffices to consider the total seniority zero case. The lowest-weight state satisfies

\[
S^-_m |0\rangle = 0; \quad m = 0, \pm 1, \pm 2, \ldots
\]

(3.26)

and

\[
S^0_m |0\rangle = -\frac{1}{2} \sum_j \epsilon_j^m \Omega_j |0\rangle.
\]

(3.27)

Furthermore, like the SSP case, it can be proved that the eigenvectors of the ESP Hamiltonian for any \( k \) can be written as

\[
|k\rangle = \mathcal{N} S^+_{a_1} S^+_{a_2} \cdots S^+_{a_k} |0\rangle,
\]

(3.28)

where

\[
S^+_{a_i} = \sum_j \frac{e^{i\delta_j}}{1 - \epsilon_j a_i} S^+_j
\]

(3.29)
and the $c$-numbers $x_i$, $i = 1, 2, ..., k$, must satisfy the relations

$$
\sum_{i=1}^{k} \frac{2}{x_i} = h_i \tag{3.30}
$$

$$
\frac{1}{x_i} = -\frac{1}{2} \sum_{j} \frac{\Omega_j}{1 - \epsilon_j x_i} - \sum_{j \neq i} \frac{x_j}{x_i - x_j}, \quad i = 1, 2, ..., k. \tag{3.31}
$$

When the $\epsilon_j$ are all equal to 0, one can obtain eigenvalues of $\hat{h}$ by simply summing over $i$ in Eq. (3.31), namely,

$$
h = -k(\Omega - k + 1), \tag{3.32}
$$

where $\Omega = \sum \Omega_j$, which is nothing but the energy eigenvalue formula for the quasi-spin approximation. If the $\epsilon_j$ are all different, (3.30) and (3.31) always give exact solutions to the problem. On the other hand, if some of the orbitals are degenerate one should reorganize the $S$ operators accordingly. For example, if the 1/2, 3/2, and 5/2 orbitals of a four orbit $j = 1/2, 3/2, 5/2, 7/2$ system are degenerate, then one should identify new $S$ operators as

$$
S^\mu_1 = S^\mu_{1/2} + S^\mu_{3/2} + S^\mu_{5/2}, \quad S^\nu_2 = S^\nu_{7/2}. \tag{3.33}
$$

where $\mu = 0, \pm$. If this is done, (3.30) and (3.31) give exact solutions for the degenerate orbit case. For realistic applications, the single-particle energy of different orbitals is rarely (if ever) exactly degenerate and therefore, (3.30) and (3.31) always give correct solutions to the ESP problem, including values of the excited eigen-energies as well as the number of excited states for given pairing number $k$.

It should be noted that the exact solutions for the ESP limit case with $c_{j\nu} = |G|$ was studied by Richardson [12, 13] in terms of boson mappings which included effects due to the Pauli Principle. That result can be obtained more simply and directly using the infinite-dimensional algebraic approach.

(2) SSP with Degenerate Single-Particle Energies

Following the quasi-spin approximation [19], consider a simpler Hamiltonian

$$
\hat{H} = -S^+_0 S^-_0, \tag{3.34}
$$

where

$$
S^+_0 = \sum j c_j^+ S^+(j), \quad S^-_0 = \sum j c_j S^-(j). \tag{3.35}
$$
To diagonalize Hamiltonian (3.34), consider an algebra generated by

\[ S_m^0 = \sum_j |c_j|^{2m} S_0(j), \]
\[ S_m^+ = \sum_j |c_j|^{2m} c_j S^+(j), \]
\[ S_m^- = \sum_j |c_j|^{2m} c_j^* S^-(j). \]  

(3.36)

It is easy to show that these generators satisfy the commutation relations

\[ [S_m^+, S_n^-] = 2S_m^0 m + n + 1, \]
\[ [S_m^0, S_n^±] = ± S_m^± n. \]  

(3.37)

The unique lowest-weight state of this algebra is simply the product of the single-j shell pairing vacua with arbitrary seniority quantum numbers. For the seniority zero case, the lowest-weight state satisfies

\[ S_m^- |0\rangle = 0; \quad m = 0, \pm 1, \pm 2, ..., \]  

(3.38)

and

\[ S_m^0 |0\rangle = -\frac{i}{2} \sum_j |c_j|^{2m} \Omega_j |0\rangle = A_m |0\rangle. \]  

(3.39)

As for the ESP case, it can be proven for this case that the eigenvectors of \( \hat{H} \) for any \( k \) and non-zero energy eigenvalue can be written as

\[ |k\rangle = S_0^+ S_1^+ S_2^+ \cdots S_{k-1}^+ |0\rangle, \]  

(3.40)

where

\[ S_n^+ = \sum_j \frac{c_j}{1 - |c_j|^2 x_n} S_j^+. \]  

(3.41)

The \( x_i \) with \( i = 1, 2, ..., k - 1 \), satisfy the relations

\[ -\frac{1}{2} \sum_{j=1}^k \Omega_j |c_j|^2 x - \frac{1}{2} |c_j|^2 = \frac{1}{y_i} + \sum_{k \neq i} \frac{1}{y_i - y_k}, \quad i = 1, 2, ..., k - 1, \]  

(3.42)

with

\[ \sum_{i=1}^{k-1} \frac{1}{y_i} = 1. \]  

(3.43)
where
\[ y_i = \frac{x_i}{x}, \quad x = -\frac{2}{h + 2A_1}, \quad h \equiv E( -|G|). \] (3.44)

A special situation arises when there are many states with eigenvalue \( h = 0 \) which cannot be determined by (3.42) and (3.43). It can be proven that the eigenvectors for the \( h = 0 \) case can be expressed as
\[ |k, 0\rangle = S_{x_1}^+ S_{x_2}^+ \cdots S_{x_k}^+ |0\rangle. \] (3.45)

Using the same technique as was used for the \( h \neq 0 \) case, one can prove that in this case the \( x_i \) with \( i = 1, 2, \ldots, k \) are determined by the following set of equations
\begin{align*}
\sum_j \Omega_j \frac{|c_j|^2}{1 - x_1 |c_j|^2} &= 0 \quad \text{for } k = 1, \quad (3.46) \\
\frac{1}{2} \sum_j \Omega_j \frac{|c_j|^2}{x_i |c_j|^2 - 1} &= \sum_{k \neq i} \frac{1}{x_i - x_k}, \quad i = 1, 2, \ldots, k, \quad \text{for } k \geq 2. \quad (3.47)
\end{align*}

When the \( c_j, j = 1, 2, \ldots, p \), are all the same, the only solution of (3.46) and (3.47) is \( x_i \rightarrow \infty \) for all \( i \). In this case, the wave function corresponding to eigenvalue \( h = 0 \) becomes a null vector and there is only one solution from (3.42) and (3.43), which is the same as for the quasi-spin approximation (QSA). These limiting cases of the SSP approximation can be summarized as follows:

<table>
<thead>
<tr>
<th>( h = 0 )</th>
<th>ESP</th>
<th>QSA</th>
</tr>
</thead>
<tbody>
<tr>
<td>SSP</td>
<td></td>
<td>QSA with ( \varepsilon_f = \epsilon ) for all ( f )</td>
</tr>
</tbody>
</table>

As in the previous section, the \( c \)-numbers \( x_i \) in the Bethe ansatz wave function (3.10), (3.28), (3.45), and (3.40) have \( S_k \) and \( S_{k-1} \) symmetries, respectively. For example, any permutation among different roots \( x_i \) for \( i = 1, 2, \ldots, k \) in (3.10), (3.28), and (3.45) gives the same \( k \)-pair excitation wave functions. Therefore, generally, there will be \( k! \) different solutions, of which the eigen-energies are the same except that the \( c \)-numbers \( x_i \) in these cases are interchanged. As before, only one of them is the solution to the problem. The roots \( \{ x_i \} \) can be arranged as \( |x_1| < |x_2| < \cdots < |x_k| \). The solutions are then all different sets of such roots \( \{ x_i \} \).

It should be noted that there are similar dispersion relations, for example, to (3.31), derived in some nuclear many-body problems. In the study of particle-hole excitations at closed shells, one can first construct the zero-order wave function based on the Hartree–Fock method. Then, the general Hamiltonian with residual interaction should be diagonalized in a \( n \)-particle-\( n \)-hole configuration, which leads to an infinite dimensional matrix eigenvalue problem and can only be solved with truncations. Two major approximations for treating the doubly closed shell nuclei are the Tamm–Dancoff approximation (TDA) and the random-phase approximation.
In the TDA approximation, which is only restricted within \(1p-1h\) configuration, one can derive similar dispersion relations for the eigenvalue problem when the separable potential is assumed \([20]\). A similar situation also occurs in the RPA, in which the Hamiltonian is diagonalized within a \(2p-2h\) configuration with a separable interaction matrix \([21, 22]\).

It is obvious that there are similarities and differences between the problems discussed in this paper and those in the TDA and RPA. In the TDA and RPA cases, the eigenvalue problems are solved within a small \(1p-1h\) and \(2p-2h\) configurations. Therefore, the problems are relatively easy; and the results can only be applied to low-lying excited states with approximations. Direct extension of those methods to higher excitation cases seems very difficult. While the infinite dimensional algebraic method outlined in this paper can be used to obtain exact solutions for the pairing problem in the whole \(n\)-pair configuration for any number of pairs. Further extension to the broken pair approximation for \(J \neq 0\) states is straightforward. The common features of the method discussed in this paper and that in the TDA and RPA are (i) separable potential is assumed in both cases; (ii) the solutions can all be expressed in terms of dispersion (secular) relations. It seems that, similar to the TDA and RPA, the general solutions within \(np-nh\) configurations with separable interaction can also be found by using the infinite dimensional algebraic method outlined in this paper, which will be our future study.

To demonstrate the effectiveness of the theory as well as the accuracy of the SSP assumption we considered the case of neutrons in the nuclear \(ds\)-shell with the \(0d_{5/2}, 0d_{3/2},\) and \(1s_{1/2}\) orbitals all active. The neutron single-particle energies \(\epsilon_j\) were taken from the energy spectra of \(^{17}\text{O}\) \((\epsilon_{1/2} = -3.273 \text{ MeV}, \epsilon_{3/2} = 0.941 \text{ MeV}, \) and \(\epsilon_{5/2} = -4.143 \text{ MeV})\). These values are all relative to the binding energy of \(^{16}\text{O}\), which was taken to be zero. The two-body terms of the general pairing Hamiltonian (3.1) were obtained from the \(J=0\) two-body matrix elements of the universal \(ds\)-shell Hamiltonian of Wildenthal \([23]\) with \(A=18\). The \(c_{jj'}\) parameters in MeV are \(c_{1/2(1/2)} = 2.125, c_{3/2(3/2)} = 1.092, c_{5/2(5/2)} = 0.940, c_{1/2(3/2)} = 0.766, c_{1/2(5/2)} = 0.765, c_{3/2(3/2)} = 1.301\). While in (3.8) the single-particle energies are taken to be the same as those of the shell model (SM), the parameters \(c_j\) were adjusted to give a best fit to the SM calculation, which yielded \(c_{1/2} = 0.70, c_{3/2} = 1.15, c_{5/2} = 1.048 \text{ MeV}\). We also calculated energy levels in the ESP approximation, with the single-particle energies also taken to be the same as those of the SM and the parameter \(|G|\) adjusted to give a best fit. The quality of the fits was determined by the quantity

\[
\sigma = \left(\frac{1}{N} \sum_{i, \text{total}} |E_{\text{SM}}(i) - E_{\text{cal}}|^2\right)^{1/2},
\]

where \(N\) is the total number of the energy levels included in the fit. All of the excited levels of the SSP, ESP, and SM calculations are shown in Fig. 1. The calculation yielded \(\sigma = 0.631 \text{ MeV}\) for SSP and \(\sigma = 0.824 \text{ MeV}\) for ESP. These results show that the SSP assumption is a rather good approximation, better than...
FIG. 1. Comparison of realistic shell-model (SM) results and spectra of the SSP and ESP Hamiltonians as a function of the number of neutron pairs in the d-shell. The results are obtained within \((J=0)\) pairing configuration only.
the ESP approximation. The method can be used to obtain exact values for the
eigen-energies as well as the exact number of the excited levels, both being consis-
tent with restrictions on the number of excited levels allowed by the Pauli Principle.

4. \(su(1,1)\) AND SOME EXACTLY SOLVABLE HAMILTONIANS

In this section we consider the \(su(1,1)\) affine algebra and related applications.
First we introduce a set of generators \(\{S^i_j\}\) of \(su(1,1)\) with \(\mu = 0, +, -\), and
\(j = 1, 2, ..., p\) which satisfy the commutation relations

\[
[S^0_j, S^\pm_j] = \pm \delta_j S^\pm_j, \quad [S^+_j, S^-_j] = -2\delta_j S^0_j.
\]  (4.1)

The Casimir operator of \(su(1,1)\) can be written as

\[
\hat{C}_2 = S^0_j (S^0_j - 1) - S^+_j S^-_j.
\]  (4.2)

Let \(|{\lambda_j}\rangle\) denote basis vectors of an irreducible representation (irrep) of \(su(1,1)\),
where \(\lambda_j\) can be any positive real number and \(\lambda_j = \lambda_j, \lambda_j + 1, \ldots\). Then

\[
\hat{C}_2(su(1,1)) |{\lambda_j}\rangle = \lambda_j (\lambda_j - 1) |{\lambda_j}\rangle, \quad S^0_j |{\lambda_j}\rangle = \lambda_j |{\lambda_j}\rangle. \quad (4.3)
\]

As for the \(SU(2)\) case, one can introduce two sets of independent parameters,
\(\{\epsilon_j\}\) and \(\{c_j\}\), where the \(\{\epsilon_j\}\) are real and the \(\{c_j\}\) may be complex. In the following,
we assume that these parameters are all non-zero. One can then construct the
\(j\)-independent operators

\[
S^+_{mn} = \sum_j \epsilon_j^m |c_j|^{2n} S^+_j, \quad S^-_{mn} = \sum_j \epsilon_j^m |c_j|^{2n} S^-_j, \quad S^0_{mn} = \sum_j \epsilon_j^m |c_j|^{2n} S^0_j.
\]  (4.4)

It can be verified that the operators (4.4) satisfy the commutation relations

\[
[S^+_{mn}, S^-_{m'n'}] = -2 S^0_{m + m'n + n' + 1}, \quad [S^0_{mn}, S^\pm_{m'n'}] = \pm S^\pm_{m + m'n + n'}. \quad (4.5)
\]

The algebra generated by the \(\{S^\mu_{mn}\}\), with \(\mu = 0, +, -\) and \(m, n = 0, 1, 2, \ldots\), is
called a two-parameter positive half-infinite affine Lie algebra \(su(1,1)\). Using these
generators, one can construct a non-trivial Hamiltonian,

\[
\hat{H} = S^0_{10} + S^+_{00} S^-_{00}.
\]  (4.6)

To diagonalize (4.6), we need a two-parameter algebraic Bethe ansatz. The
lowest weight state, \(|{\lambda}\rangle\), of \(su(1,1)\) satisfies

\[
S^-_{mn} |{\lambda}\rangle = 0 \quad \forall m, n, \quad (4.7)
\]
where

\[ |w\rangle = |\kappa_1, \mu_1 = \kappa_2, \mu_2 = \kappa_3, \ldots, \kappa_p, \mu_p = 0\rangle. \quad (4.8) \]

One also has that

\[ S^0_{mn} |w\rangle = P^0_{mn} |w\rangle \quad (4.9) \]

with

\[ P^0_{mn} = \sum_j \varepsilon_j^n |c_j|^{2n} \kappa_j. \quad (4.10) \]

Obviously, the lowest weight state is the ground state of Hamiltonian (4.6), which is called the level 0 state. Excited states are classified according to the number of \( S^+ \) type raising operators that are applied on the ground state. If a state is constructed by applying \( S^+ \) type operators on the ground state \( k \) times, this state is called the level \( k \) state. As for the SU(2) case, the level \( k \) eigenvectors of Hamiltonian (4.6) are assumed to be of the form

\[ |k\rangle = \mathcal{N} S^+(x_1) S^+(x_2) \cdots S^+(x_k) |0\rangle, \quad (4.11) \]

where \( \mathcal{N} \) is the normalization constant. It can be proven that

\[ S^+(x_i) = \sum_j \frac{c_j}{1 - \varepsilon_j x_i} S_j^+. \quad (4.12) \]

Using a similar procedure to that employed in the SU(2) case, one can easily obtain that the \( x_i \) and eigenvalues \( E^{(k)} \) of the Hamiltonian (4.6) should satisfy

\[ h^{(k)} = \sum_{i=1}^k \frac{1}{x_i}, \quad (4.13a) \]

\[ \frac{1}{x_i} = 2 \sum_{j=1}^k \frac{|c_j|^2 \kappa_j}{1 - \varepsilon_j x_i} + 2 \sum_{j \neq i} \left( \sum_m \frac{b_m}{x_j - z_m} + 1 \right) x_j/(x_j - x_i) \quad (4.13b) \]

for \( i = 1, 2, \ldots, k \) where

\[ h^{(k)} = E^{(k)} - A^0_{10} \quad (4.14) \]

under conditions (3.14) and (3.16). Equation (4.13) gives solutions to the eigenvalue problem (4.6), which can be applied to some interacting boson systems.

In the interacting boson model (IBM), for example, the generators of the SU(1, 1) algebra can be expressed in terms of \( s \) - and \( d \)-boson pairing operators with

\[ S^+(s) = \frac{1}{2} s^+ s^2, \quad S^+(d) = \frac{1}{2} d^+ d^+. \quad (4.15) \]
Indeed, it can be verified that Hamiltonian (4.6) with \( p = 2 \) can be used to describe nuclei in the \( U(5) \leftrightarrow SO(6) \) transitional region, which in geometrical terms corresponds to a vibration to the \( \gamma \)-unstable transitional region. The corresponding geometrical problem and its solution were proposed in [24] and discussed further in [25]. Reference [26] reports on a simple IBM application, namely \[ |c_j|^2 = x \epsilon_j, \]
where \( x \) is a real parameter. In this case the \( |c_j|^2 \) serve as transition parameters. Hamiltonian (4.6) with the \( p = 4 \) case [26] may also prove to be useful in describing \( U(5) \leftrightarrow O(6) \) transitional nuclei in an IBM-2 framework.

5. DISCUSSION

In this paper exact solutions for eigenvalue problems of some nuclear many-body systems, found by using an infinite-dimensional Lie algebra approach based on the corresponding Bethe ansatz, are reported. The advantage of this new method is that the building blocks of the Bethe ansatz wave functions can be derived when the interaction matrix is assumed separable. Hence, the method outlined in this paper provides a general procedure for deriving exact solutions of some nuclear many-body problems with separable interaction matrix. The building blocks of the Bethe ansatz wave functions satisfy nonlinear algebras of the Gaudin type, that is, an infinite-dimensional algebra, which, in general, is an affine Lie algebra without central extension. Applications to some boson and fermion many-body problems, including nuclear pairing and solutions of \( U(5) \leftrightarrow SO(6) \) transitional Hamiltonians in the interacting boson model for nuclei, are discussed. The method can also be applied to other physical problems with \( SU(2) \) and \( SU(1, 1) \) symmetry, such as \( N \)-coupled interacting rotors [27], spin-glass systems [28, 29], and many other spin-interaction models [30]. It should be clear that the results of this paper are also useful for other fermion and boson interacting systems where realizations of \( SU(2) \) and \( SU(1, 1) \) are encountered.

A longstanding challenge has been to find algebras and their finite-dimensional representations which can be used to represent physical Hamiltonians in the form

\[ \hat{H} = \sum_i \epsilon_i B_i + \sum_{i<j} c_{ij} B_i B_j, \quad (5.1) \]

where \( B_i \), with \( i = 1, 2, ..., d \), are generators of a finite-dimensional Lie algebra or quantum algebra, and to derive the corresponding exact solutions. The results reported demonstrate that a special case of (5.1) with \( c_{ij} = c_i c_j \) can be solved using the infinite-dimensional algebra approach. The q-deformed extension of (5.1) may also be solvable by using this method. It is known that Hamiltonians expressed by q-deformed algebra operators can be regarded as certain higher order corrections to the problem [31, 32]. On the other hand, quantum algebras appear naturally in solving various integrable models with classical Lie algebra symmetry. It will be interesting to see what the algebraic structure is when fundamental symmetries are
replaced by quantum algebras. Also, it has been found [33–35] that solutions of nonlinear Schrödinger equations, Sine-Gordon models, and nonlinear σ models can be derived from a discrete spin interaction theory by using the Bethe ansatz method and taking the limit as \( p \to \infty \). It is therefore expected that solutions of other interesting and useful nonlinear equations may be found by using the infinite-dimensional algebraic approach.

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