Exact boson mapping of the nuclear pairing Hamiltonian

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Abstract
An exact boson mapping of the deformed mean-field plus equal strength pairing Hamiltonian is considered. In the mapping, fermion pair operators are mapped exactly to the corresponding bosons. The mapping occurs at the level of the Richardson–Gaudin equations. The image of the mapping results is a Bose–Hubbard model with level-dependent hopping. Although the resultant Bose–Hubbard Hamiltonian is non-Hermitian, a part of spectrum of the Bose–Hubbard Hamiltonian with $U/t = 1$ determined by the corresponding Richardson–Gaudin (Bethe ansatz) equations corresponds exactly to the whole spectrum of the pairing Hamiltonian.

Pairing is an important residue interaction in many areas of physics, especially in the study of superconductors [1, 2], nuclear systems [3], metallic clusters [4, 5] and liquids [6]. Limitations of the Bardeen–Cooper–Schrieffer (BCS) and Hartree–Fock–Bogolyubov (HFB) methods [1, 7] for finding approximate solutions of finite nuclear systems and nanoscale metallic grains where pairing is known to play a significant role are well understood [4, 8]. Fortunately, a spherical or deformed mean-field plus equal strength pairing model has been shown to be exactly solvable following the early work of Richardson [9, 10] and studies based on the Gaudin algebraic Bethe ansatz method [11], which has received considerable attention in recent years [12–15].

Juxtaposed on this is a long and rich history that involves the search for appropriate boson mapping methods or boson expansions in nuclear many-body systems [16], using concepts that can also be applied to other many-fermion systems. The linkage comes about because pairs of fermions in these systems often exhibit boson-like behavior. In these analyses, certain degrees of freedom of fermion pairs are replaced by boson degrees of freedom. This approach is helpful in describing low-lying collective motion in terms of boson degrees of freedom as the latter allows one to circumvent difficult fermionic formulations, because boson operators have their counterparts in classical canonical variables and thus provide a direct link, for...
example, between microscopic nuclear models and phenomenological collective models. As with BCS-type theories, a lot of attention has been paid to these boson-type expansions in recent years [16, 17], especially following successes of the interacting boson model for nuclei [18].

The purpose of this paper is to report on an exact boson mapping of a deformed mean-field plus equal strength pairing Hamiltonian. When \( m \) unpaired (single) valence nucleons occupy a set of \( j_1, j_2, \ldots, j_m \) levels in a deformed mean-field, the mean-field plus equal strength pairing Hamiltonian for a deformed nuclear system is given by

\[
\hat{H}_{BCS}/G = \sum_{\tau=1}^{m} (\epsilon_{j_\tau}/G) \left( c_{j_\tau}^{\dagger} c_{j_\tau} + c_{j_\tau} c_{j_\tau}^{\dagger} \right) + \sum_{j} \eta_j \hat{k}_j - \sum_{i,j} S^+_i S^-_j,
\]

where \( c_{j_\tau}^{\dagger} (c_{j_\tau}) \) are fermion creation (annihilation) operators, \( S^+_j = c_{j_\uparrow}^{\dagger} c_{j_\downarrow} \) (\( S^-_j = c_{j_\downarrow} c_{j_\uparrow} \)) are pair creation (annihilation) operators, \( \hat{k}_j = (c_{j_\uparrow} c_{j_\downarrow} + c_{j_\downarrow} c_{j_\uparrow})/2, \) \( \epsilon_j \) are single-particle energies taken from any deformed mean-field theory, \( G > 0 \) is the equal strength pairing parameter, \( \eta_j = 2\epsilon_j/G, \) and the prime on the summations indicates that the sum is restricted to levels other than those occupied by single fermions. This type of pairing interaction is referred to as equal strength pairing interaction because the pairing interaction strength described by the parameter \( G > 0 \) between any two pairs of valence nucleons is all the same.

Because solutions of \( m \neq 0 \) cases are similar to those of the seniority zero case, in the following we only consider the case with \( m = 0 \) (no single fermions). For \( k \)-particle excitations, the eigenstates of (1) for \( p \) levels can be written as [9, 10]

\[
|k; \xi \rangle = S^* (E_1^{(\xi)}) S^* (E_2^{(\xi)}) \cdots S^* (E_p^{(\xi)}) |0 \rangle,
\]

where \( |0 \rangle \) is the pairing vacuum state satisfying \( S^-_j |0 \rangle = 0 \) for \( 1 \leq j \leq p, \)

\[
S^* (E_{\mu}^{(\xi)}) = \sum_{j=1}^{p} \frac{1}{\eta_j - E_{\mu}^{(\xi)}} S^+_j,
\]

with the corresponding eigenenergy \( E_{k}^{(\xi)} = G \sum_{\mu=1}^{k} E_{\mu}^{(\xi)}. \)

The pair energies \( E_{\mu}^{(\xi)} \) should satisfy the \( k \)-coupled Bethe ansatz or Richardson–Gaudin equations

\[
1 = \sum_{j=1}^{p} \frac{1}{\eta_j - E_{\mu}^{(\xi)}} + \sum_{\mu \neq \mu'}^{k} \frac{2}{E_{\mu}^{(\xi)} - E_{\mu'}^{(\xi)}},
\]

for \( \mu = 1, 2, \ldots, k. \) It is understood that the additional quantum number \( \xi \) in (2)–(4) is introduced to label the \( \xi \)th set of roots \( \{E_{\mu}^{(\xi)}\} \) of equations (4). A similar procedure also applies to spherical nuclear mean-field plus \( J = 0 \) pairing interaction [9–11] or \( T = 1 \) pairing interaction case considered in [14–15].

To map the Hamiltonian (1) into a boson Hamiltonian, we first use a mapping that maps the fermion pair operators \( \hat{k}_j, S^+_j \) into the corresponding real boson operators with

\[
\hat{k}_j \mapsto n_j = b_{j}^{\dagger} b_{j}, \quad S^+_j \mapsto b_{j}^{\dagger}, \quad S^-_j \mapsto b_{j} \quad \forall j,
\]

in which the images satisfy the usual commutation relations of boson operators: \( [b_{j}, b_{j}'] = \delta_{jj} \) and \( [b_{j}, b_{j}] = 0. \) It is clear that this mapping is different from the one based on a group structure [16], because the images of \( S^\pm \) no longer satisfy the commutation relations of the original \( SU(2) \) algebra. Furthermore, the mapping is unitary and number conserving, which is also different from the Dyson mapping adopted in [19]. We then seek a Bose Hamiltonian constructed from those boson images which should keep the eigenstate (2) consistent after the
mapping. We found that the one-body term in (1) maintains the same form after the mapping. The pairing interaction term, however, cannot be mapped onto a one-body form. It requires an additional non-Hermitian two-body interaction term in order to keep the energy spectrum of a Bose Hamiltonian after the mapping exactly the same as the one of the Hamiltonian (1). This is quite natural because the fermion pairing interaction, as for hard-core boson hopping, cannot be replaced by the usual boson hopping. The final image of (1) with $m = 0$ after the mapping (5) is of the following form:

$$
\hat{H}_{\text{Bose}} = G \sum_{j=1}^{p} (\epsilon_j - 1) n_j - \sum_{i \neq j}^{p} b_i^\dagger b_j + \sum_{i,j=1}^{p} \eta_{ij} n_i b_j^\dagger b_j.
$$

(6)

To understand the dynamics of the Hamiltonian, we consider a more general form of (6),

$$
\hat{H}_{\text{BH}} = \sum_{j} \left(2 \epsilon_j - t - U\right) n_j - \sum_{i \neq j} \left(t - n_j U\right) b_i^\dagger b_j + U \sum_{j} n_j^2,
$$

(7)

where $2 \epsilon_j - t - U$ in the first term can be regarded as a contribution from an external potential or on-site disorder, the second term describes boson hopping among all sites with a site-dependent hopping parameter $t - n_j U$ and the third term is the on-site repulsion. Since the two-body interaction term usually contributes with the same order of magnitude as the one-body term, the on-site repulsion parameter may be set as $U = U_0 / k$, where $k$ is the total number of bosons and $U_0$ and $t$ are of the same order of magnitude. Hence, the Hamiltonian (1) is mapped into a Bose–Hubbard model with a site-dependent hopping parameter $t - U_0 (n_j / k)$. Therefore, the more the bosons on the $j$th level, the less the hopping strength of other bosons hopping onto the $j$th level if $t \geq U_0$. It should be pointed out that the Hamiltonian (7) including hopping among all sites differs from the conventional Bose–Hubbard model [20]. Only nearest-neighbor hopping is considered in the latter case. However, the model with the Hamiltonian (7) is quasi-exactly solvable with the help of the Richardson–Gaudin method, namely only a part of the spectrum can be obtained exactly, while the conventional Bose–Hubbard model with nearest-neighbor hopping for finite $U$ can only be exactly solved in some special cases [21–24]. In the Bose Hamiltonian (6) with $t = U = G$, however, the condition $1 \geq n_j$ is no longer satisfied if $n_j \neq 0$ or 1, which means that the fermion pairing interaction looks extremely repulsive after the boson mapping (5).

To prove that (6) is indeed the exact boson image of (1), one can simply verify that the eigenstates of (7), at least a part of them, can indeed be written as the boson image of (2) with

$$
|k, \xi\rangle = B^\dagger \left( E^{(\xi)}_1 \right) B^\dagger \left( E^{(\xi)}_2 \right) \cdots B^\dagger \left( E^{(\xi)}_k \right) |0\rangle_B,
$$

(8)

where $|0\rangle_B$ is the corresponding boson vacuum, and

$$
B^\dagger \left( E^{(\xi)}_\mu \right) = \sum_{j=1}^{p} \frac{1}{2 \epsilon_j / t - E^{(\xi)}_\mu} b_j^\dagger
$$

(9)

with the corresponding eigenenergies

$$
E^{(\xi)} = t \sum_{\mu=1}^{k} E^{(\xi)}_\mu
$$

(10)

and the Bethe ansatz equations

$$
1 = \sum_{j=1}^{p} \frac{1}{2 \epsilon_j / t - E^{(\xi)}_\mu} + \sum_{\nu \neq \mu} \frac{2(U/t)}{E^{(\xi)}_\mu - E^{(\xi)}_\nu}
$$

(11)

for $\mu = 1, 2, \ldots, k$. 

Therefore, the Bose Hamiltonian (6) should be projected onto the physical subspace for given $p$ and $k$ when $Uk/t \leq 1$. When $U = t = G$, (7) reduces to (6) which is the image of the Hamiltonian (1). In this case, the condition $Uk/t \leq 1$ is no longer satisfied for $k \geq 2$. Therefore, in this case only a part of real eigenvalues of (6), $p!/(p-k)!k!$ in number, can be obtained from (10) and (11), which correspond exactly to those of the Hamiltonian (1). It is well known that the boson Hilbert subspace for given $p$ and $k$ is $(k + p - 1)!/(p-k)!k!$ dimensional, while the Richardson–Gaudin equations (11) only provide $p!/(p-k)!k!$ solutions. Such a situation is often called quasi-exactly solvable. Other eigenvalues of (6) which may be either real or complex are not provided by (10) and (11). Although not all of the eigenvalues of (6) are guaranteed to be real, especially for large values of $k$, a part of them, $p!/(p-k)!k!$ in number, that satisfy (10) and (11) are, and these are the same as those given by (4) and therefore correspond exactly to those of the Hamiltonian (1). There are additional $[(k + p - 1)!/(p-k)!k! - p!/(p-k)!k!]$ eigenvalues that are not provided for by (10)–(11), and the associated eigenvectors cannot be written in the Bethe ansatz form (8). This is due to the well-known fact that the Hilbert space constructed from boson operators $b_1^+$ is much larger than that spanned by the counterpairing fermionic pairs $S_j^+$ due to Pauli blocking.

Similar to the Dyson mapping [16, 17] and, for example, the iterative boson expansion approach [17], the resultant Bose Hamiltonian (6) is non-Hermitian. In addition, there will be spurious states involved in the boson space. The spurious states of (6) are those with $[(k + p - 1)!/(p-k)!k!] - p!/(p-k)!k!$ in number that cannot be written in the Bethe ansatz form (8). Therefore, the Bose Hamiltonian (6) should be projected onto the physical subspace spanned by (8). Let $\hat{P}$ be the projection operator, which can be expressed as

$$\hat{P} = \sum_{k\xi} |k, \xi \rangle \langle \xi, k|,$$

(12)
where \(|k, \xi\rangle\) are normalized eigenvectors given by (8), and the sum runs over all possible values according to the number of solutions of (11) with \(U = t = G\). It should be clear that by construction the projection operator \(\hat{P}\) annihilates the unphysical subspace spanned by those spurious states [16]. It follows that the projected Bose Hamiltonian with

\[
\hat{H}_{\text{Bose}} = \hat{P} \hat{H}_{\text{Bose}} \hat{P} \tag{13}
\]

is diagonalizable under the physical subspace spanned by \(|k, \xi\rangle\) with results shown by (8)–(11). Because the total number of eigenstates described by (8)–(11) is \(p!/(p-k)!k!\) for given \(p\) and \(k\), the physical boson subspace for given \(p\) is \(2^p\) dimensional, which is exactly equal to that spanned by fermion pair operators. Hence, we obtain the Bose Hamiltonian \(\hat{H}_{\text{Bose}}\) exactly equivalent to the Hamiltonian (1) in the physical boson subspace. Not only the dimension of physical boson subspace is the same as that spanned by fermion pair operators, but also the eigenenergies given by (10) and (11) of the Bose Hamiltonian (13) are exactly the same as those of (1) with \(m = 0\).

In summary, an exact boson mapping of a mean-field plus equal strength pairing Hamiltonian has been obtained under the guidance of the Richardson–Gaudin exact solutions of the pairing model. Though non-Hermitian, a part of solutions of the resultant Bose–Hubbard Hamiltonian provided by the Richardson–Gaudin type eigenstates and the corresponding Bethe ansatz equations are real when \(U = t = G\). The physical Bose image of the pairing Hamiltonian must then be projected onto the physical boson subspace, which results in exactly the same number of eigenstates with corresponding eigenenergies exactly the same as those of the pairing Hamiltonian (1).

In addition, because our construction assumes a deformed shell-model like basis, the boson operators \(\{b_j, \ b_j^\dagger\}\) do not associate with an angular momentum quantum number which must therefore be restored by angular momentum projection, as noted in [25]. In the Nilsson mean-field, for example, the boson operator \(b_i^\dagger\) can be rewritten in terms of spherical pairs as

\[
b_i^\dagger = x_0^i s_0^\dagger + x_2^i d_0^\dagger + x_4^i g_0^\dagger + \cdots, \tag{14}
\]

where \(x_L^i\) are transformation coefficients between the \(i\)th Nilsson level and the spherical basis, and \(s_0^\dagger, d_0^\dagger,\) etc are boson operators with \(L = 0, 2, \ldots,\) and \(M_L = 0\). After the angular momentum projection, one can better understand the links between mean-field plus pairing models and the interacting boson model. Further study on this problem will be carried out in the near future. Anyway, the total number of bosons in the system described by the Hamiltonian (13) is a conserved quantity which equals exactly the number of valence fermion pairs in the mean-field plus pairing model of nuclei described by the Hamiltonian (1). It is well known in the interacting boson model that the total number of bosons corresponding exactly to the number of valence nucleon pairs is merely an assumption. Hence, this work shows that this correspondence is indeed the case for shell model like mean-field plus equal strength pairing interaction models projected onto a boson subspace based on the Richardson–Gaudin equations, which may provide another way of finding the shell-model foundation of the interacting boson model.

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