COUPLING COEFFICIENTS
AND MATRIX ELEMENTS OF ARBITRARY TENSORS
IN THE ELLIOTT PROJECTED ANGULAR MOMENTUM BASIS

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Abstract: An explicit expression for the transformation brackets between the basis for an arbitrary irreducible representation of U(3) labeled in the U(3) $\supset$ R(3) scheme introduced by Elliott and the Gel'fand U(3) $\supset$ U(2) $\otimes$ U(1) scheme is obtained. The results are applied in deriving an expression for both the U(3) $\supset$ R(3) coupling coefficients and projected tensor matrix elements.

1. Introduction

The fundamental properties of the unitary groups and their representations have been available for many years in the works of Weyl ¹ and Murnaghan ², but only within the past decade have physicists diligently examined, exploited and extended these ideas to other than the special unitary group in two dimensions and the locally isomorphic rotation group in three dimensions. In fact the unfortunate occurrence of the terminology "accidental degeneracy" associated with the energy levels of a particle moving in an isotropic harmonic oscillator potential, which has long been used in nuclear shell theory, serves as a reminder of the failure of most nuclear physicists to recognize the invariance of the corresponding Hamiltonian to unitary transformations in three dimensions U(3). Although this invariance was pointed out by Jauch and Hill ³ in 1940, it remained for Elliott ⁴ in 1958 to firmly establish the importance of U(3) in nuclear structure calculations. Some four years later U(3) was also recognized as being of importance in the classification of elementary particles ⁵,⁶).

More recently physicists have devoted ⁷ considerable effort to investigating the properties of the unitary group in N-dimensions U(N). The most complete comprehension of U(N) has been obtained by the establishment of the connection between the abstract Weyl ¹ basis vectors and the boson calculus familiar from the work of Schwinger ⁸ and Bargmann ⁹). This together with the adaptation by Nagel and Moshinsky ¹⁰) of the work of Gel'fand and Zetlin ¹¹) gives a canonical setting to the state labeling problem for U(N). This labeling scheme involves the natural chain of subgroups U(N) $\supset$ U(N-1) $\supset$ . . . $\supset$ U(2) $\supset$ U(1), where in each step of the chain the

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reduction $U(M) \supset U(M-1) \otimes U(1)$ must be considered. For $U(3)$, one must therefore consider $U(3) \supset U(2) \otimes U(1)$ or equivalently in terms of the unitary unimodular groups $SU(3) \supset SU(2) \otimes U(1)$. This decomposition of $SU(3)$ is particularly useful in elementary particle theory since the $SU(2)$ and $U(1)$ refer respectively to the isospin and hypercharge of a given $SU(3)$ multiplet. For use in nuclear physics, however, it is essential to recognize that this $SU(2)$ is not equivalent to the physical angular momentum subgroup $R(3)$. The physically different requirements of $SU(3) \supset SU(2) \otimes U(1)$ for elementary particle theory and $SU(3) \supset R(3)$ for nuclear theory have led to the development of the properties of $U(3)$ along seemingly different paths.

The required reduction of $SU(3) \supset R(3)$ in nuclear physics involves a well-known multiplicity problem since a given irreducible representation (IR) of $R(3)$ may be contained more than once in a given IR of $SU(3)$. Elliott resolved this problem by a projection technique which introduced a quantum number $K$ that may be given the physical interpretation of being analogous to the projection of the total physical orbital angular momentum on an intrinsic axis of a nuclear many-body system. Bargmann and Moshinsky also resolved this problem by introducing an auxiliary operator $\Omega$ whose eigenvalue $\omega$ may be given the physical interpretation of the projection of the mass quadrupole moment along the direction of the total orbital angular momentum also in a nuclear many-body system. In both solutions to the multiplicity problem complications arise as a manifestation of the fact that it is impossible to define an auxiliary quantum number in the reduction $SU(3) \supset R(3)$ which yields both simple selection rules and state orthogonality. Because of the physical interpretation and extremely simple selection rules of $K$, the somewhat more intuitive approach of Elliott is often preferred to that of Bargmann and Moshinsky.

Whether one works in the $SU(3) \supset SU(2) \otimes U(1)$ scheme or in the $SU(3) \supset R(3)$ scheme, the quantities of interest are coupling coefficients and tensorial matrix elements. If therefore the transformation brackets between the two state labeling schemes for a given IR of $SU(3)$ are known, a duplication of effort may be avoided. In a previous paper, we exploited the connection between the Gel'fand basis and the natural $SU(2) \otimes U(1)$ basis used by Elliott for projection of the $R(3)$ basis states in order to derive an expression for the normalization and overlap integrals of these projected states. In this paper, we shall further utilize this connection to derive an expression for the transformation brackets between the $SU(3) \supset R(3)$ and $SU(3) \supset SU(2) \otimes U(1)$ state labeling schemes. We shall begin by briefly reviewing the results of ref. in order to establish both notation and definitions. Then in sect. 3 we shall develop an explicit expression for the transformation brackets. These will then be used in sect. 4 to give a completely general discussion of $U(3) \supset R(3)$ coupling coefficients. In sect. 5, we define tensorial projections and use the results of sect. 4 to derive their matrix elements. In sect. 6, we shall discuss briefly the implications of the theory developed in sects. 3-5.
2. Notation and definitions

The infinitesimal generators of U(3) defined in terms of oscillator creation and annihilation operators are

\[ C_{\alpha\beta} = \sum_{s=1}^{A} a_s^\dagger a_\beta, \]

where \( A \) denotes the number of particles and \( \alpha \) and \( \beta \) the Cartesian tensor indices. The corresponding U(3) basis vectors in the Gel'fand scheme may be denoted by

\[ |G\rangle = \begin{pmatrix} h_{13} \\ h_{12} \\ h_{23} \\ h_{22} \\ h_{33} \end{pmatrix}, \]

in which the \( h_{\alpha\beta} \) are integers satisfying the betweeness conditions

\[ h_{\alpha,\beta} \geq h_{\alpha,\beta-1} \geq h_{\alpha+1,\beta} \geq 0, \]

which are made evident by the geometrical construction of the state vector. The eigenvalues of \( C_{\alpha\beta} \) are the number of oscillator quanta \( N_\alpha \) along each of the three Cartesian directions and may be given in terms of the \( h_{\alpha\beta} \) by

\[ N_1 = h_{11}, \]
\[ N_2 = h_{12} + h_{22} - h_{11}, \]
\[ N_3 = h_{13} + h_{23} + h_{33} - h_{12} - h_{22}. \]

The eigenvalue of the reduced Hamiltonian,

\[ H = \sum_{s=1}^{3} C_{ss}, \]

is equal to the number of oscillator quanta \( N \), where

\[ N = h_{13} + h_{23} + h_{33}. \]

Conversely, given a particular permutation symmetry and the total number of oscillator quanta, it is possible to determine the allowable values of \( h_{33} \). The allowable values of the remaining \( h_{\alpha\beta} \) for a fixed set of the \( h_{33} \) may then be determined by using eq. (3) and are related to the \( N_\alpha \) by eqs. (4).

The \( h_{33} \) are related to the Young pattern method \(^1\) for labeling the IR of U(3) by

\[ \begin{array}{c}
\hline
h_{13} \\
\hline
h_{23} \\
\hline
h_{33}
\end{array} \]
from which the corresponding SU(3) pattern is obtained by subtracting $h_{33}$ from each row. The SU(3) IR labels $\lambda$ and $\mu$ used by Elliott are then given by

$$\lambda = h_{13} - h_{23},$$  \hspace{1cm} (7a)
$$\mu = h_{23} - h_{33}. \hspace{1cm} (7b)$$

The U(2) patterns

$$\begin{bmatrix}
  h_{12} \\
  h_{22}
\end{bmatrix}$$

consistent with a given U(3) pattern are determined by eq. (3). The SU(2) pattern corresponding to a given U(2) pattern is then obtained by subtracting $h_{22}$ from each row and is simply related to the SU(2) IR label $A$ used by Elliott. In particular

$$2A = h_{12} - h_{22}, \hspace{1cm} (8)$$

and the projection $A_0$ of $A$ is given by

$$2A_0 = N_1 - N_2$$
$$= 2h_{11} - h_{12} - h_{22}. \hspace{1cm} (9)$$

The U(1) IR label $\epsilon$ used by Elliott is the eigenvalue of $2C_{33} - C_{11} - C_{22}$, therefore

$$\epsilon = 2N_3 - N_1 - N_2$$
$$= 2(h_{13} + h_{23} + h_{33}) - 3(h_{12} + h_{22}). \hspace{1cm} (10)$$

For completeness, we quote the rules for determining the allowable values of $\epsilon$, $A$, and $A_0$ given the values of $\lambda$ and $\mu$

$$\epsilon = 2\lambda + \mu - 3(p + q), \hspace{1cm} (11a)$$
$$A = \frac{1}{2}(\mu + p - q), \hspace{1cm} (11b)$$
$$A_0 = -A + r, \hspace{1cm} (11c)$$

where $p$, $q$ and $r$ may take on the values

$$p = 0, 1, 2, \ldots, \lambda, \hspace{1cm} (12a)$$
$$q = 0, 1, 2, \ldots, \mu, \hspace{1cm} (12b)$$
$$r = 0, 1, 2, \ldots, 2A. \hspace{1cm} (12c)$$

These rules are of course nothing more than the betweenness conditions of eq. (3) expressed in terms of the state labels used by Elliott.
In ref.\(^{14}\), it was shown that \(U(3) \supset R(3)\) states may be obtained from extremal Gel'fand states through the projections

\[
|G_{E, KLM}\rangle \equiv \int d\Omega D_{M\ell}^{\mu}(\Omega)R(\Omega)|G_E\rangle,
\]

where if \(G_E = G_{\text{HW}}\) is the highest weight state \((h_{11} = h_{12} = h_{13}, h_{22} = h_{23})\) the rule for determining \(K\) and the corresponding \(L\)-values is

\[
K = \lambda, \lambda - 2, \ldots, 1 \text{ or } 0; \\
L = K, K + 1, \ldots, K + \mu; \quad K \neq 0, \\
L = \mu, \mu - 2, \ldots, 1 \text{ or } 0; \quad K = 0,
\]

whereas if \(G_E = G_{\text{LW}}\) is the lowest weight state \((h_{11} = h_{22} = h_{33}, h_{12} = h_{23})\) the rule for determining \(K\) and the corresponding \(L\)-values is

\[
K = \mu, \mu - 2, \ldots, 1 \text{ or } 0; \\
L = K, K + 1, \ldots, K + \lambda; \quad K \neq 0, \\
L = \lambda, \lambda - 2, \ldots, 1 \text{ or } 0; \quad K = 0.
\]

Rules (14) and (15) are also applicable to the extremal states with \(G_E = G_{\text{HW}}(h_{11} = h_{22} = h_{33}, h_{12} = h_{13})\) and \(G_E = G_{\text{LW}}(h_{11} = h_{12} = h_{23}, h_{22} = h_{33})\), respectively.

In table 1, we summarize by relating all of these to the familiar labels \(e, 2A\) and \(y = 2A_0\) introduced by Elliott\(^{4}\) and Elliott and Harvey\(^{16}\).

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
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<tbody>
<tr>
<td>Relationship between the Elliott and Gel'fand state labels</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>(G_{\text{HW}})</th>
<th>(h_{11})</th>
<th>(h_{12})</th>
<th>(h_{22})</th>
<th>(e)</th>
<th>(2A)</th>
<th>(2A_0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(G_{\text{HW}})</td>
<td>(h_{11})</td>
<td>(h_{12})</td>
<td>(h_{22})</td>
<td>(-\lambda - 2\mu)</td>
<td>(\lambda)</td>
<td>(\lambda)</td>
</tr>
<tr>
<td>(G_{\text{LW}})</td>
<td>(h_{11})</td>
<td>(h_{12})</td>
<td>(h_{22})</td>
<td>(2\lambda + \mu)</td>
<td>(\mu)</td>
<td>(\mu)</td>
</tr>
</tbody>
</table>

Therefore the highest weight state of Elliott and Harvey \((e = e_{\text{max}}, y = y_{\text{max}})\) corresponds to \(G_E = G_{\text{HW}}\) and their lowest weight state \((e = e_{\text{min}}, y = y_{\text{max}})\) would correspond to \(G_E = G_{\text{LW}}\).

Although a complete basis may be obtained from any of the \(G_E\), by choosing the extremal states for which \(e = e_{\text{max}}\) if \(\lambda \geq \mu\) and \(e = e_{\text{min}}\) if \(\lambda < \mu\) one can minimize \(|y|\). Such a minimization is physically significant since as may be seen from the fact that the \(N_a\) are a measure of the linear extension of the system,

\[
N_a = \langle \sum_{i=1}^{A} (x_i^a)^2 \rangle,
\]

which
minimizing $|v|$ minimizes the deviation from axial symmetry \(^1\). For such $v$, however, there still remains a choice between $v = v_{\text{max}}$ and $v = v_{\text{min}}$. Although the results we shall derive are general enough to include both cases, it is to be noted that group theoretically it is the states $G_E = G_{1\text{W}}$ and $G_E = G_{1\text{LW}}$ which are to be preferred. The reason being that such a choice allows standard phase conventions for coupling coefficients (which can cause considerable difficulty in groups having rank greater than one) to be introduced simply and directly into the $SU(3) \supseteq R(3)$ scheme.

We note for future reference that eigenstates of the total orbital angular momentum may equally well be projected from any of the Gel'fand states using any of the values of $K$ such that $|K| \leq L$ by defining

$$|G; KLM\rangle = \int d\Omega D_{LM}^{K\ast}(\Omega) R(\Omega)|G\rangle. \quad (17)$$

In eq. (17), we have retained the complete $G$ symbol on the left as a reminder of the Gel'fand state from which the projected vectors were obtained; only the $U(3)$ IR labels $h_\lambda$, however, remain valid state labels. In many cases, the $|G; KLM\rangle$ will turn out to be identically zero. Only for the special cases of eq. (13) taken together with the appropriate rule for determining the $K$-label and the corresponding $L$-value is one guaranteed that the $U(3) \supseteq R(3)$ basis vectors span the representation space.

3. Transformation brackets

Since the Gel'fand basis vectors $|G\rangle$ for a given IR of $U(3)$ form an orthonormal set which spans the representation space, an arbitrary $U(3) \supseteq R(3)$ basis vector $|G; KLM\rangle$ belonging to the IR may be expanded in terms of the $|G\rangle$ as

$$|G; KLM\rangle = \sum_{G'} \langle G'|G; KLM\rangle |G'\rangle, \quad (18)$$

where it is to be understood that $h'_{\lambda_1} = h_{\lambda_2}$. The $\langle G'|G; KLM\rangle$ in eq. (18) are the transformation brackets between the Gel'fand $U(3) \supseteq U(2) \oplus U(1)$ scheme and the $U(3) \supseteq R(3)$ scheme introduced by Elliott \(^2\). The inverse of the transformation is only guaranteed to exist if the $|G; KLM\rangle$ are the extremal $U(3) \supseteq R(3)$ basis vectors defined in sect. 2. We shall derive an expression for the general transformation brackets $\langle G'|G; KLM\rangle$; those for the extremal $U(3) \supseteq R(3)$ basis vectors follow as a special case.

From eq. (17), we have

$$\langle G'|G; KLM\rangle = \int d\Omega D_{LM}^{K\ast}(\Omega) \langle G'|R(\Omega)|G\rangle. \quad (19)$$

Therefore the problem reduces to one of finding an explicit expression \(^\dagger\) for the matrix

\(^\dagger\) The form of the expression on the right-hand side of eq. (19) is a direct consequence of the fact that we have used the projection operator technique. This form would occur for any continuous group. See for example ref. \(^{18}\).
elements $\langle G'|R(\Omega)|G \rangle$. Following the convention of Rose\(^{19}\), $R(\Omega)$ may be written as

$$R(\Omega) = R_3(\theta_1)R_x(\theta_2)R_y(\theta_3),$$

(20)

where the $\theta$'s are Euler angles and the $R_x(\phi)$ are given by

$$R_x(\phi) = e^{-i\phi L_x}.$$  

(21)

The $L_\alpha$ are the Cartesian components of the physical orbital angular momentum and are expressible in terms of the $C_{\alpha\beta}$ of eq. (1) as

$$L_\alpha = -ie_{\alpha\beta\gamma} C_{\beta\gamma},$$

(22)

In eq. (22), the $e_{\alpha\beta\gamma}$ is the third rank Levi-Civita tensor. The validity of this result is most easily established by expressing the creation and annihilation operators of the $C_{\beta\gamma}$ in terms of Cartesian coordinates and their conjugate momenta.

Now, the infinitesimal generators of SU(2) may be defined by

$$A_+ = C_{12},$$

(23a)

$$A_- = C_{21},$$

(23b)

$$A_0 = \frac{1}{2}(C_{11} - C_{22})$$  

(23c)

where

$$A = A_1 \pm iA_2.$$  

(24)

Therefore, we have that

$$L_3 = 2A_2.$$  

(25)

Since any Cartesian component of the angular momentum can be related to $L_3$ by the appropriate axis permutation, we may write

$$L_2 = (23)(-L_3)(23),$$

(26)

or by using eq. (25)

$$L_2 = (23)(-2A_2)(23).$$

(27)

Thus the rotation operator $R(\Omega)$ may be written in terms of the $A_2$ as

$$R(\Omega) = \mathcal{R}(2\theta_1)(23)\mathcal{R}(-2\theta_2)(23)\mathcal{R}(2\theta_3),$$

(28)

where

$$\mathcal{R}(\phi) = e^{-i\phi A_2}.$$  

(29)

In the following, for notational convenience we shall replace $h_{13}$, $h_{23}$, $h_{33}$, $h_{12}$, $h_{22}$ and $h_{11}$ by $h_1$, $h_2$, $h_3$, $s$, $t$, and $r$, respectively. Furthermore, we shall suppress the U(3) IR labels $h_\alpha$ in the Gel'fand basis vectors. That is,

$$\begin{pmatrix} h_{13} & h_{23} & h_{33} \\ h_{12} & h_{22} & h_{11} \end{pmatrix} \equiv \begin{pmatrix} h_1 & h_2 & h_3 \\ s & t & r \end{pmatrix}$$

(30a)

$$\equiv \begin{pmatrix} s & t \\ r & \end{pmatrix}.$$  

(30b)
It will also prove to be convenient to replace \( A \) by \( j \) and \( A_0 \) by \( m \). Then it follows from eqs. (8) and (9) together with eq. (29) that

\[
\begin{align*}
\langle s' r' | t' \rangle & \langle s r | t \rangle = \delta s s' \delta \tau t' \mu m m' \rho \rho' \eta \eta' \varphi (\varphi) ; \\
\end{align*}
\]

\( j = \frac{1}{2}(s + t) \), \( m = r - \frac{1}{2}(s + t) \), \( m' = r' - \frac{1}{2}(s' + t') \).

In addition, using the result given by Chacón and Moshinsky \( ^{20} \) for the matrix elements of the transposition (23) in the Gel'fand basis in terms of Racah \( ^{21} \) coefficients, we have

\[
\begin{align*}
\langle s' t' | (23) | s r \rangle &= (-1)^{j' + \sigma + \tau + \rho - s + t - r} \sqrt{(2j' + 1)(2j + 1)} W(abcd; ef) ; \\
\end{align*}
\]

\( a = \frac{1}{2}(h_1 + h_2 - s - t') \), \( b = \frac{1}{2}(h_2 + r - t - t') \), \( c = \frac{1}{2}(s + s' - h_2 - r) \), \( d = \frac{1}{2}(s' + t - r - h_3 - r) \), \( e = \frac{1}{2}(h_1 + r - t - s') \), \( f = \frac{1}{2}(h_1 + r - s - t') \)

and where \( h \) is the total number of quanta.

From the eqs. of eqs. (28)-(32), it then follows that

\[
\begin{align*}
\langle s' t' | R(\Omega) | s r \rangle &= \sum_{\sigma \rho \sigma' \rho'} \langle s' t' | R(\Omega) | s' t' \rangle \\
& \times \langle s' \rho' | (23) | s' \rho' \rangle \\
& \times \langle s' \rho' | R(\Omega) | s' \rho' \rangle \\
& \times \langle s' \rho' | (23) | s' \rho' \rangle.
\end{align*}
\]

or

\[
\begin{align*}
\langle s' t' | R(\Omega) | s r \rangle &= \sqrt{(2j + 1)(2j + 1)} \sum_{\sigma \rho} \sum_{\sigma \rho} \sum_{\sigma \rho} \sum_{\sigma \rho} \sum_{\sigma \rho} \sum_{\sigma \rho} d_{\sigma \rho}^{j'} d_{\sigma \rho}^{j-1} \langle 2\theta_1 \rangle \langle 2\theta_2 \rangle \langle 2\theta_3 \rangle \\
& \times W(abcd; ef) \delta_{s + t - r - s - t} \delta_{s + t - r - s - t} \delta_{s + t - r - s - t}.
\end{align*}
\]

\( j' = \frac{1}{2}(s' - t') \), \( m' = r' - \frac{1}{2}(s' + t') \), \( \mu' = \rho' - \frac{1}{2}(s + t) \),

\( j = \frac{1}{2}(s + t) \), \( m = r - \frac{1}{2}(s + t) \), \( \mu = \rho - \frac{1}{2}(s + t) \),

\( k = \frac{1}{2}(\sigma - \tau) \), \( \nu = \rho - \frac{1}{2}(\sigma + \tau) \), \( \nu' = \rho - \frac{1}{2}(\sigma + \tau) \),

\( a' = \frac{1}{2}(h_1 + h_2 - s' - t') \), \( b' = \frac{1}{2}(h_2 + \rho - t - t') \), \( c' = \frac{1}{2}(\sigma + s' - h_2 - r) \),

\( d' = \frac{1}{2}(s' + t' - h_3 - \rho) \), \( e' = \frac{1}{2}(h_1 + \rho - t - \sigma) \), \( f' = \frac{1}{2}(h_1 + \rho - t - \sigma) \),

\( a = \frac{1}{2}(h_1 + h_2 - \sigma - \tau) \), \( b = \frac{1}{2}(h_2 + \rho - t - \sigma) \), \( c = \frac{1}{2}(s + \sigma - h_2 - \rho) \),

\( d = \frac{1}{2}(\sigma + \tau - h_3 - \rho) \), \( e = \frac{1}{2}(h_1 + \rho - t - \sigma) \), \( f = \frac{1}{2}(h_1 + \rho - \sigma - \tau) \).
The transformation brackets of eq. (19) are then given by

\[
\langle G' | G ; KLM \rangle = \sqrt{(2j'+1)(2j+1)} \sum_{\rho'} I_1(m'j'\mu'M)
\times \sum_{\rho} I_1(\mu j m K) \sum_{\delta} (2k+1) I_2(\nu'k'vMLK) W(a'bc'd'; e'f')
\times W(abcd; ef) \delta_{\alpha' + \nu' - \rho', h - \epsilon - \rho, h - \epsilon - t,}
\]

where we define

\[
I_1(m'jml) = \frac{1}{2\pi} \int_{0}^{2\pi} d\phi d_{m'm}(2\phi)e^{i\phi},
\]

\[
I_2(m'jmn'ln) = \frac{1}{2} \int_{0}^{\pi} \sin \phi d\phi d_{m'm}(2\phi)d_{n'n}(\phi).
\]

The evaluation of the integrals \(I_1\) and \(I_2\) is performed in appendices A and B, respectively. Since \(U(3) \supset SU(3) \supset R(3)\) these transformation brackets are of course invariant under the unimodular restriction. In addition, if one chooses to perform the projection of eq. (17) on the state \(|G\rangle \equiv i^{-w} |G\rangle\), where \(|G\rangle\) is the Gel'fand basis state defined by Chacón and Moshinsky \(^{20}\) and \(w = s + t - r\), it can be shown that the transformation brackets are real.

In general these transformation brackets \(\langle G'|G; KLM \rangle\) relate the set of non-orthogonal basis vectors \(|G; KLM \rangle\) to the set of orthonormal basis vectors \(|G\rangle\) and are therefore the elements of a non-unitary matrix \(A\). As already pointed out, the inverse expansion of the \(|G\rangle\) in terms of the \(|G_E; K'L'M' \rangle\) exists and the coefficients \(\langle G_E; K'L'M'|G\rangle\) can be obtained by inverting the appropriate \(A\)-matrix. The dimensionality of this matrix \(A\) is equal to the dimensionality \(d(\lambda \mu)\) of the IR, given by \(^{22}\)

\[
d(\lambda \mu) = (\lambda + 1)(\mu + 1)\lambda + \mu + 2.
\]

An equivalent but perhaps somewhat simpler evaluation of these coefficients can be obtained by considering directly the expansion

\[
|G\rangle = \sum_{K'L'M'} \langle G_E; K'L'M'|G\rangle |G_E; K'L'M'\rangle.
\]

By inserting this expression for \(|G\rangle\) into eq. (17), we find that

\[
|G; KLM \rangle = \sum_{K'L'M'} \langle G_E; K'L'M'|G\rangle \int d\Omega D_{MK}^{L*}(\Omega) R(\Omega) |G_E; K'L'M'\rangle
= \sum_{K'L'M'M'} \langle G_E; K'L'M'|G\rangle \int d\Omega D_{MK}^{L*}(\Omega) D_{M'M'}^{E*}(\Omega) |G_E; K'L'M'\rangle
= \sum_{K'L'M'M'} \langle G_E; K'L'M'|G\rangle (2L+1)^{-1} \delta_{MM'} \delta_{LL'} \delta_{KM'} |G_E; K'L'M'\rangle
= \sum_{K'} \langle G_E; K'LK|G\rangle (2L+1)^{-1} |G_E; K'L'M\rangle.
\]
That is, the \( \langle G; K'L'M'|G \rangle \) are not only the coefficients in the expansion of the \( |G \rangle \) in terms of the \( |G; K'L'M' \rangle \) but they are also \((2L+1)\) times the coefficients in the expansion of the \( |G; KLM \rangle \) in terms of the \( |G; K'L'M' \rangle \). Using this result, a unique solution for the \( \langle G; K'L'M'|G \rangle \) can be determined from the set of simultaneous equations

\[
\langle G|G; KLM \rangle = \sum_k (2L+1)^{-1} \langle G; K'LK|G \rangle \langle G|G; KLM \rangle,
\]

in which the coefficients of the unknowns are a particularly simple special case of eq. (35). Explicitly,

\[
\langle G|G; KLM \rangle = \sum_{\rho} I_1(j jmM)I_1(mjK)I_2(\rho kKMLK);
\]

\[
j = \frac{1}{2}(h_1 - h_2), \quad k = \frac{1}{2}(h_1 + h_2), \quad m = \rho - \frac{1}{2}(h_1 + h_2); \quad G_E = G_{1W},
\]

\[
j = \frac{1}{2}(h_2 - h_3), \quad k = \frac{1}{2}(h_1 + \rho), \quad m = \rho - \frac{1}{2}(h_2 + h_3); \quad G_E = G_{1L}.
\]

(42)

A corresponding expression for \( G_E = G_{1W} \) and \( G_E = G_{1L} \) may be obtained by simply replacing \( m \) by \(-m\). The dimensionality of the matrix equation to be solved in this case is equal to the number of allowable \( K \)-values corresponding to a given \( L \)-value. Since \( K \) resolves the degeneracy in the \( U(3) \to R(3) \) scheme, this must be equal to the total number of occurrences \( d(\lambda; L) \) of \( L \) in the \( IR \) which from ref. \(^{14}\) is given by

\[
d(\lambda; L) = \left[ \frac{1}{2}(\lambda + \mu - L) \right] - \left[ \frac{1}{2}(\lambda + 1 - L) \right] - \left[ \frac{1}{2}(\mu + 1 - L) \right] + 1,
\]

(43)

where \( [\,] \) means largest integer contained in the argument and is to be interpreted as zero if the argument is negative.

It is to be noted that the transformation brackets are equivalent to the normalization and overlap integrals of the \( U(3) \to R(3) \) basis vectors \( |G; KLM \rangle \). This may most easily be seen by considering

\[
\langle G'; K'L'M'|G; KLM \rangle = \int d\Omega D_{MK}^{l*}(\Omega)R^{-1}(\Omega)|G; KLM \rangle
\]

\[
= \sum_{M'} \int d\Omega D_{MK}^{l*}(\Omega)D_{M'M'}^{l*}(\Omega)\delta_{MM'} \langle G'|G; KLM' \rangle
\]

\[
= (2L+1)^{-1} \delta_{MM'} \langle G'|G; KLM' \rangle
\]

(44)

Apart from a factor due to normalization, the special case of eq. (42) with \( G_E = G_{1L} \) is then simply \((2L+1)\) times the result derived by Elliott and Harvey \(^{19}\). For \( G_E = G_{1W} \) it is equal to \((2L+1)\) times the result derived in ref. \(^{14}\).
4. The U(3) ⊃ R(3) coupling coefficients

The U(3) ⊃ R(3) coupling coefficients are defined by

\[
|G_1; K_1 L_1 M_1, K_2 L_2 M_2\rangle = \sum_{\alpha \mathcal{E}, \mathcal{E}, \mathcal{E}} \langle G_3; K_3 L_3 M_3 | G_1; K_1 L_1 M_1, G_2; K_2 L_2 M_2\rangle_{\alpha} |G_3; K_3 L_3 M_3\rangle_{\mathcal{E}},
\]

(45)

where \(\alpha\) distinguishes the multipole occurrences of the IR \(G_3\) in the reduction of the direct product \(G_1 \otimes G_2\). These may be related to the SU(3) ⊃ SU(2) ⊃ U(1) coupling coefficients \(^{23}\) by using the transformation brackets of sect. 3. Explicitly, if \(R(\Omega; \beta)\) is the rotation operator in the space labeled by \(\beta\), then from the definition of eq. (17) we have

\[
|G_1; K_1 L_1 M_1, K_2 L_2 M_2\rangle = \int d\Omega D_{M_1 K_1}^{L_1^*}(\Omega) R(\Omega; 1)|G_1\rangle |G_2; K_2 L_2 M_2\rangle
\]

\[
= \int d\Omega D_{M_1 K_1}^{L_1^*}(\Omega) R(\Omega; 1)|G_1\rangle R(\Omega; 2) R^{-1}(\Omega; 2)|G_2; K_2 L_2 M_2\rangle
\]

\[
= \sum_{K'} \int d\Omega D_{M_1 K_1}^{L_1^*}(\Omega) D_{M_2 K_2}^{L_2^*}(\Omega) R(\Omega; 1) R(\Omega; 2)|G_1\rangle |G_2; K_2 L_2 K'_2\rangle.
\]

(46)

From the well-known R(3) Clebsch-Gordan series for the rotation matrices and the fact that \(R(\Omega; 1) R(\Omega; 2) = R(\Omega; 3)\), this may be written as

\[
|G_1; K_1 L_1 M_1, K_2 L_2 M_2\rangle = \sum_{L_3 K_3} C(L_1 L_2 L_3; M_1 M_2 M_3) C(L_1 L_2 L_3; K_1 K_2 K'_3)
\]

\[
\times \int d\Omega D_{M_1 K_1}^{L_1^*}(\Omega) R(\Omega; 3)|G_1\rangle |G_2; K_2 L_2 K'_2\rangle.
\]

(47)

By applying eq. (18), we find that the right-hand side of eq. (47) becomes

\[
\sum_{L_3 K_3} C(L_1 L_2 L_3; M_1 M_2 M_3) C(L_1 L_2 L_3; K_1 K_2 K'_3)
\]

\[
\times \langle G'_2; G_2; K_2 L_2 K'_2 | \int d\Omega D_{M_1 K_1}^{L_1^*}(\Omega) R(\Omega; 3)|G_1\rangle |G'_2\rangle.
\]

(48)

Now, if we use the SU(3) ⊃ SU(2) ⊃ U(1) coupling coefficients to express

\[
|G_1\rangle |G'_2\rangle = \sum_{\alpha\mathcal{E}} \langle G_3; G_1\rangle |G'_2\rangle |G_3\rangle_{\alpha},
\]

(49)

together with eq. (40), we obtain the final results
\[ |G; K_1 L_1 M_1, G_1; K_2 L_2 M_2, G_2\rangle = \sum_{a G; K_3} C(L_1 L_2 L_3; M_1 M_2 M_3) C(L_1 L_2 L_3; K_1 K_2 K_3) \times\langle G; K_2 L_2 K_3^*| G_1; G_2^*\rangle \cdot d\Omega D_{M_1, K_3^*}^{L_3}(\tilde{\Omega}) \cdot R(\Omega; 3)|G_3\rangle \]

\[ = \sum_{a G; K_3} C(L_1 L_2 L_3; M_1 M_2 M_3) C(L_1 L_2 L_3; K_1 K_2 K_3) \times\langle G; K_2 L_2 K_3^*| G_1; G_2^*\rangle \cdot (2L_3 + 1)^{-1} \langle G; K_3 L_3 K_3^*|G_3\rangle \]

where it is to be understood that the \(h_{a}\) of \(G_3\) are equal to the \(h_{a}\) of \(G_{3E}\).

From this development, we see that the \(U(3) \supset R(3)\) coupling coefficients are given by

\[ \langle G_{3E}; K_3 L_3 M_3| G_1; K_1 L_1 M_1, G_2, K_2 L_2 M_2\rangle = \sum_{a G; K_3} C(L_1 L_2 L_3; M_1 M_2 M_3) C(L_1 L_2 L_3; K_1 K_2 K_3) \times\langle G_3| G_2; K_2 L_2 K_3^*\rangle \cdot \langle G_3| G_1; G_2^*\rangle \cdot (2L_3 + 1)^{-1} \langle G_{3E}; K_3 L_3 K_3^*|G_3\rangle. \]  

(51)

The reduced \(U(3) \supset R(3)\) coupling coefficients follow directly as

\[ \langle G_{3E}; K_3 L_3| G_1, K_1 L_1; G_2, K_2 L_2\rangle = \sum_{a G; K_3} C(L_1 L_2 L_3; K_1 K_2 K_3) \times\langle G_3| G_2; K_2 L_2 K_3^*\rangle \cdot \langle G_3| G_1; G_2^*\rangle \cdot (2L_3 + 1)^{-1} \langle G_{3E}; K_3 L_3 K_3^*|G_3\rangle. \]  

(52)

Since Resnikoff\(^{29}\) has given a complete general solution for the SU(3) \(\supset\) SU(2) \(\otimes\) U(1) coupling coefficients including a resolution of the multiplicity problem in an orthogonal way, eqs. (51) and (52) yield the complete general solution for the \(U(3) \supset R(3)\) coupling coefficients.

Although the \(|G_3; K_3 L_3 M_3\rangle\) are a set of non-linearly independent basis vectors which over-span the representation space so that no unique expansion of the direct product in terms of these basis vectors exists, the particular expansion coefficients of eq. (50a) are well-defined and for this reason are useful for computational purposes. It is to be emphasized, however, that the expansion coefficients of eqs. (51) and (52) are unique to within the resolution of the multiplicity problem as given by Resnikoff [ref. 29].

Pursey\(^{24}\) has pointed out that a more symmetric form for the \(U(3) \supset R(3)\) coup-
Coupling coefficients may be given in terms of the product of two transformation brackets by using the unitarity properties of the Clebsch-Gordan coefficients. Explicitly, if we expand

\[ |G_1; K_1 L_1 K_1'\rangle |G_2; K_2 L_2 K_2'\rangle = \sum_{g_1', g_2'} \langle G_1'; |G_1; K_1 L_1 K_1'\rangle \]
\[ \times \langle G_2'; |G_2; K_2 L_2 K_2'\rangle \langle G_1' |G_1; K_1 L_1 K_1'\rangle \langle G_2 |G_2; K_2 L_2 K_2'\rangle \langle G_3 |G_3; K_3 L_3 K_3'\rangle, \]  
\[ \text{(53)} \]

then

\[ \int \text{d} \Omega D_{M_2}^{I_2*} (\Omega) R(\Omega; 3) |G_1; K_1 L_1 K_1'\rangle |G_2; K_2 L_2 K_2'\rangle \]
\[ = \sum_{M_1 M_2} \int \text{d} \Omega D_{M_1}^{I_1*} (\Omega) D_{M_1, K_1}^{L_1} (\Omega) D_{M_2, K_2}^{L_2} (\Omega) |G_1; K_1 L_1 M_1\rangle |G_2; K_2 L_2 M_2\rangle \]
\[ = (2L_3 + 1)^{-1} \sum_{M_1 M_2} C(L_1 L_2 L_3; M_1 M_2 M_3) C(L_1 L_2 L_3; K_1' K_2' K_3') \]
\[ \times |G_1; K_1 L_1 M_1\rangle |G_2; K_2 L_2 M_2\rangle, \] 
\[ \text{(54)} \]

which from eq. (53) is equal to

\[ \sum_{g_1', g_2'} \langle G_1'; |G_1; K_1 L_1 K_1'\rangle \langle G_2'; |G_2; K_2 L_2 K_2'\rangle \langle G_3 |G_3; K_3 L_3 K_3'\rangle a. \] 
\[ \text{(55)} \]

Multiplying both eqs. (54) and (55) by \( C(L_1 L_2 L_3; K_1' K_2' K_3') \) and summing over \( K_1' \) and \( K_2' \), we find by using the unitarity of the \( R(3) \) Clebsch-Gordan coefficients that

\[ \sum_{M_1 M_2} C(L_1 L_2 L_3; M_1 M_2 M_3) |G_1; K_1 L_1 M_1\rangle |G_2; K_2 L_2 M_2\rangle \]
\[ = \sum_{g_1', g_2'} (2L_3 + 1) C(L_1 L_2 L_3; K_1' K_2' K_3') \langle G_1 |G_1; K_1 L_1 K_1'\rangle \]
\[ \times \langle G_2 |G_2; K_2 L_2 K_2'\rangle \langle G_3 |G_3; K_3 L_3 K_3'\rangle a. \] 
\[ \text{(56)} \]

Multiplying this result by \( C(L_1 L_2 L_3; M_1 M_2 M_3) \) and summing over \( L_3 \) and once again using the unitarity of the \( R(3) \) Clebsch-Gordan coefficients, we find the following results analogous to eqs. (50a) and (50b):

\[ |G_1; K_1 L_1 M_1\rangle |G_2; K_2 L_2 M_2\rangle = \sum_{g_1', g_2'} C(L_1 L_2 L_3; M_1 M_2 M_3) \]
\[ \times C(L_1 L_2 L_3; K_1' K_2' K_3')(2L_3 + 1) \langle G_1 |G_1; K_1 L_1 K_1'\rangle \langle G_2 |G_2; K_2 L_2 K_2'\rangle \]
\[ \langle G_3 |G_3; K_3 L_3 K_3'\rangle a. \] 
\[ \text{(57a)} \]

\[ = \sum_{g_1', g_2'} C(L_1 L_2 L_3; M_1 M_2 M_3) C(L_1 L_2 L_3; K_1' K_2' K_3') \]
\[ \times \langle G_1 |G_1; K_1 L_1 K_1'\rangle \langle G_2 |G_2; K_2 L_2 K_2'\rangle \langle G_3 |G_3; K_3 L_3 K_3'\rangle a. \] 
\[ \text{(57b)} \]
where again it is to be understood that \( h'_{33} \) of \( G_3' \) must be equal to the \( h_{33} \) of \( G_{3E} \). An alternative expression for the \( U(3) \supset R(3) \) coupling coefficients is then given by
\[
\langle G_{3E}, K_3 L_3 M_3|G_1, K_1 L_1 M_1; G_2, K_2 L_2 M_2\rangle_x = \sum_{G_3' K_3' G_3} C(L_1, L_2, L_3; M_1, M_2, M_3) \\
\times C(L_1, L_2, L_3; K_3' K_2 K_1') \langle G_1'|G_1; K_1 L_1 K_1'\rangle \langle G_2'|G_2; K_2 L_2 K_2'\rangle \\
\times \langle G_3'|G_3'; G_3'\rangle \langle G_{3E}; K_3 L_3 K_3'\rangle\langle G_3\rangle.
\]
(58)

An alternative expression for the reduced \( U(3) \supset R(3) \) coupling coefficients follow directly as
\[
\langle G_{3E}, K_3 L_3 M_3|G_1, K_1 L_1; G_2, K_2 L_2\rangle_x = \sum_{G_3' K_3' G_3} C(L_1, L_2, L_3; K_1' K_2 K_3') \\
\times \langle G_1'|G_1; K_1 L_1 K_1'\rangle \langle G_2'|G_2; K_2 L_2 K_2'\rangle \langle G_3'|G_3; G_3'\rangle \langle G_{3E}; K_3 L_3 K_3'\rangle\langle G_3\rangle.
\]
(59)

Although this form for the \( U(3) \supset R(3) \) coupling coefficients is convenient for a study of their symmetry properties, the results of eqs. (51) and (52) are more convenient for computational purposes since they involve fewer sums.

5. The \( U(3) \) tensors

In nuclear physics, when working within the framework of the full invariance group of the Hamiltonian for the isotropic harmonic oscillator, it is necessary to consider the \( U(3) \) tensorial properties of physical operators. It is again convenient to consider the reduction \( U(3) \supset R(3) \) rather than the reduction \( U(3) \supset U(2) \otimes U(1) \) for tensor labeling because the \( U(3) \supset R(3) \) reduction involves the physical orbital angular momentum properties of the operators which are so important to their physical interpretation. Such a tensor labeling scheme is possible by defining in analogy with eq. (17) the generalized spherical tensors
\[
T(G; KLM) = \int d\Omega D_{MK}^{L*}(\Omega)R(\Omega)T(G)R^{-1}(\Omega),
\]
(60)

where \( T(G) \) is a \( U(3) \supset U(2) \otimes U(1) \) tensor defined by
\[
[C_{sp}, T(G)] = \sum_G \langle G'|C_{sp}|G\rangle T(G').
\]
(61)

In eq. (61), the \( C_{sp} \) are the infinitesimal generators of \( U(3) \). The requisite quantum number \( K \) resolves the multiplicity in the reduction \( U(3) \supset R(3) \) for the generalized spherical tensors in precisely the same manner as found to be true for the basis vectors.

By simply using the definition of eq. (60), it is not difficult to show that the general-
COUPLING COEFFICIENTS

ized spherical tensors possess the following properties:

\[
R(\Omega)T(G; KLM)R^{-1}(\Omega) = \sum_{K'M'} D_{M'M}^L(\Omega) T(G; KLM'), \tag{62}
\]

\[
[T(G; KLM)]^* = (-1)^{K-M} T(G; -K, L, -M), \tag{63}
\]

\[
T(G; KLM) = \sum_{K'} D_{KK'}^L(\Omega) T'(G; K'LM), \tag{64}
\]

where

\[
T'(G; KLM) = [R^{-1}(\Omega)T(G)R(\Omega)](G; KLM). \tag{65}
\]

Eq. (62) gives the transformation properties of the generalized spherical tensors under rotations. Eq. (63) relates the set of hermitian conjugate generalized spherical tensors to the set of generalized spherical tensors, and eq. (64) relates generalized spherical tensors obtained from rotated U(3) ⊇ U(2) ⊗ U(1) tensors to the generalized spherical tensors of eq. (60).

It remains to determine the matrix elements of the generalized spherical tensors in the U(3) ⊇ R(3) basis. This may be accomplished by using eq. (60) to write

\[
T(G_1; K_1 M_1 K_1') G_2; K_2 L_2 M_2) = \int d\Omega D_{M_1 K_1}^{L_1}(\Omega) R(\Omega) T(G_1) R^{-1}(\Omega) G_2; K_2 L_2 M_2)
\]

\[
= \sum_{K_2} \int d\Omega D_{M_1 K_1}^{L_1}(\Omega) D_{M_2 K_2}^{L_2}(\Omega) R(\Omega) T(G_1) G_2; K_2 L_2 K_2'). \tag{66}
\]

From the well-known R(3) Clebsch-Gordan series for the rotation matrices and the result expressed by eq. (18), we obtain

\[
T(G_1; K_1 M_1 K_1') G_2; K_2 L_2 M_2) = \sum_{L_3 K_3} C(L_1 L_2 L_3; M_1 M_2 M_3) C(L_1 L_2 L_3 K_1 K_2 K_3')
\]

\[
\times \int d\Omega D_{M_1 K_1}^{L_1}(\Omega) R(\Omega) T(G_1) G_2; K_2 L_2 K_2')
\]

\[
= \sum_{L_3 K_3} C(L_1 L_2 L_3; M_1 M_2 M_3) C(L_1 L_2 L_3; K_1 K_2 K_3') G_2; K_2 L_2 K_2')
\]

\[
\times \int d\Omega D_{M_1 K_1}^{L_1}(\Omega) R(\Omega) T(G_1) G_2; K_2 L_2 K_2'). \tag{67}
\]

Now, if we use the SU(3) ⊇ SU(2) ⊗ U(1) tensorial result

\[
T(G_1) G_2') = \sum_{G_3} \langle G_3 | T(G_1) | G_2') \rangle \langle G_3 | G_3 \rangle
\]

\[
= \sum_{G_3} \langle G_3 | G_1 \rangle \langle G_2') | G_3 \rangle \langle G_3 | T(G_1) | G_2') \rangle \langle G_3 | G_3 \rangle, \tag{68}
\]

where \( \langle G_3 | T(G_1) | G_2') \rangle \) is the reduced U(3) ⊇ U(2) ⊗ U(1) tensorial matrix element corresponding to the state \( | G_2 > \), we obtain the final results analogous to eqs. (50a)
and (50b)
\[
T(G_1; K_1 L_1 M_1) | G_2; K_2 L_2 M_2 \rangle = \sum_{a_1 b_1 c_1 d_1} C(L_1 L_2 L_3; M_1 M_2 M_3) \\
\times C(L_1 L_2 L_3; K_1 K'_1 K_1') \langle G_1' | G_2 \rangle \langle G_3 | T(G_1) | G_2' \rangle \langle G_3' | G_2 \rangle \\
\times \int d\Omega D^{a_1 a_2}_{b_1 b_2}(\Omega) R(\Omega) | G_2 \rangle \\
= \sum_{a_1 b_1 c_1 d_1} C(L_1 L_2 L_3; M_1 M_2 M_3) C(L_1 L_2 L_3; K_1 K'_1 K_1') \langle G_1' | G_2 \rangle \langle G_3 | T(G_1) | G_2' \rangle \langle G_3' | G_2 \rangle \\
\times \langle G_3 | G_1 \rangle \langle G_2' | G_2 \rangle \langle G_2 || T(G_1) || G_2' \rangle \langle G_3 | G_2 \rangle \langle G_3' | G_2 \rangle \\
\times \langle G_3 || T(G_1) || G_2' \rangle \langle G_3' | G_2 \rangle (2L_3 + 1)^{-1} \langle G_3' E; K_3 L_3 K'_3 \rangle \\
\times | G_3 E; K_3 L_3 M_3 \rangle. \tag{69a}
\]

The matrix elements of the generalized spherical tensors are then given by
\[
\langle G_3; K_3 L_3 M_3 | T(G_1; K_1 L_1 M_1) | G_2; K_2 L_2 M_2 \rangle \\
= \sum_{G_2' K_2'} C(L_1 L_2 L_3; M_1 M_2 M_3) C(L_1 L_2 L_3; K_1 K'_1 K_1') \langle G_1' | G_2 \rangle \langle G_3 | T(G_1) | G_2' \rangle \langle G_3' | G_2 \rangle \\
\times \langle G_3 | G_1 \rangle \langle G_2' | G_2 \rangle \langle G_2 || T(G_1) || G_2' \rangle \langle G_3 | G_2 \rangle \langle G_3' | G_2 \rangle \\
\times \langle G_3 || T(G_1) || G_2' \rangle \langle G_3' | G_2 \rangle (2L_3 + 1)^{-1} \langle G_3' E; K_3 L_3 K'_3 \rangle \\
\times \langle G_3; K_3 L_3 M_3 | G_3 E; K_3' L_3 M_3 \rangle. \tag{70}
\]

By using eq. (51) together with the result that \( \langle G_3' || T(G_1) || G_2' \rangle \), depends only upon the \( h_{33}^*, h_{33} \), and \( h_{33} \), we find that
\[
\langle G_3; K_3 L_3 M_3 | T(G_1; K_1 L_1 M_1) | G_2; K_2 L_2 M_2 \rangle \\
= \langle G_3 || T(G_1) || G_2 \rangle \sum_{K_3'} \langle G_3 E; K'_3 L_3 M_3 | G_1, K_1 L_1 M_1; G_2, K_2 L_2 M_2 \rangle \\
\times \langle G_3; K_3 L_3 M_3 | G_3 E; K_3' L_3 M_3 \rangle. \tag{71}
\]

The reduced matrix elements of the generalized spherical tensor follow from eq. (52) and are
\[
\langle G_3; K_3 L_3 || T(G_1; K_1 L_1) || G_2; K_2 L_2 \rangle \\
= \langle G_3 || T(G_1) || G_2 \rangle \sum_{K_3'} \langle G_3 E; K'_3 L_3 || G_1, K_1 L_1; G_2, K_2 L_2 \rangle \\
\times \langle G_3; K_3 L_3 | G_3 E; K_3' L_3 \rangle. \tag{72}
\]
where we have made use of the fact that the normalization and overlap integrals defined by eq. (44) are independent of the \( M \) projection.

Since by analogy the \( T(G_K;KLM) \) taken together with the appropriate rule for determining the \( K \)-label and the corresponding \( L \)-values form a complete set in terms of which any of the \( T(G) \) may be expanded, one need only consider the matrix elements of these spherical tensors. The corresponding simplifications in the formalism follow as a special case of the preceding development.

6. Conclusion

We have derived an explicit expression for the transformation brackets between the \( U(3) \supseteq R(3) \) states obtained by using the projection technique introduced by Elliott \(^4\) and the Gel'fand \( U(3) \supseteq U(2) \otimes U(1) \) basis states. These were then shown to play a central role in determining both \( U(3) \supseteq R(3) \) coupling coefficients in terms of \( U(3) \supseteq U(2) \otimes U(1) \) coupling coefficients and \( U(3) \supseteq R(3) \) tensorial matrix elements in terms of \( U(3) \supseteq U(2) \otimes U(1) \) tensorial matrix elements. The results are a first step in the development of a complete \( U(3) \supseteq R(3) \) tensorial algebra. Such an algebra is of course desirable if one is to use the \( U(3) \supseteq R(3) \) states to study the relationship between collective and individual particle degrees of freedom in nuclear many-particle systems.

The form of some of the intermediate results is of course familiar from the work of other authors. As pointed out, this is simply a manifestation of the fact that the projection technique introduced by Elliott \(^4\) has been used. Important special cases of some of the final results are also available in the literature \(^{25,26}\). It is important to note, however, the generality of the results we have obtained and the fact that if judiciously applied, they can result in an avoidance of a duplication of effort by nuclear and elementary particle physicists.

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Appendix A

To evaluate the integral \( I_1 \) defined by eq. (36) consider

\[
I(p, q, l) = \frac{1}{2\pi i} \int_0^{2\pi} d\phi (\cos \phi)^\delta (\sin \phi)^\delta e^{il\phi}.
\]  

(A.1)

By applying Cauchy’s residue theorem, one can obtain the result

\[
I(p, q, l) = \frac{(-i)^{\delta}}{2^{p+q}} \sum_s (-1)^s \binom{p}{s-n} \binom{q}{n},
\]  

(A.2)
where \( s = \frac{1}{2}(p+q+l) \) must be integral for \( I \) to be non-zero. Then by using the result given by Edmonds \(^{27}\) for the \( d_{m;m}^{(1)}(\phi) = d_{m;m}^{(2)}(\phi) \), one obtains

\[
I_{1}(m'jml) = (-1)^{j-m} \left[ \frac{(j+m')!(j-m')!}{(j+m)!(j-m)!} \right]^{\frac{1}{2}} \sum_{n} (-1)^{n} \left( \begin{array}{c}
\frac{j-m}{n} \\
\frac{j+m}{n}
\end{array} \right) \sum_{n} (-1)^{n} \left( \begin{array}{c}
\frac{j-m}{n} \\
\frac{j+m}{n}
\end{array} \right) \\
\times \int_{0}^{2\pi} d\phi (\cos \phi)^{2j+m+n} (\sin \phi)^{2j-m-m'-2n} e^{i\phi}
\]

\[
= (-1)^{j-m} \left[ \frac{(j+m')!(j-m')!}{(j+m)!(j-m)!} \right]^{\frac{1}{2}} \sum_{n} (-1)^{n} \left( \begin{array}{c}
\frac{j-m}{n} \\
\frac{j+m}{n}
\end{array} \right) \\
\times I(2j+m+m', 2j-m-m'-2n, l). \tag{A.3}
\]

**Appendix B**

To evaluate the integral \( I_{2} \) defined by eq. (37) consider

\[
I(p, q) = \frac{1}{2} \int_{0}^{2\pi} d\phi (\cos \phi)^{p} (\sin \phi)^{q} \tag{B.1}
\]

In terms of gamma functions, one has \(^{28}\)

\[
I(p, q) = \frac{\Gamma \left( \frac{p+1}{2} \right) \Gamma \left( \frac{q+1}{2} \right)}{4\Gamma \left( \frac{p+q+2}{2} \right)} \tag{B.2}
\]

Then by once again using the results given by Edmonds \(^{27}\), one obtains

\[
I_{2}(m'jmn'I) = (-1)^{j-m+i-s} \left[ \frac{(j+m')!(j-m')!(l+n')!(l-n)!}{(j+m)!(j-m)!(l+n)!(l-n)!} \right]^{\frac{1}{2}} \\
\times \sum_{r} (-1)^{r} \left( \begin{array}{c}
\frac{j-m}{r} \\
\frac{j+m}{r}
\end{array} \right) \sum_{t} (-1)^{t} \left( \begin{array}{c}
\frac{l-n}{t} \\
\frac{l+n}{t}
\end{array} \right) \\
\times \frac{1}{2} \int_{0}^{2\pi} d\phi (\cos \phi)^{2j+m+n'} (\sin \phi)^{2j-m-m'-2s+1} (\cos \frac{1}{2}\phi)^{2j+s+n'} (\sin \frac{1}{2}\phi)^{2j-n-n'-2t} \\
\times \sum_{r} (-1)^{r} (2j+m+m') \left[ \frac{(l+m)!(l-m')!(l+n)!(l-n)!}{(l+m)!(l-m)!(l+n)!(l-n)!} \right]^{\frac{1}{2}} \\
\times \sum_{r} (-1)^{r} \left( \begin{array}{c}
\frac{j-m}{r} \\
\frac{j+m}{r}
\end{array} \right) \sum_{t} (-1)^{t} \left( \begin{array}{c}
\frac{l-n}{t} \\
\frac{l+n}{t}
\end{array} \right) \\
\times \sum_{r} (-1)^{r} \left( \begin{array}{c}
2s+m+m' \end{array} \right) \times \left[ \frac{(j+s+l-r)+m+m'+n+n'+1, 2(j-s+l-t+r)-m-m'-n-n'+1} \right]. \tag{B.3}
\]
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