LETTER TO THE EDITOR

Connection between macroscopic and microscopic Hamiltonians for nuclear rotational motion

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Abstract. Analytic expressions are given for mapping between the macroscopic Hamiltonian of an initially symmetric rotor and a microscopic SU(3) × SO(3) theory with both interaction. No special approximations are made regarding the nature of the rotation; it could be rigid-rotor, a symmetric-top or an intermediate form. Under the mapping, invariants of the collective model are carried over into Casimir invariants of the algebraic symmetry. The results are applicable to both axially and fermion quadrupole of the SU(3) × SO(3) system. The effectiveness of the mapping is illustrated by a comparison of associated spectra for ground and excited band states of 115/Er.

A long-time goal of nuclear theory has been to provide a shell-model interpretation of nuclear collective phenomena [1]. Algebro-group theories such as the interacting boson model (IBM) [2] and the microscopic collective model (MCN) [3] have been developed with this objective in mind for they are built around operators that represent fundamental collective modes of excitation in nuclei. Both of these models, as well as the SU(3) model of Elliott [4] which works for light nuclei and its extension, the so called pseudo-SU(3) model, which applies for heavier deformed systems [5], contain a root SU(3) × SO(3) group-structure; see table 1. That structure has been associated with rotational motion in each of these models, and for each there is a prescription for determining the SU(3) content of the model space and the representations of SU(3) that are expected to be the dominant ones in a description of the low-lying states of a system.

Here, in an attempt to gain a deeper understanding of low rotational behaviour underlying in such theories, we make the extreme simplifying assumption that SU(3) is a good symmetry. That is, we shall assume that the most important features of the low-lying states of systems whose spectra show rotational characteristics can be found in a subspace of the full space consisting of the single leading representation of SU(3). It follows from this that the corresponding effective interaction can be expressed solely in terms of the generators of SU(3). In reality, of course, there will be representation mixing (induced by terms in the full-space interaction that break the SU(3) symmetry). Indeed, calculations which take this into account have been carried out for both the IBM and MCN realizations of the SU(3) × SO(3).

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Table 1. Shell-model states which share the SU(3)×SO(3) root structure.

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<th>Boson</th>
<th>Fermion</th>
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<tbody>
<tr>
<td>SU(3)</td>
<td>SO(3)</td>
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<tr>
<td>U(6)</td>
<td>S(3,0)</td>
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<td>SO(3)</td>
<td>SO(3)</td>
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Our purpose here is to not distract from the significance of those efforts; rather, it is to bare more fully the beauty and simplicity of the SU(3)×SO(3) algebras by showing explicitly how it supports the dynamics of rotations. The picture that emerges is very simple: whenever a model space can be partitioned into representations of SU(3), regardless of the structure in which it is embedded or whether the statistics are those of a boson or fermion system, and a simple linear combination of the SU(3)×SO(3) integrity basis set is representative of a major part of the interaction, the dynamics will be that of a collection of interacting rotes.

The SU(3) algebra is generated by eight elements: three angular momentum operators associated with the SO(3) subgroup, \( L_z \); \( \mu = 0, \pm 1 \), and five quadrupole operators, \( Q \); \( \mu = 0, \pm 1, \pm 2 \). Now the Hamiltonian of a system that is invariant under spatial rotations must be an SO(3) scalar operator. We are therefore interested in the SO(3) scalars that can be built out of the eight SU(3) generators. This is a classic problem of group theory that is addressed, for example, in the work of Molten, Nother, and Weyl [8]. The solution is that all the SO(3) scalar operators that can be built out of the generators of SU(3) can be expressed as simple polynomial functions of a finite subset of the SU(3)×SO(3) scalars.

That set of basic scalars is called the SU(3)×SO(3) integrity basis. It is known to consist of six operators [9], which can be chosen to be the second- and third-order Casimir invariants of SU(3), which we label \( C_2 \) and \( C_3 \), respectively, the Casimir invariant of SO(3), which is \( L^2 \), and three non-Casimir invariant SO(3) scalar operators: one of degree three in the generators of SU(3), which we label \( X_3 \), another of degree four which we label \( X_4 \), and an antisymmetrization of degree six that can be represented by the commutator of \( X_3 \) and \( X_4 \), \( X_3 \). The square of \( X_3 \), which is necessarily hermitian, has a polynomial representation in terms of the other integrity basis operators, so in any polynomial expansion \( X_3 \) should only enter linearly.

Indeed, since it is an antisymmetric operator, it can be eliminated altogether if time reversal invariance applies.

Since the Casimir invariants of SU(3) act as multiples of the identity within a single representation of SU(3), our interaction can be written as a polynomial function of the operators \( L^2, X_3, X_4 \). In a previous paper [10] we presented numerical results which showed that there exists a fourth-order SU(3)×SO(3) integrity basis interaction that reproduces the eigenvalue spectrum of an axially symmetric rotor Hamiltonian,

\[
\frac{1}{2} L^2 \frac{1}{2} L^2 + \frac{1}{2} X_3^2 + \frac{1}{2} X_4^2 + \frac{1}{2} X_3 X_4 + a L^4.
\]  

Actually, as will be shown below, the mapping can be established without including the
Consider \( Q'^2 \) to be the mass quadrupole moment of a deformed system. (The superscript 'c' will be appended, as necessary, to denote collective-model operators.) In a principal-axis system \( Q'^2 \) is diagonal with cartesian components \( \lambda_{\alpha \beta}; \alpha, \beta = 1, 2, 3 \):

\[
\lambda_{\alpha \beta} = \int (3x_\alpha x_\beta - \delta_{\alpha \beta}) \rho(\theta) \, d\theta \quad \text{under} \quad \lambda_{\alpha \beta} \rho_{\alpha \beta}.
\]

(2)

Since \( Q'^2 \) is traceless, the \( \lambda_{\alpha \beta} \) satisfy the constraint relation \( \lambda_1 + \lambda_2 + \lambda_3 = 0 \), which further implies that \( 2\lambda_2 - \lambda_1 - \lambda_2 - \lambda_3 \) and \( \lambda_2 \lambda_3 = \lambda_1 (\lambda_1 - \lambda_2 - \lambda_3) + 2\lambda_2 \). We can now write down the collective-model analogues of the integrity basis operators,

\[
\begin{align*}
I_2 & = \sum_{a,b} I_a I_b = I_a I_a + \lambda_1 I_1^2 + \lambda_2 I_2^2 + \lambda_3 I_3^2, \\
X_1 & = \sum_{a,b} I_a Q_{ab} I_b = \lambda_1 I_1^2 + \lambda_2 I_2^2 + \lambda_3 I_3^2, \\
X_2 & = \sum_{a,b} I_a Q_{ab} I_b = 2I_2^2 + \lambda_1 I_1^2 + \lambda_2 I_2^2.
\end{align*}
\]

(3)

Here \( I_a \) denotes the \( a \)th component of the angular momentum in the principal-axis system. The last form given for \( X_1 \) and \( X_2 \) follows because they are rotational scalars and can therefore be evaluated without loss of generality in the principal-axis system. It follows that there is a combination of \( X_1 \) and \( X_2 \) that is a linear function of \( I_1^2 \) and \( I_2^2 \):

\[
X_1 + \frac{1}{\lambda_2} X_2 = \frac{\lambda_3 - \lambda_2}{\lambda_2} I_1^2 + \frac{\lambda_1 - \lambda_2}{\lambda_2} I_2^2.
\]

(4)

Alternatively, in terms of the shape parameters \( \beta \) and \( \gamma \) of the hydrodynamic model,

\[
X_1 + \frac{3}{\beta \cos \gamma} X_2 = -\frac{\beta (\cos^2 \gamma - 1)}{3 \cos \gamma} I_1^2 + \frac{\beta (\cos^2 \gamma - 1)}{3 \cos \gamma} I_2^2
\]

(5)

where \( \langle Q_1 \rangle = \beta \cos \gamma, \langle Q_2 \rangle = 0 \) and \( \langle Q_3 \rangle = (1/\sqrt{3}) \beta \) sin \( \gamma \).

These results can be used to establish a mapping between \( H_{\text{RO}} \) and \( H_{\text{RO}} \). But first, in order to establish a relationship between the \( \lambda_{\alpha \beta} \) and the representation labels \( \alpha \) and \( \beta \) of \( SU(3) \), note that under the Elliott prescription for projecting states of good angular momentum from the intrinsic highest weight \( SU(3) \) state,

\[
\langle Q_1 \rangle = \langle N_1 - N_2 \rangle \sim 2\lambda + \mu
\]

and

\[
\langle Q_2 \rangle = \langle N_1 - N_2 \rangle \sim 2\mu.
\]

(6)

In (6), \( N_\alpha \) counts the number of oscillator quanta in the \( \alpha \)th direction. Since \( \langle x_\alpha \rangle \sim \langle N_\alpha \rangle \)
for the oscillator, it follows that \( \langle N_a \rangle \sim \lambda_a \) and, accordingly,

\[
\lambda_1 = k(-\lambda + \mu) + \sigma_1
\]

\[
\lambda_2 = k(-\lambda + 2\mu) + \sigma_2
\]

\[
\lambda_3 = k(\lambda + \mu) + \sigma_3
\]

(7)

In (7) the \( \sigma_a \) are constants that satisfy \( \sigma_1 + \sigma_2 + \sigma_3 = 0 \). To fix the \( \sigma_a \) and hence uniquely specify the relationship between the \( \lambda_a \) and \( \lambda \) and \( \mu \), we require that the collective-model invariants \( \text{tr}[Q^2] \) and \( \text{tr}[Q^3] \) go over into invariants of the SU(3) algebra. For this \( \sigma_1 = 0, \sigma_2 = -1, \sigma_3 = 1 \), in which case

\[
\text{tr}[Q^2] = A_1^2 - A_2^2 + A_3^2 - 3(A_1^2 + A_2^2 + A_3^2) \equiv |C_2| + 2
\]

(8)

\[
\text{tr}[Q^3] = A_1^3 + A_2^3 + A_3^3 - 3(A_1 A_2 A_3) \equiv |C_3|
\]

The microscopic equivalent of (4), above, can now be determined. However, care must be exercised in relating the cartesian operators of the rotational model to the spherically coupled tensor operators of the integrity basis: \( X_2 = \sqrt{\frac{6}{10}} X^2 \) and \( X_4 = \sqrt{\frac{2}{5}} X^4 \). Taking

![Figure 1](image-url)

**Figure 1.** Experimental and theoretical spectra for the ground and giamik level states of 58Fe. The eigenvalue spectrum labelled S is for \( H_{ext} \) (1) with the parameters \( \alpha = 1 \) and \( \beta = 2 \) determined from the macroscopic-microscopic correction formula (11) with \( \lambda = 30, 8 \), which is the needed representation as a pseudo SU(5) description of 58Fe. These results are indicative of what can be expected under the simplest application of the theory. Best fit results, which verify the correction formula, show to about the 1% level, yield spectra which for all practical purposes are identical to the experimental one.
these differences into account we finally have

\[ X_0 = \frac{\sqrt{10}}{2(2\lambda + \mu + 3)} \quad X = \frac{2(\lambda - \mu)(\lambda + 2\mu + 3)}{\sqrt{10}(2\lambda + \mu + 3)} \quad Y = \frac{18(\lambda + 1)(\lambda + \mu + 2)}{\sqrt{10}(2\lambda + \mu + 3)} \quad T. \]  

(9)

It follows from this that \( H_{R(3)} \) will be the image of \( H_{SU(3)} \) when

\[ a = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \quad b = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \quad c = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \]

(10)

As a simple application we note that for \(^{169}\)Er the \( 2^+ \) state lies at an excitation energy of 79.3 keV above the \( 0^+ \) ground state, which translates into a value for \( 1/2, \frac{1}{2} \) of 13.3 keV. The \( 2^+ \) state lies at an excitation energy of 821.1 keV. This yields a value of 185.3 keV for \((1/2, 1/2, 1/2)\). The leading pseudo-SU(3) representation for \(^{169}\)Er is \((2, 2, 2)\). Using (10) one finds the values 31.2, 1.86 and \(-0.042\) keV, respectively, for \( a, b \) and \( c \). In figure 1, the eigenvalues of \( H_{R(3)} \) and \( H_{SU(3)} \) are compared opposite results for the experimental spectrum [11]. It is clear that, as improved agreement with experiment could be obtained by fine-tuning the parameters of each theory. Indeed, if the parameters are assigned their best-fit values, the resulting spectra, for all practical purposes, are identical to the experimental ones [12]. And for those best-fit parameter values, the connection formulas (10) are verified down to about the 1% level. It is important to note how accurately the mapping reproduces the K-band splitting, a feature that has always presented practitioners of SU(3) shell-model theories with one of their greatest challenges. These results for \(^{169}\)Er are indicative of what should be expected in other cases when the macroscopic–microscopic connection formulae are used.

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References