Complementary group resolution of the SU(3) outer multiplicity problem. II. Recoupling approach for SU(3)⊃U(2) reduced Wigner coefficients

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A general procedure for the derivation of SU(3)⊃U(2) reduced Wigner coefficients (RWCs) for the coupling (λ₁μ₁)×(λ₂μ₂)↓(λμ), where η is the outer multiplicity label required in the decomposition, is proposed based on a recoupling approach that follows the complementary group technique for a resolution of the outer multiplicity of SU(n) introduced in Part I of this series. RWCs of SU(n) are not unique under a canonical resolution of the outer multiplicity; the transformation from one set to another are elements of SO(m), where m is the number of occurrences of the (λμ) irrep in the decomposition (λ₁μ₁)×(λ₂μ₂)↓(λμ). A special resolution of the multiplicity is identified that leads to a recursive procedure for the determination of RWCs. New features of these special RWCs and differences from those obtained with other choices are discussed. The method can be applied to the derivation of general SU(n) Wigner or RWCs. Algebraic expressions for another kind of RWCs, the so-called reduced auxiliary Wigner coefficients for SU(3)⊃U(2), are also obtained. © 1998 American Institute of Physics.

I. INTRODUCTION

Wigner coefficients (WCs) or reduced Wigner coefficients (RWCs) of SU(3) in the SU(3)⊃U(2) reduction have been discussed by many authors, including, for example, Biedenharn et al. using a canonical unit tensor operator method, Moshinsky et al. using an infinitesimal generator approach along with a complementary group prescription, and most recently Ališauskas et al. using a symmetric group approach and paracanonical and pseudocanonical coupling schemes. Certain classes of RWCs have been considered by Hecht, Resnikoff, Shelepin and Karasev, Klimyk and Gavrilik, Le Blanc and Rowe, and others. Among these approaches, only the unit tensor operator method is canonical in the sense that it leads directly to the usual RWC orthogonality relations. Other methods for labeling the outer multiplicity are noncanonical, i.e., the RWCs produced are in general nonorthogonal with respect to the outer multiplicity label. In this case a Gram–Schmidt process must be adopted, which then depends upon an arbitrary choice of order for carrying out the orthogonalization. In this case a numerical algorithm is required such as the one used by Draayer and Akiyama or Kaeding and Williams. In Ref. 33, RWCs associated with the 27-plet operator, which has a maximum multiplicity of three, are studied in full detail. Recently, Parkash and Sharatchandra worked out an algebraic formula for general WC for SU(3) in the canonical basis. However, the final result, which involves a summation over 33 variables with some restrictions, requires a separate normalization procedure. This means the scheme is neither algebraically or numerically easy to implement.

In this paper, the complementary group representation proposed in (I) is used to determine RWCs of SU(3)⊃U(2) from known multiplicity-free RWCs through simple recoupling procedures. This new approach, in principle, is labeling scheme independent. For example, the RWCs of SU(3)⊃U(2) in the scheme proposed by Biedenhan et al. can be derived using the present method. However, values of the RWCs with different canonical outer multiplicity labeling
II. RELATIONS AMONG DIFFERENT CHOICES FOR THE MULTIPURITY LABELS

Let \([\lambda]\xi, \rho)\) be coupled basis vectors of \(U(n) \times U(n) \times U(n)\), where \(\xi\) denotes multiplicity labels that are needed in the decomposition \(\lambda_1 \times [\lambda_2, \lambda]\), and \(\rho\) denotes sublabels of the resultant irrep \([\lambda]\). The completeness condition for \([\lambda]\xi, \rho)\) is

\[
\sum_{\xi, \rho} |\lambda\xi, \rho)\rangle \langle \lambda\xi, \rho| = 1. \tag{2.1}
\]

The choice for the multiplicity labels is not unique. If there are \(m\) distinct \(\xi\) values, denoted \(\xi = \xi_1, \xi_2, \ldots, \xi_m\), the transformation from one scheme to another is given by

\[
|\lambda\eta, \rho) = \sum_{\xi} y(\xi, \eta) |\lambda\xi, \rho)\langle, \tag{2.2}
\]

where \(\eta\) denotes another set of multiplicity labels for \(U(n)\) and \(y(\xi, \eta)\) is an element of an SU\((m)\) transformation between the two. This can be seen by noting that

\[
\sum_{\xi} y(\xi, \eta)y^*(\xi, \eta') = \delta_{\eta\eta'}, \tag{2.3}
\]

\[
\sum_{\eta} y(\xi, \eta)y^*(\xi', \eta) = \delta_{\xi\xi'}. \tag{2.3}
\]

The \(\{y(\xi, \eta)\}\) define a unitary transformation of the basic representation of SU\((m)\). A unitary transformation \(Y \in SU(m)\), where \(m\) is the number of occurrences of \([\lambda]\) in the decomposition \(\lambda_1 \times [\lambda_2]\), that transforms from one set of outer multiplicity labels to another always exists. Usually, the WCs of \(U(n)\) are taken to be real in which case the internal symmetry group for the transformation of the outer multiplicity labels can then be chosen to be SO\((m)\). It is clear that the dimension of the transformation group is the number of occurrences of the resultant irrep \([\lambda]\) in the decomposition \(\lambda_1 \times [\lambda_2]\), and the basis vectors of \([\lambda]\) remain orthonormal with respect to one another under the transformation. This freedom accounts for the fact that there exists different forms for RWCs or WCs with respect to the outer multiplicity labels. In what follows, RWCs are considered to be real.

Although the discussion will now be restricted to SU\((3) \times U(2)\), it can be extended easily to the general SU\((n)\) case. The common notation for SU\((3)\) irreps \((\lambda\mu) = [\lambda + \mu, \mu]\) will also be used, where \([\lambda + \mu, \mu]\) is a usual two-rowed label corresponding to the irrep described by the two-rowed Young diagram with \(\lambda + \mu\) boxes in the first row, and \(\mu\) boxes in the second row. In (I) it was shown that there is only one multiplicity label needed in the decomposition \((\lambda_1, \mu_1) \times (\lambda_2, \mu_2)\) \([m_1m_2m_3]\). The multiplicity is labeled \(\xi\) with \(\xi = \xi_1, \xi_2, \ldots, \xi_m\), where \(m\) is the number of occurrences of \([m_1m_2m_3]\) in the decomposition.

The key relation in our evaluation of SU\((3)\) WCs or RWCs is the transformation
\[
\sum_\xi \begin{pmatrix}
\lambda_1 \mu_1 \\
\rho_1
\end{pmatrix}
\begin{pmatrix}
\lambda_2 \mu_2 \\
\rho_2
\end{pmatrix}
\begin{pmatrix}
\xi [m_1 m_2 m_3] \\
\rho
\end{pmatrix}
y(\xi, \eta) = \begin{pmatrix}
\lambda_1 \mu_1 \\
\rho_1
\end{pmatrix}
\begin{pmatrix}
\lambda_2 \mu_2 \\
\rho_2
\end{pmatrix}
\begin{pmatrix}
\eta [m_1 m_2 m_3] \\
\rho
\end{pmatrix},
\tag{2.4}
\]

where \(\rho_1, \rho_2,\) and \(\rho\) are sublabels of SU(3) if

\[
\begin{pmatrix}
\lambda_1 \mu_1 \\
\rho_1
\end{pmatrix}
\begin{pmatrix}
\lambda_2 \mu_2 \\
\rho_2
\end{pmatrix}
\begin{pmatrix}
\xi [m_1 m_2 m_3] \\
\rho
\end{pmatrix}
\]

is a WC or if a RWC then the corresponding U(2) labels, and the \(y(\xi, \eta)\) are a special set of matrix elements of SO(m) chosen in a manner that will now be specified.

Assume

\[
\begin{pmatrix}
\lambda_1 \mu_1 \\
\rho_1
\end{pmatrix}
\begin{pmatrix}
\lambda_2 \mu_2 \\
\rho_2
\end{pmatrix}
\begin{pmatrix}
\xi [m_1 m_2 m_3] \\
\rho
\end{pmatrix}, \quad \xi = \xi_1, \xi_2, \ldots, \xi_m, \tag{2.5}
\]

is a set of WCs satisfying the orthogonality relation

\[
\sum_{\rho_1 \rho_2} \begin{pmatrix}
\lambda_1 \mu_1 \\
\rho_1
\end{pmatrix}
\begin{pmatrix}
\lambda_2 \mu_2 \\
\rho_2
\end{pmatrix}
\begin{pmatrix}
\xi [m_1 m_2 m_3] \\
\rho
\end{pmatrix}
\begin{pmatrix}
\lambda_1 \mu_1 \\
\rho_1
\end{pmatrix}
\begin{pmatrix}
\lambda_2 \mu_2 \\
\rho_2
\end{pmatrix}
\begin{pmatrix}
\xi' [m_1' m_2' m_3'] \\
\rho'
\end{pmatrix} = \delta_{m m'} \delta_{\xi \xi'} \delta_{\rho \rho'}.
\tag{2.6a}
\]

or RWCs satisfying

\[
\sum_{\xi \eta m_i} \begin{pmatrix}
\lambda_1 \mu_1 \\
\rho_1
\end{pmatrix}
\begin{pmatrix}
\lambda_2 \mu_2 \\
\rho_2
\end{pmatrix}
\begin{pmatrix}
\xi [m_1 m_2 m_3] \\
\rho
\end{pmatrix}
\begin{pmatrix}
\lambda_1 \mu_1 \\
\rho_1
\end{pmatrix}
\begin{pmatrix}
\lambda_2 \mu_2 \\
\rho_2
\end{pmatrix}
\begin{pmatrix}
\xi' [m_1' m_2' m_3'] \\
\rho'
\end{pmatrix} = \delta_{\rho_1 \rho_1'} \delta_{\rho_2 \rho_2'},
\tag{2.6b}
\]

or

\[
\sum_{\rho_1 \rho_2} \begin{pmatrix}
\lambda_1 \mu_1 \\
\rho_1
\end{pmatrix}
\begin{pmatrix}
\lambda_2 \mu_2 \\
\rho_2
\end{pmatrix}
\begin{pmatrix}
\xi [m_1 m_2 m_3] \\
\rho
\end{pmatrix}
\begin{pmatrix}
\lambda_1 \mu_1 \\
\rho_1
\end{pmatrix}
\begin{pmatrix}
\lambda_2 \mu_2 \\
\rho_2
\end{pmatrix}
\begin{pmatrix}
\xi' [m_1' m_2' m_3'] \\
\rho'
\end{pmatrix} = \delta_{m m'} \delta_{\xi \xi'}.
\tag{2.6d}
\]

According to the Schur–Weyl duality relation, the RWCs for SU(3) ⊕ U(2) given by (2.5) are also RWCs for SU(4) ⊕ U(3) for the same coupling with the same set of outer multiplicity labels \(\{\xi_i\}\). Also, according to the complementary group technique outlined in (I), the complementary group for the SU(3) coupling is \(U(4)\). Therefore, consider the same coupling of \(U(4)\) in the special Gel’fand basis assigned according to the Littlewood rule, namely,

\[
\begin{pmatrix}
\lambda_1 + \mu_1, \mu_1 \\
\lambda_2 + \mu_2, 0
\end{pmatrix}
\begin{pmatrix}
\xi [m_1 m_2 m_3] \\
\lambda_1 + \mu_1, \lambda_2 + \mu_2
\end{pmatrix}
\]

\[
= \eta_{\min} \leq \eta \leq \eta_{\max}
\tag{2.8a}
\]

where

\[
\eta_{\min} = \min(m_1 - \lambda_1 - \mu_1, \mu_2, m_2 - \mu_1, \lambda_2 + \mu_2 - m_3, \mu_1 + \mu_2 - m_3, m_2 - m_3),
\tag{2.8b}
\]

Next recall that there is an \(m \times m\) orthonormal matrix \(Y\) that transforms the RWCs or WCs between two sets of multiplicity labels \(\xi\) and \(\eta\), where the range of \(\xi\) and \(\eta\) is the same as the sublabels \(\eta\) in the SU(3) subreps \([m_1, m_2 - \eta, m_3 - \mu_2 + \eta]\). Specifically, in (I) it is shown that the number of occurrences of \([m_1, m_2, m_3]\) in the Kronecker product \([\lambda_1 + \mu_1, \mu_1] \times [\lambda_2 + \mu_2, 0]\) can be described exactly by \(\eta\) within the range

\[
\eta_{\min} \leq \eta \leq \eta_{\max}
\tag{2.8a}
\]

where

\[
\eta_{\max} = \min(m_1 - \lambda_1 - \mu_1, \mu_2, m_2 - \mu_1, \lambda_2 + \mu_2 - m_3, \mu_1 + \mu_2 - m_3, m_2 - m_3),
\tag{2.8b}
\]
\[ \eta_{\text{min}} = \max(0, \mu_2 - m_3, m_2 - \lambda_1 - \mu_1). \]

Thus, we require

\[
\sum_{\xi} y(\xi, \eta) \left[ \begin{array}{c} [\lambda_1 + \mu_1, \mu_1] \\ [\lambda_1 + \mu_2, \mu_2] \\ [\lambda_2 + \mu_2, \mu_2] \\ [\lambda_2 + \mu_2, 0] \\ [m_1, m_2 - \eta_3, m_3 - \mu_2 + \eta] \\ [m_1, m_2 - \eta_3, m_3 - \mu_2 + \eta] \\ [m_1, m_2 - \eta_3, m_3 - \mu_2 + \eta] \end{array} \right] \xi \left[ m_1 m_2 m_3 \right] = \left[ \begin{array}{c} \bar{\eta}[m_1 m_2 m_3] \\ \bar{\eta}[m_1 m_2 m_3] \\ \bar{\eta}[m_1 m_2 m_3] \end{array} \right],
\]

where \( \bar{\eta}_i = \eta_1, \eta_2, \ldots, \eta_m \), and the prime indicates a new RWC.

The RWCs,

\[
\left( \begin{array}{c} \xi \\ \eta \end{array} \right) = \left( \begin{array}{c} [\lambda_1 + \mu_1, \mu_1] \\ [\lambda_2 + \mu_2, \mu_2] \\ [\lambda_2 + \mu_2, 0] \\ [m_1, m_2 - \eta_3, m_3 - \mu_2 + \eta] \end{array} \right) \xi \left[ m_1 m_2 m_3 \right],
\]

with fixed \( \eta \) can be regarded as a vector in \( \mathbb{R}^m \) space.

\[
\left( \begin{array}{c} \xi \\ \eta \end{array} \right), \quad \xi = \xi_1, \xi_2, \ldots, \xi_m.
\]

Now consider a special transformation,

\[
Y \left( \begin{array}{c} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_m \\ \eta_2 \\ \eta_3 \\ \eta_m \\ \eta_2 \\ \eta_3 \\ \eta_m \end{array} \right) = \left( \begin{array}{c} \eta_1 \\ 0 \\ \vdots \\ 0 \end{array} \right),
\]

so all of the components of the old vectors \( \left( \xi / \eta_1 \right) \) can be expressed by the following relation:

\[
\left( \begin{array}{c} \xi \\ \eta_1 \end{array} \right) = y(\xi, \eta) \left( \begin{array}{c} \eta_1 \\ \eta_1 \end{array} \right).
\]

The other \( m - 1 \) vectors \( \left( \xi / \eta_i \right), \quad i = 2, 3, \ldots, m \), undergo the same transformation. The complete transformation is specified by requiring

\[
Y \left( \begin{array}{c} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_m \\ \eta_2 \\ \eta_3 \\ \eta_m \\ \eta_2 \\ \eta_3 \\ \eta_m \end{array} \right) = \left( \begin{array}{c} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_m \end{array} \right), \quad Y \left( \begin{array}{c} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_m \\ \eta_2 \\ \eta_3 \\ \eta_m \\ \eta_2 \\ \eta_3 \\ \eta_m \end{array} \right) = \left( \begin{array}{c} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_m \end{array} \right), 
\]

\[
\ldots.
\]

\[
(2.14)
\]
\[
\begin{pmatrix}
\frac{\eta_1}{\eta_{m-1}} \\
\frac{\eta_2}{\eta_{m-1}} \\
\vdots \\
\frac{\eta_m}{\eta_{m-1}} \\
0
\end{pmatrix} = \begin{pmatrix}
\frac{\eta_1}{\eta_{m-1}} \\
\frac{\eta_2}{\eta_{m-1}} \\
\vdots \\
\frac{\eta_m}{\eta_{m-1}} \\
0
\end{pmatrix} = \begin{pmatrix}
\frac{\eta_1}{\eta_{m-1}} \\
\frac{\eta_2}{\eta_{m-1}} \\
\vdots \\
\frac{\eta_m}{\eta_{m-1}} \\
0
\end{pmatrix},
\]

where the zero components (the special RWCs) have clearly been written out after the transformation. Other components are in general nonzero.

In the following, an example for the \( m = 3 \) case is given to show that such a transformation is always possible. In the \( m = 3 \) case, one can make the following special rotation \( A_1 \) around the third axis such that

\[
\begin{pmatrix}
\cos \theta_1 & -\sin \theta_1 \\
\sin \theta_1 & \cos \theta_1
\end{pmatrix}
\begin{pmatrix}
\frac{\eta_1}{\eta_1} \\
\frac{\eta_2}{\eta_1} \\
\frac{\eta_3}{\eta_1}
\end{pmatrix}
= \begin{pmatrix}
\frac{\eta_1}{\eta_1} \\
0 \\
0
\end{pmatrix},
\]

\[
\begin{pmatrix}
\cos \theta_1 & -\sin \theta_1 \\
\sin \theta_1 & \cos \theta_1
\end{pmatrix}
\begin{pmatrix}
\frac{\eta_1}{\eta_2} \\
\frac{\eta_2}{\eta_2} \\
\frac{\eta_3}{\eta_2}
\end{pmatrix}
= \begin{pmatrix}
\frac{\eta_1}{\eta_2} \\
\frac{\eta_2}{\eta_2} \\
0
\end{pmatrix}, \tag{2.15}
\]

\[
\begin{pmatrix}
\cos \theta_1 & -\sin \theta_1 \\
\sin \theta_1 & \cos \theta_1
\end{pmatrix}
\begin{pmatrix}
\frac{\eta_1}{\eta_3} \\
\frac{\eta_2}{\eta_3} \\
\frac{\eta_3}{\eta_3}
\end{pmatrix}
= \begin{pmatrix}
\frac{\eta_1}{\eta_3} \\
\eta_2 \\
\frac{\eta_3}{\eta_3}
\end{pmatrix}.
\]

Then, make another rotation \( A_2 \) around the second axis with

\[
\begin{pmatrix}
\cos \theta_2 & -\sin \theta_2 \\
0 & 1 \\
\sin \theta_2 & \cos \theta_2
\end{pmatrix}
\begin{pmatrix}
\frac{\eta_1}{\eta_1} \\
\frac{\eta_2}{\eta_1} \\
\frac{\eta_3}{\eta_1}
\end{pmatrix}
= \begin{pmatrix}
\frac{\eta_1}{\eta_1} \\
0 \\
0
\end{pmatrix}.
\]
Finally, make a special rotation $A_3$ around the first axis with

$$
\begin{bmatrix}
1 & 0 & 0 \\
\cos \theta_3 & -\sin \theta_3 & 0 \\
\sin \theta_3 & \cos \theta_3 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\eta_1 \\
\eta_2 \\
\eta_3 \\
\end{bmatrix}
=
\begin{bmatrix}
\eta_1 \\
\eta_2 \\
\eta_3 \\
\end{bmatrix},
$$

(2.16)

It should be clear that the angle $\theta_1$ is fixed by $\langle \eta_1 / \eta_1 \rangle$ and $\langle \eta_2 / \eta_1 \rangle$, $\theta_2$ by $\langle \eta_1 / \eta_1 \rangle'$ and $\langle \eta_3 / \eta_1 \rangle$, and $\theta_3$ by $\langle \eta_2 / \eta_2 \rangle''$ and $\langle \eta_3 / \eta_2 \rangle''$. One can extend this procedure to the $m$-dimensional space and prove that there are indeed unique solutions for $m$ rotational angles if the final form (2.14) is selected.

It is important to note that the choice of the rotation is not unique because there is an infinite number of solutions for RWCs or WCs with outer multiplicity, all related through SO($m$) transformations. However, once the special rotation given in (2.14) is chosen, the resolution of the outer multiplicity is fixed; there remains no other arbitrariness in specifying the RWCs with multiplicity except for an overall phase factor. We will now show that the overall phase factor can be chosen as

$$
Y = A_3 A_2 A_1,
$$

(2.18)
for $i = 1, 2, \ldots, m$. The RWCs are then uniquely determined.

It is easy to prove that the orthogonality conditions for the WCs or RWCs given by (2.6) remain valid after the transformation for both the $\mathcal{U}(4) \triangleright \mathcal{U}(3)$ and $SU(3) \triangleright U(2)$ cases. First of all note that the $\mathcal{U}(4) \triangleright \mathcal{U}(3)$ or $SU(3) \triangleright U(2)$ RWCs all undergo the same transformation $Y$. One can easily verify that the orthogonality conditions are valid for the $\mathcal{U}(4) \triangleright \mathcal{U}(3)$ case. Since, according to the Schur–Weyl duality relation, $SU(3)$ WCs or $SU(3) \triangleright U(2)$ RWCs are a subset of those of $\mathcal{U}(4)$ or $\mathcal{U}(4) \triangleright \mathcal{U}(3)$, the same conclusion applies to WCs of $SU(3)$ or RWCs of $SU(3) \triangleright U(2)$ as well.

Although the orthogonality conditions survive, other properties such as Biedenharn’s $1 \leftrightarrow 2$ exchange symmetry, is lost in this new scheme. Questions of this type will be considered further in Sec. IV.

Finally, we want to show what is actually achieved by the special rotation $Y$. If the RWCs of $\mathcal{U}(4) \triangleright \mathcal{U}(3)$ in terms of an $m \times m$ matrix are arranged such that the columns are set by the outer multiplicity label $\eta = \eta_1, \eta_2, \ldots, \eta_m$ and the rows by the label $\tilde{\eta}$ in the (irrep) $[m_1, m_2 - \eta, m_3 - \mu_2 + \eta]$ for $\mathcal{U}(3)$, the RWCs have the following structure:

$$
\begin{pmatrix}
   \times & 0 & \cdots & \cdots & 0 \\
   \times & \times & 0 & \cdots & 0 \\
   \times & \times & \times & 0 & \cdots & 0 \\
   \cdots & \cdots & \cdots & \cdots & \cdots \\
   \times & \times & \cdots & \times & 0 \\
   \times & \times & \cdots & \cdots & \times
\end{pmatrix},
$$

i.e., one achieves a lower triangular structure for the RWCs with the multiplicity postulated by Braunschweig.\textsuperscript{35} This development shows that it is always possible to choose such a structure. Hecht, in Ref. 22, argued that one can resolve the $SU(3)$ multiplicity problem simply by requiring a similar lower triangular structure. Le Blanc and Rowe also pointed out that such a resolution becomes \textit{ipso facto} equivalent to a canonical labeling scheme.\textsuperscript{27} Actually, one can also choose an upper triangular structure for the RWCs. (Under row-column interchange there are clearly a total of $m(m - 1)/2$ “equivalent” choices.) The structure of the RWCs is not unique, it depends on what kind of special transformation is chosen.

Now recall Biedenharn’s definition for a canonical resolution to the outer multiplicity problem. In Ref. 6, “canonical” implies there no free choices involved in the resolution of the multiplicity. This definition must be understood within the context of the procedure introduced for resolving the multiplicity problem. As shown here, there is not a single “canonical” resolution of the multiplicity; “canonical” has to be understood within the context of the methodology offered for a resolution of the multiplicity. The same applies to the definition of a canonical basis for $U(n)$. A “canonical” construction has to be understood in terms of an equivalent class associated with a particular $U(1)$, out of the set of all equivalent $U(1)$ subgroups, at each stage of the decomposition.

The special transformation given by (2.14) makes it possible to evaluate RWCs of $SU(3)$ in any basis using only recoupling procedures. In the following RWCs for $SU(3)$ in the canonical $SU(3) \triangleright U(2)$ basis will be considered; RWCs for $SU(3)$ in the noncanonical $SU(3) \triangleright O(3)$ basis will be discussed elsewhere.

### III. RECOUPLING APPROACH FOR $SU(3) \triangleright U(2)$ RWCs

In this section, a new procedure for evaluating RWCs with multiplicity for $SU(3) \triangleright U(2)$, or $SU(n) \triangleright U(n - 1)$ in general, is introduced. It uses known results, including the analytic expression for RWCs of $U(n) \triangleright U(n - 1)$ with one symmetric irrep\textsuperscript{15} given by Ališauskas et al., and a special choice for the multiplicity transformation. $U(3) \triangleright U(2)$ RWCs of the type given by Ališauskas et al. can also be found in the work of Chacon et al.,\textsuperscript{5} who derived their results using the
A. A recoupling approach

Using analytic expressions for multiplicity free RWCs given, for example, in Ref. 15, one can construct an expression for the product of general $\mathcal{U}(4)$ or SU(3) RWCs and their respective special recoupling coefficients,

$$\sum \xi \mathcal{U}(\xi) \begin{pmatrix} \lambda_1, \mu_1 & \lambda_2, \mu_2 \cr \mu_2 & m_1m_2m_3 & \lambda_2, \mu_2 \end{pmatrix} = \sum \begin{pmatrix} \lambda_1, \mu_1 & \lambda_2, \mu_2 \cr \mu_2 & m_1m_2m_3 & \lambda_2, \mu_2 \end{pmatrix} \rho_1 \rho_2 \bar{\rho}$$

$$= \sum \begin{pmatrix} \lambda_1, \mu_1 & \lambda_2, \mu_2 \cr \mu_2 & m_1m_2m_3 & \lambda_2, \mu_2 \end{pmatrix} \rho_1 \rho_2 \bar{\rho}$$

$$\times \begin{pmatrix} \lambda_2, \mu_2 \cr \mu_2 & m_1m_2m_3 & \lambda_2, \mu_2 \end{pmatrix}.$$  \hspace{1cm} (3.1)

In this equation, $U$ is the unitary form of a Racah coefficient for $\mathcal{U}(4)$ or SU(3), respectively, and the sum on the rhs is over $\rho_1', \rho_2''$, and $\bar{\rho}$.

It is convenient to introduce the following abbreviated notation:

$$U(\xi) = \mathcal{U}(\xi) \begin{pmatrix} \lambda_1 \mu_1 & \lambda_2 + \mu_2 \cr \mu_2 & m_1m_2m_3 & \lambda_2 \mu_2 \end{pmatrix}$$

for the Racah coefficients,

$$\begin{pmatrix} \bar{\eta} \\ \eta \end{pmatrix} = \begin{pmatrix} \lambda_1 + \mu_1, \mu_1 & \lambda_2 + \mu_2, \mu_2 \\ \lambda_1 + \mu_1, \mu_1 & \lambda_2 + \mu_2, 0 \end{pmatrix} \begin{pmatrix} m_1m_2m_3 \\ m_1m_2 - \eta m_3 - \mu_2 + \eta \end{pmatrix}$$

for a special set of $\mathcal{U}(4) \supset \mathcal{U}(3)$ RWCs,

$$\begin{pmatrix} \bar{\eta} \\ \eta \end{pmatrix} = \begin{pmatrix} \lambda_1 + \mu_1, \mu_1 & \lambda_2 + \mu_2, \mu_2 \cr \rho_1 & \rho_2 & \bar{\rho} \end{pmatrix}$$

for SU(3) WCs, or SU(3) $\supset \mathcal{U}(2)$ RWCs if $\rho_1$, $\rho_2$, and $\bar{\rho}$ are the corresponding $\mathcal{U}(2)$ labels,

$$\begin{pmatrix} \bar{\eta} \\ \eta \end{pmatrix} = \begin{pmatrix} \lambda_1 + \mu_1, \mu_1 & \lambda_2 + \mu_2, \mu_2 \cr \rho_1 & \rho_2 & \bar{\rho} \end{pmatrix}$$

for SU(3) $\supset \mathcal{U}(3)$) reduced coupling coefficients, and

$$\begin{pmatrix} \bar{\eta} \\ \eta \end{pmatrix} = \begin{pmatrix} \lambda_1 + \mu_1, \mu_1 & \lambda_2 + \mu_2, \mu_2 \cr \rho_1 & \rho_2 & \bar{\rho} \end{pmatrix}$$

for $\mathcal{U}(4) \supset \mathcal{U}(3)$ coupling coefficients.

For the $\mathcal{U}(4) \supset \mathcal{U}(3)$ case, only the simpler case

$$\sum \xi \mathcal{U}(\xi) \begin{pmatrix} \xi \\ \eta \end{pmatrix} = G([\bar{\eta}], \eta),$$  \hspace{1cm} (3.2)

needs to be considered, where
\[ G([\vec{m}], \eta) = \sum_{\rho_1, \rho_2} \left[ \begin{array}{c} \lambda_1 + \mu_1 \mu_1 \\ \lambda_2 + \mu_2 \mu_2 \\ \lambda_1 + \mu_1 \mu_1 \\ \lambda_2 + \mu_2 \mu_2 \\ \lambda_3 + \mu_3 \mu_3 \end{array} \right] \left[ \begin{array}{c} \vec{m} \\ \vec{m} \prime \\ \vec{m} \\ \vec{m} \prime \\ \vec{m} \prime \prime \end{array} \right] \left[ \begin{array}{c} \rho_1 \\ \rho_2 \\ \rho_1 \\ \rho_2 \\ \rho \end{array} \right] \]

While for SU(3), the following expression is important:

\[ \sum_{\xi} U_\xi([\vec{m}]) \left[ \begin{array}{c} \lambda_1 + \mu_1 \mu_1 \\ \lambda_2 + \mu_2 \mu_2 \\ \lambda_3 + \mu_3 \mu_3 \end{array} \right] \left[ \begin{array}{c} \rho_1 \\ \rho_2 \end{array} \right] \vec{m} \vec{m} \vec{m} = G([\vec{m}], \rho_1, \rho_2, \rho). \]  \hspace{1cm} (3.4)

Using these \(G\) polynomials, one can easily construct the following coupling coefficients:

\[ \left[ \begin{array}{c} \eta' \\ \eta \end{array} \right] = \sum_{[\vec{m}]} G([\vec{m}], \eta) G([\vec{m}], \eta'). \]  \hspace{1cm} (3.5)

As pointed out by Biedenharn et al.,\(^\text{7}\) there are close relations between RWCs of U(n) and the coupling coefficients of U(n) \(\times\) U(n). They have proved that such coupling coefficients can be expressed in terms of the product of two U(n) WCs with summation over the outer multiplicity labels. One can directly verify by using the recoupling technique that

\[ \left[ \begin{array}{c} \eta' \\ \eta \end{array} \right] = \sum_{[\vec{m}]} G([\vec{m}], \eta) G([\vec{m}], \eta') = \sum_{\xi} \xi \left[ \begin{array}{c} \xi \\ \eta \end{array} \right] \left[ \begin{array}{c} \xi \\ \eta' \end{array} \right] \]  \hspace{1cm} (3.9)
\[
\left( \frac{\eta}{\rho_1 \rho_2 \rho} \right) = \sum_{\tilde{m}} G(\tilde{m}, \eta) G(\tilde{m}, \rho_1 \rho_2 \rho) = \sum_{\xi} \left( \frac{\xi}{\eta} \right) \left( \frac{\xi}{\rho_1 \rho_2 \rho} \right).
\]  

(3.10)

**B. Explicit expressions of \( G \) polynomials**

Using the recoupling technique, it is possible to evaluate explicit expressions for the \( G \) polynomials defined in the above subsection from the known multiplicity-free RWCs of \( U(n) \uplus U(n-1) \) given by Ališauskas et al.,

\[
G(k_2; \lambda) = \left[ \frac{(m_3 - \mu_2 + \eta)!}{(m_2 - \eta + 1)!} \right] \left[ \frac{(m_3 - \mu_2)!}{(m_2 - \eta)!} \right] \frac{(\lambda - \mu_2 + \eta)!}{(\lambda - \mu_2)!} \frac{(\mu_2 - m_3 - \eta)!}{(\mu_2 - m_3)!} \frac{(\lambda - m_3 - \eta)!}{(\lambda - m_3)!} \frac{(m_3 - \eta)!}{(m_3 - \eta)!} \left( \begin{array}{c} \lambda - \mu_2 + \eta + 1 \\ \lambda - m_3 + 1 \end{array} \right) \left( \begin{array}{c} \mu_2 - m_3 + 1 \\ \mu_2 - m_3 \end{array} \right) \left( \begin{array}{c} \lambda - m_3 + 1 \\ \lambda - m_3 \end{array} \right)
\]

(3.11)

where

\[
\lambda = \lambda_1 + \mu_1 + \lambda_2 + \mu_2,
\]

(3.12a)

\[
[\tilde{m}] = [\lambda - k_1 - k_2, \mu_1 + k_1, k_2].
\]

(3.12b)

Boundary conditions for \( m_{23} \) and \( m_{33} \) in these expressions can be obtained by using the Littlewood rule for the Kronecker products involved and the betweenness conditions for the decomposition \( U(n) \uplus U(n-1) \),

\[
\max(m_{23}) = \min(\lambda_1, k_1, \lambda_2 + \mu_2 - p, m_2 - \eta - \mu_1),
\]

(3.12c)

\[
\min(m_{23}) = \max(0, m_2 - \eta - \mu_1 - p, m_3 - \mu_1 - \mu_2 + \eta),
\]

for fixed \( p \), while

\[
\max(m_{33}) = \min(\lambda - p - m_2 - m_2 + \eta, m_3 - \mu_2 + \eta, \mu_1),
\]

\[
\lambda_2 + \mu_2 - p - m_2, \quad \lambda - p - m_2 - \mu_1 - k_1, \quad \lambda - p - m_2 - \lambda_1 - \mu_1),
\]

(3.12d)

\[
\min(m_{33}) = \max(0, m_2 + m_3 + \mu_1 - \mu_2 - p + m_2, k_2 - p - m_2, k_2)
\]

for fixed \( p \) and \( m_{23} \). Furthermore,
\[
F_2 \left[ \begin{array}{c} h_1 h_3 \\ q_1 q_2 \\ n_{1/2} n_{1/2} \\
-\end{array} \right] \begin{array}{c} m_{1/2} m_{1/2} \\ m_{1/2} m_{1/2} \\ n_{1/2} n_{1/2} \\
\end{array} \right] \\
= \sum_{s,t} (-1)^{s+y+z-t_1-t_2-t_3} \\
\times \frac{(x-y+1)(x-z+1)(y-z+1)(x-h_1+1)}{(h_2-y)(n_{1/2}+1)(h_3-z+1)(h_3-z+2)(h_1-y+1)} \\
\times \frac{(q_1-y)(q_1-z+1)(q_1-z)(n_{1/2}+1)(n_{1/2}+1)(y-n_{1/2})(y-h_1)}{(x-q_1)(x-q_2+1)(x-q_2)(y-q_2+1)(y-q_3+1)(z-q_3)(n_{1/2}-y)(n_{1/2}+1)} \\
\times \frac{(m_1-y)(m_1-y+1)(n_{1/2}+1)(m_{1/2}+1)(m_{1/2}-z+1)(m_{1/2}-z+2)(h_3)}{(n_{1/2}-y)(n_{1/2}-z+1)(n_{1/2}-z+1)(x-m_1)!} \\
. \\
(3.13)
\]

Similarly,

\[
G(k_1 k_2; [m_{12} m_{22}][m'_{12} m'_{22}][m''_{12} m''_{22}]) \\
= \sum_{\xi} U_\xi (k_1 k_2) \begin{array}{c} \lambda_1 \mu_1 \\ \lambda_2 \mu_2 \\ \xi_{m_{1/2} m_{1/2}} \\
\end{array} \begin{array}{c} m_{12} m_{22} \\ m'_{12} m'_{22} \\ m''_{12} m''_{22} \\
\end{array} \\
= \left[ \begin{array}{c} (\lambda_2 + 1)! \mu_2 - m_{12}! (m_1 - m_2 + 1) (m_1 - 3m_2 + 1) (m_2 - m_3 + 1) (m_{12} - m_2)! \\
(\lambda_2 + 2)! \mu_3 - m_{12}! (m_1 - m_2 + 1) (m_1 - 3m_2 + 1) (m_2 - m_3 + 1) (m_{12} - m_2)! \\
(\lambda_2 - k_1 - k_2 - m_2)! (\lambda_1 + k_3) - m_3! (\lambda_2 - k_1 - k_2 - m_3 + 1)! (m_{12} - m_2)! (m''_{12} - m_3)! \\
(\lambda_2 - k_1 - k_2 - m_2)! (\lambda_1 + k_3) - m_3! (\lambda_2 - k_1 - k_2 - m_3 + 1)! (m_{12} - m_2)! (m''_{12} - m_3)! \\
(\mu_1 - k_2)! (\lambda_1 + \mu_1 - m_2)! (\lambda_1 + \mu_1 - m_2)! (\lambda_1 + \mu_1 - m_2 + 1)! \\
(\lambda_1 + k_3) - m_3! (\lambda_2 - k_1 - k_2 - m_3 + 1)! \\
\end{array} \right]^{1/2} \\
\times \sum_{p = 0}^{\max(q)} \sum_{q = \min(q)} (-1)^{p} (m_{12} - m_2 - p - m_{12} - m_2)! \\
\times ((m'_{12} - p)! (m''_{12} - m_2 - p - q - m_{12})! (q - m_{12})!^{1/2} \\
\times \left( (m''_{12} - p - q - m_{12} + 1)! (m''_{12} - p - q - m_{12} + 1)! (m''_{12} - p - q + 1)! (p + q - m_{12} + 1)! (m''_{12} - q - 1)! \\
\left( (p - m_{12}) (m''_{12} - p - q - \mu_1 - k_1) (m''_{12} - q)! \\
\end{array} \right) \\
\times U\left[ \begin{array}{c} m_{12} m_{22} \\ m'_{12} m'_{22} \\ m''_{12} m''_{22} \\
\end{array} \right] \\
\times F_2 \left[ \begin{array}{c} \lambda - k_1 - k_2, \mu_1 + k_3, k_2 \\ m_{12} m_{12} + m_{22} - p - \mu_1 - k_1 \\
\end{array} \right] \\
\times F_2 \left[ \begin{array}{c} \lambda + \mu_1 - 1 \\ m_{12} + m_{22} - p - \mu_1 - k_1 \\
\end{array} \right] \\
(3.14)
\]

where

\[
\max(q) = \min(\mu_1 + k_3, m''_{12} - m_2 - p - \mu_1 - k_1), \\
\min(q) = \max(k_2, m''_{12} - m_2 - p - \lambda + k_1 + k_2), \\
\]

\(U\) is an SU(2) Racah coefficient in unitary form, and
\[ F_{2} \left[ \begin{bmatrix} h_{1} h_{2} h_{3} \\ q_{1} q_{2} \end{bmatrix} \right] \mid \begin{bmatrix} m_{1} m_{2} m_{3} \\ n_{1} n_{2} \end{bmatrix} \] = \sum_{x y} (-)^{x+y} q_{1} q_{2} (x-y+1) \times \\
(\frac{(x-n_{2})!(m_{1}-x)!(x-h_{2})!(x-h_{3}+1)!}{(x-q_{1})!(x-q_{2}+1)!(n_{1}-x)!(x-m_{2})!(x-m_{3}+1)!(h_{1}-x)!}) \\
\times (\frac{(q_{1}-y)!(m_{1}-y+1)!(m_{2}-y)!(y-h_{3})!}{(y-q_{2})!(n_{1}-y)!(y-m_{2})!(y-m_{3})!(h_{1}-y+1)!(h_{2}-y)!})^{3}. \\
\] (3.15b)

Using these expressions, (3.9) and (3.10) can be expressed explicitly as

\[
\begin{bmatrix} \eta^{'} \\ \eta \end{bmatrix} = \sum_{k_{1} = \min(k_{1})}^{\max(k_{1})} \sum_{k_{2} = \min(k_{2})}^{\max(k_{2})} G(k_{1}, k_{2}, \eta) G(k_{1}, k_{2}, \eta^{'}), \\
\begin{bmatrix} \eta \\ \rho_{1} \rho_{2} \rho \end{bmatrix} = \sum_{k_{1} = \min(k_{1})}^{\max(k_{1})} \sum_{k_{2} = \min(k_{2})}^{\max(k_{2})} G(k_{1}, k_{2}, \eta) G(k_{1}, k_{2}, \rho_{1} \rho_{2} \rho),
\] (3.16a, 3.16b)

where

\[
\begin{align*}
\max(k_{1}) &= \min(m_{2} - \mu_{1}, \lambda_{1}, \lambda_{2} + \mu_{2}), \\
\min(k_{1}) &= \max(m_{3} - \mu_{1}, m_{2} - \mu_{1} - \mu_{2}, 0), \\
\max(k_{2}) &= \min(\mu_{1}, \lambda_{2} + \mu_{2} - k_{1}, \lambda_{2} + \mu_{2} - k_{1} + \lambda_{1} + \mu_{1} - m_{2}), \\
\min(k_{2}) &= \max(m_{2} + m_{3} - \mu_{1} - \mu_{2} - k_{1}).
\end{align*}
\] (3.16c)

C. Recursive procedure for the evaluation of all SU(3)/U(2) RWCs

The \( G \) polynomials can be used to construct \( \mathfrak{l}(4) \times \mathfrak{l}(4) \) as well as \( \mathfrak{l}(4) \times \text{SU}(3) \) reduced coupling coefficients as given by (3.16a) and (3.16b), and these, in turn, can be used to evaluate all RWCs of SU(3)/U(2) after the special transformation given in Sec. II. It can be proved that

\[ \sum_{\xi} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \neq 0 \] (3.17)

for any \( \eta \). First, if there exists an \( \eta \) such that

\[ \sum_{\xi} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = 0, \] (3.18)

then

\[ \begin{bmatrix} \xi \\ \eta \end{bmatrix} = 0 \] (3.19)

for \( \xi = \eta_{1}, \eta_{2}, \ldots, \eta_{m} \) because the condition (3.18) can be written as \( \eta \cdot \eta = 0 \) in \( \mathbf{R}^{m} \) space, which then implies its components are all zero, i.e.,

\[ \begin{bmatrix} \lambda_{1} + \mu_{1}, \mu_{1} \\ \lambda_{2} + \mu_{2}, \mu_{2} \end{bmatrix} \begin{bmatrix} m_{2} m_{3} m_{1} \\ m_{1} m_{2} - \eta, m_{3} - \mu_{2} + \eta \end{bmatrix} = 0 \] (3.20)

for any multiplicity label \( \xi \) and fixed \( \eta \). However, using the build-up principle, one can deduce that (3.20) is valid if and only if

\[ G([\bar{m}], \eta) = 0, \] (3.21)
where the expression of \( G([\hat{m}], \eta) \) is given by (3.3). In this case, the unitarity condition for \( U(n) \bowtie U(n-1) \) RWCs to prove that (3.21) is satisfied only when one type of the WC or RWCs involved are zero. However, one can verify from the explicit expressions for these WC or RWCs given by Ališauskas et al.\(^{15} \) that (3.21) is satisfied only when the irreps involved in the coupling do not satisfy the Littlewood rule for the Kronecker products involved or betweenness conditions for the decomposition. But these are all trivial cases and need not be considered. Hence, (3.17) is valid in general for all nontrivial cases.

Hence, after applying the special transformation given in Sec. II, the rhs of (3.9) has, in general, \( k \) nonzero terms in the summation when a smaller \( \eta \) which is a label in \([m_1,m_2 - \eta,m_3 - \mu_2 + \eta]\) equals to \( \eta_k \), namely

\[
\left\langle \frac{\eta_k}{\eta_l} \rightangle = \sum_{\xi = \eta_l}^{\eta_k} \left\langle \frac{\xi}{\eta_k} \right\rangle \left\langle \frac{\xi}{\eta_l} \right\rangle, \quad \text{for } k < l, \ k \leq m. \tag{3.22}
\]

Using (3.22), it can be shown that

\[
\left\langle \frac{\eta_k}{\eta_l} \right\rangle = \left[ \left\langle \frac{\eta_k}{\eta_k} \right\rangle - \sum_{\xi = \eta_l}^{\eta_k-1} \left\langle \frac{\xi}{\eta_k} \right\rangle \left\langle \frac{\xi}{\eta_l} \right\rangle \right]^{1/2}, \tag{3.23}
\]

where the sign is chosen positive for any \( k \), fixing the overall phase. Then,

\[
\left\langle \frac{\eta_k}{\eta_l} \right\rangle \left\langle \frac{\eta_k}{\eta_l} \right\rangle = \left\langle \frac{\eta_k}{\eta_k} \right\rangle - \sum_{\xi = \eta_l}^{\eta_k-1} \left\langle \frac{\xi}{\eta_k} \right\rangle \left\langle \frac{\xi}{\eta_l} \right\rangle \tag{3.24}
\]

for \( l > k \). It can also be proven that (3.23) cannot be zero. First, \( \langle \eta_k/\eta_l \rangle^2 \) is a square of the component of the vector \( \eta_k \) in \( \mathbb{R}^m \). Therefore,

\[
\left\langle \frac{\eta_k}{\eta_k} \right\rangle^2 \geq 0, \quad \text{for } k \leq m. \tag{3.25a}
\]

Second, if (3.23) is zero, (3.24) should also be zero for any \( l \) values. This only occurs when the multiplicity equals to \( k - 1 \). However, an initial assumption was that \( m \geq k \). Hence, (3.24) cannot be zero for \( k \leq m \). Hence,

\[
\left\langle \frac{\eta_k}{\eta_k} \right\rangle^2 > 0, \quad \text{for } k \leq m. \tag{3.25b}
\]

Thus, (3.23) and (3.24) allow us to calculate all of the special RWCs of \( \mathfrak{u}(4) \bowtie \mathfrak{u}(3) \) recursively,

\[
\left\langle \frac{\eta_1}{\eta_1} \right\rangle = \left( \frac{\eta_1}{\eta_1} \right)^{1/2}, \tag{3.26a}
\]

\[
\left\langle \frac{\eta_1}{\eta} \right\rangle = \left( \frac{\eta_1}{\eta} \right)^{-1/2}, \tag{3.26b}
\]

\[
\left\langle \frac{\eta_2}{\eta} \right\rangle = \left[ \left( \frac{\eta_2}{\eta} \right) - \left( \frac{\eta_1}{\eta_1} \right)^{-1} \left( \frac{\eta_1}{\eta_2} \right) \right] \left\langle \frac{\eta_2}{\eta_2} \right\rangle, \tag{3.26b}
\]

where

\[
\left\langle \frac{\eta_2}{\eta_2} \right\rangle = \left[ \left( \frac{\eta_2}{\eta_2} \right) - \left( \frac{\eta_1}{\eta_1} \right)^{-1} \left( \frac{\eta_1}{\eta_2} \right) \right]^{21/2}. \tag{3.26c}
\]

\[
\ldots \ldots
\]
Once the \( \langle \eta_{k-1} / \eta \rangle \) for any \( \eta \) are known from the \((k-1)\)th step, \( \langle \eta_k / \eta \rangle \) can be obtained by using (3.23) and (3.24). Thus, one obtains all the special RWCs of \( \mathfrak{u}(4) \oplus \mathfrak{u}(3) \), which are important in determining the RWCs of \( SU(3) \supset U(2) \).

Using (3.10) and known special RWCs of \( \mathfrak{u}(4) \oplus \mathfrak{u}(3) \), SU(3) WC or SU(3) \( \supset U(2) \) RWCs can be determined from

\[
\left( \frac{\eta_1}{\rho_1 \rho_2} \right) \left( \frac{\eta_2}{\rho_1 \rho_2} \right) = \left( \frac{\eta_1}{\eta_1} \right)^{-1} \left( \frac{\eta_1}{\rho_1 \rho_2} \right),
\]

\[
\left( \frac{\eta_2}{\rho_1 \rho_2} \right) = \left[ \left( \frac{\eta_2}{\rho_1 \rho_2} \right) - \left( \frac{\eta_1}{\eta_1} \right)^{-1} \left( \frac{\eta_1}{\rho_1 \rho_2} \right) \right] / \left( \frac{\eta_2}{\eta_2} \right),
\]

(3.27)

\[
\ldots \ldots
\]

\[
\left( \frac{\eta_k}{\rho_1 \rho_2} \right) = \left( \eta_k \right)^{-1} \sum_{\eta' = \eta_1}^{\eta_k-1} \left( \frac{\eta'}{\rho_1 \rho_2} \right) \left( \frac{\eta'}{\eta_k} \right) / \left( \frac{\eta_k}{\eta_k} \right), \quad \text{for } k \leq m.
\]

(3.28)

It should be noted that (3.28) determines not only SU(3) WC in the canonical basis, and SU(3) \( \supset U(2) \) RWCs, but also SU(4) \( \supset U(3) \) RWC for the same coupling.

### D. Some algebraic expressions

In this subsection, some algebraic expressions for SU(3) \( \supset U(2) \) RWCs and related Racah coefficients of SU(3) are worked out. Starting from \( \eta = \eta_1 \), and using (3.23) and (3.24), one obtains

\[
\left( \frac{\eta_1}{\eta_1} \right) = \sum_{k_1 k_2} G^2(k_1, k_2, \eta_1),
\]

\[
\left( \frac{\eta_{k+1}}{\eta_k} \right) = \left( \frac{\eta_1}{\eta_1} \right)^{-1} \sum_{k_1 k_2} G(k_1 k_2, \eta_{k+1})G(k_1 k_2, \eta_1),
\]

\[
\left( \frac{\eta_2}{\eta_2} \right)^2 = \left( \frac{\eta_1}{\eta_1} \right)^{-2} \sum_{i=0}^{1} \sum_{k_1 k_2 k_2'} (-)^{[i/2]} G(k_1 k_2, \eta_2)G(k_1 k_2, \eta_{2-i})G(k_1 k_2', \eta_{1+i})G(k_1 k_2', \eta_1),
\]

(3.29)

\[
\ldots \ldots
\]

\[
\left( \frac{\eta_3}{\eta_3} \right)^2 = \left( \frac{\eta_1}{\eta_1} \right)^{-5} \left( \frac{\eta_2}{\eta_2} \right)^{-1} \sum_{i=0}^{2} \sum_{j=0}^{1} \sum_{k=0}^{1} \sum_{k_1 p_1, q_1, \eta} (-)^{[i/2] + [j/2] + [k/2]} G(k_1 k_2, \eta_3)
\]

\[
\times G(p_1 p_2, \eta_{1+i})G(p_1 p_2, \eta_1)G(q_1 q_2, \eta_{2+i})
\]

\[
\times G(q_1 q_2, \eta_{2-k})G(n_1 n_2, \eta_{1+k})G(n_1 n_2, \eta_1),
\]

where the primes on the summation signs indicate that the sums should be restricted by

\[
i + j + k \leq 2,
\]

(3.30)

and

\[
[x] = \begin{cases} 
  x & \text{if } x \text{ is an integer}, \\
  2x & \text{if } x \text{ is a half-integer}.
\end{cases}
\]

(3.31)
The expression becomes more complicated with increasing $k$ in $\eta_k$. Once $(\eta_k/\eta_1)$ for $k = 1,2,\ldots,m$ are known, one can similarly get the WCs or RWCs of SU(3)$\supset$U(2) with multiplicity,

$$
\left\langle \frac{\eta_k}{\rho_1 \rho_2} \right\rangle = \frac{k!}{\eta_k} \sum_{i=0}^{k-1} \sum_{i_1, \ldots, i_t} G(k_1, k_2, \rho_1 \rho_2) G(k_1, k_2, \eta_{k_i} - \eta_{k_{i+1}})
\times (-1)^{k-1} \prod_{q=1}^{k-1} \left( \frac{\eta_q}{\eta_q + i_p} \right),
$$

where the prime on the summation sum indicates that the condition

$$
\sum_{i=0}^{k-1} i_t \leq k - 1
$$

should be satisfied. Similarly, define

$$
V \left( \frac{\eta_i}{\eta_j} \right) = \left\langle \frac{\eta_i}{\eta_j} \right\rangle^{-1} \left\langle \frac{\eta_i}{\eta_j} \right\rangle.
$$

Then the Racah coefficient of SU(3) can be expressed as

$$
\begin{align*}
R_{\eta_1}([\bar{m}]) \left( \frac{\eta_1}{\eta_1} \right) &= G([\bar{m}], \eta_1), \\
R_{\eta_2}([\bar{m}]) \left( \frac{\eta_2}{\eta_2} \right) &= G([\bar{m}], \eta_2) - V \left( \frac{\eta_1}{\eta_2} \right) G([\bar{m}], \eta_1), \\
R_{\eta_3}([\bar{m}]) \left( \frac{\eta_3}{\eta_3} \right) &= G([\bar{m}], \eta_3) + \sum_{i=1}^{j-1} \sum_{k=1}^{j-1} (-1)^k \sum_{i_1 \leq i_2 \leq \ldots \leq i_k} G([\bar{m}], \eta_{i_k}) V \left( \frac{\eta_{i_1}}{\eta_{i_{k+1}}} \right) \prod_{l=1}^{k-1} V \left( \frac{\eta_{i_l}}{\eta_{i_{l+1}}} \right).
\end{align*}
$$

**IV. Some features of the new SU(3)$\supset$U(2) RWCs**

In this section, symmetry properties of WCs of SU(3) in the canonical basis will be considered. In Sec. III, the phase

$$
\begin{bmatrix}
[\lambda_1 + \mu_1, \mu_1] & [\lambda_2 + \mu_2, \mu_2] \\
[\lambda_1 + \mu_2, \mu_1] & [\lambda_2 + \mu_2, 0]
\end{bmatrix}
\begin{bmatrix}
\eta[m_1 m_2 m_3] \\
\eta[m_1 m_2 m_3 - \eta - \mu_2]
\end{bmatrix}
$$

was chosen. Then, the relative phase is completely determined by the recursion relations (3.23) and (3.24). When $\mu_2 = 0$ one can check that this phase choice is consistent with that of (Ref. 5), which is the same as that of (Ref. 5) with the same phase structure defined by Biedenharn et al.,

as it should be because the multiplicity-free RWCs of (Refs. 5 and 15) are used. Hence, in order to discuss symmetry properties of the SU(3) WCs, one can expand the WCs obtained in this paper in terms of those defined in (Ref. 10),

$$
\begin{align*}
\left\langle \frac{(\lambda_1 \mu_1)}{\rho_1} \left\langle (\lambda_2 \mu_2) \eta[m_1 m_2 m_3] \right\rangle \right. & = \sum_{\xi} y(\xi, \eta) \left\langle \frac{(\lambda_1 \mu_1)}{\rho_1} \left\langle (\lambda_2 \mu_2) \xi[m_1 m_2 m_3] \right\rangle \right.,
\end{align*}
$$

where the coupling coefficients on the rhs in square brackets are the WCs defined in (Ref. 10), and $y(\xi, \eta)$ is the corresponding special transformation matrix element. Using the orthogonality relations for $y(\xi, \eta)$ and symmetry properties of the WCs discussed in (Ref. 10), one can prove that
\[
\begin{pmatrix}
\lambda_1 \mu_1 & \lambda_2 \mu_2 \\
\rho_1 & \rho_2
\end{pmatrix}
\begin{pmatrix}
\eta'[m_1m_2m_3] \\
\rho
\end{pmatrix}
= \sum_y \sum_{\xi} y(\xi, \eta)y(\xi, \eta')(\-)^{\phi(\xi)}
\begin{pmatrix}
\lambda_2 \mu_2 & \lambda_1 \mu_1 \\
\rho_2 & \rho_1
\end{pmatrix}
\begin{pmatrix}
\eta'[m_1m_2m_3] \\
\rho
\end{pmatrix},
\] (4.2b)

where the phase factor \((-)^{\phi(\xi)}\) comes from
\[
\begin{pmatrix}
\lambda_1 \mu_1 & \lambda_2 \mu_2 \\
\rho_1 & \rho_2
\end{pmatrix}
\begin{pmatrix}
\xi[m_1m_2m_3] \\
\rho
\end{pmatrix} = (-)^{\phi(\xi)}
\begin{pmatrix}
\lambda_2 \mu_2 & \lambda_1 \mu_1 \\
\rho_2 & \rho_1
\end{pmatrix}
\begin{pmatrix}
\xi[m_1m_2m_3] \\
\rho
\end{pmatrix},
\] (4.3)
given in Ref. 10,
\[
\phi(\xi) = \Gamma_{12} - \Gamma_{11} + \mu_1 + \mu_2 - m_2 + m_1.
\] (4.4)

The multiplicity label \(\xi\) in this case can be regarded as
\[
\xi = \Gamma_{12} - \Gamma_{11},
\] (4.5)
where \(\Gamma_{ij}\) are multiplicity labels from the upper SU(3) pattern defined by Biedenharn et al.

It is obvious that
\[
Z_{\eta'\eta} = \sum_\xi y(\xi, \eta)y(\xi, \eta')(\-)^{\phi(\xi)},
\] (4.6)
where \(Z_{\eta'\eta}\) is a special \(Z\) coefficient\(^{38}\) defined by Millener, which transforms the coupling coefficients from the coupling \(\lambda_1 \mu_1 \times \lambda_2 \mu_2\) to \(\lambda_2 \mu_2 \times \lambda_1 \mu_1\). Equation (4.2) can only be simplified when the coupling is multiplicity-free. In this case, the transformation matrix \(Y\) is 1\(\times\)1 with \(y(\xi, \eta) = 1\) for fixed \(\xi\) and \(\eta\), and
\[
Z_{\eta'\eta} = \delta_{\eta'\eta}(\-)^{\phi(\xi)},
\] (4.7)
where \(\xi\) can be expressed in terms of \(\lambda_1, \mu_1, \lambda_2, \mu_2,\) and \(m_i\) with \(i = 1, 2, 3\).

Similarly, one can deduce the following symmetry properties for the WCs of SU(3):
\[
\begin{pmatrix}
\mu_1 \lambda_1 & \mu_2 \lambda_2 \\
\bar{\rho}_1 & \bar{\rho}_2
\end{pmatrix}
\begin{pmatrix}
\eta'[-m_3-m_2-m_1] \\
\bar{\rho}
\end{pmatrix}
= \sum_\eta Z_{\eta'\eta}(\-)^{-\lambda_1 \mu_1}
\begin{pmatrix}
\mu_1 \lambda_1 & \mu_2 \lambda_2 \\
\rho_1 & \rho_2
\end{pmatrix}
\begin{pmatrix}
\eta'[m_1m_2m_3] \\
\rho
\end{pmatrix},
\] (4.8a)
where \(\bar{\rho}\) is the conjugation of \(\rho\) defined by
\[
|\bar{\rho}| = |\bar{m}| = \begin{pmatrix}
-m_{22} - m_{12} \\
-m_{11}
\end{pmatrix},
\] (4.8b)
and
\[
\begin{pmatrix}
\lambda_1 \mu_1 & \lambda_2 \mu_2 \\
\rho_1 & \rho_2
\end{pmatrix}
\begin{pmatrix}
\eta'[m_1m_2m_3] \\
\rho
\end{pmatrix} = \frac{\dim([m_1m_2m_3])^{1/2}}{\dim([\lambda_1 \mu_1])} (-)^{\phi_3 - \phi_1}
\times \sum_\eta Z_{\eta'\eta}
\begin{pmatrix}
[m_1m_2m_3] \\
\rho
\end{pmatrix}
\begin{pmatrix}
\mu_2 \lambda_2 & \lambda_1 \mu_1 \\
\bar{\rho}_2 & \bar{\rho}_1
\end{pmatrix}
\begin{pmatrix}
\eta(\lambda_1 \mu_1) \\
\bar{\rho}_1
\end{pmatrix},
\] (4.9)
where
\[
\phi_3 = m_{11}^\mu - m_{12}^\mu - m_{22}^\mu + m_1,
\] (4.10)
Using the recursion relations given by

\[ \left[ q_{1}q_{2}\right]_{m_{1}m_{2}m_{3}} \begin{bmatrix} \lambda_{1} \mu_{1} \\ \lambda_{2} \mu_{2} \end{bmatrix}, \begin{bmatrix} m_{1}m_{2}m_{3} \\ n_{1}n_{2}n_{3} \end{bmatrix}, \begin{bmatrix} \tilde{u}_{1}\tilde{u}_{2}\tilde{u}_{3} \end{bmatrix} = \delta_{m_{1}m_{3}} \delta_{m_{2}m_{1}}, \]

\[ \left[ q_{1}q_{2}\right] \begin{bmatrix} \lambda_{1} \mu_{1} \\ \lambda_{2} \mu_{2} \end{bmatrix}, \begin{bmatrix} m_{1}m_{2}m_{3} \\ n_{1}n_{2}n_{3} \end{bmatrix}, \begin{bmatrix} \tilde{u}_{1}\tilde{u}_{2}\tilde{u}_{3} \end{bmatrix}, \left[ q_{1}q_{2} \right] \begin{bmatrix} \lambda_{1} \mu_{1} \\ \lambda_{2} \mu_{2} \end{bmatrix}, \begin{bmatrix} m_{1}m_{2}m_{3} \\ n_{1}n_{2}n_{3} \end{bmatrix}, \begin{bmatrix} \tilde{u}_{1}\tilde{u}_{2}\tilde{u}_{3} \end{bmatrix} = \delta_{m_{1}m_{3}} \delta_{m_{2}m_{1}}, \]

\[ \phi_{1} = m_{11} - m_{12} - m_{22} + \lambda_{1} + \mu_{1}, \]

and

\[ \rho = \begin{bmatrix} m_{12}m_{22} \\ m_{11} \end{bmatrix}, \rho_{1} = \begin{bmatrix} m_{12}m_{22} \\ m_{11} \end{bmatrix}. \]

Explicit examples for the coupling \([21] \times [21], [321]\), which illustrate main features of the new RWCs, will now be given. The following notation will be used:

\[ \begin{bmatrix} 0 \\ \eta = 0 \end{bmatrix} = \begin{bmatrix} [21] \\ [21] \end{bmatrix}, \begin{bmatrix} 1 \\ \eta = 1 \end{bmatrix} = \begin{bmatrix} [21] \\ [21] \end{bmatrix} \]

\[ \begin{bmatrix} 0 \\ \eta = 0 \end{bmatrix} = \begin{bmatrix} [21] \\ [21] \end{bmatrix}, \begin{bmatrix} 1 \\ \eta = 1 \end{bmatrix} = \begin{bmatrix} [21] \\ [21] \end{bmatrix} \]

Using the recursion relations given by (3.23) and (3.24), it is simple to show that

\[ \begin{bmatrix} 0 \\ 0 \\ 0 \\ \eta = 0 \end{bmatrix} = \begin{bmatrix} \sqrt{7} \\ 10 \\ -\sqrt{4} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \eta = 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \eta = 0 \end{bmatrix} = \begin{bmatrix} \sqrt{10} \\ 21 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} \sqrt{4} \\ \sqrt{10} \end{bmatrix} \begin{bmatrix} 1 \\ \eta = 1 \end{bmatrix} \begin{bmatrix} \eta = 0 \end{bmatrix} \]

With this result in hand, Tables I and II can be worked out by using (3.28). In the tables, the upper parts are taken from de Swart, while the lower parts are derived by the new method. These two types of RWCs with multiplicity two can be transformed with each other by a two dimensional rotation. Furthermore, one can check that the new RWCs satisfy the orthogonality conditions for SU(3) RWCs given by (2.6).

**V. REDUCED AUXILIARY WIGNER COEFFICIENTS FOR SU(3)⊗U(2)**

The complementary U(4) group is also used in Ref. 11 to label the multiplicity of SU(3). However, in that case the U(4) group is labeled in a noncanonical U(2)⊗U(2) chain. The authors also point out that the so-called auxiliary Wigner coefficient (AWC) of SU(3) can be calculated from a U(4)⊗SU(3) scalar product. The reduced AWCS satisfy

\[ \sum_{\lambda_{2}\mu_{2}q_{1}q_{2}} \begin{bmatrix} \lambda_{1}\mu_{1} \\ \lambda_{2}\mu_{2} \end{bmatrix}, \begin{bmatrix} m_{1}m_{2}m_{3} \\ n_{1}n_{2}n_{3} \end{bmatrix}, \begin{bmatrix} \tilde{u}_{1}\tilde{u}_{2}\tilde{u}_{3} \end{bmatrix} = \delta_{m_{1}m_{3}} \delta_{m_{2}m_{1}}, \]

\[ \prod_{t} \delta_{u_{t}u_{t}}, \]

(5.1)
where the sum is over $\rho_1, \rho'_2, \rho''_2, \bar{\rho}$, and the WCs involved in the summation are all given by Ališauskas et al.\textsuperscript{15} and Chacon et al.\textsuperscript{5}

The AWCs can be evaluated by the overlap
One can check that the labels $u_1, u_2, u_3$ take the same values as those given by Brody et al.\textsuperscript{11} One can also get reduced AWCs as follows:

\[
\begin{align*}
\left\langle \frac{\lambda_1\mu_1}{p_1} ; \frac{\lambda_2\mu_2}{p_2} \right| \left[ m_1 m_2 m_3 \right] u_1 u_2 u_3 \right| \left. \right\rangle &= \sum_{\rho_0 \rho_0^*} \delta_{\rho_0 \rho_0^*} \left\langle \frac{\lambda_1\mu_1}{p_1} \right| \left[ m_1 m_2 m_3 \right] \left\langle \frac{\lambda_2\mu_2}{p_2} \right| u_1 u_2 u_3 \right| \left. \right\rangle \times \left\langle \frac{\lambda_2' + \mu_2' - p_0}{p_0} \right| \left[ m_1 m_2 m_3 \right] \left\langle \frac{\mu_2' + p_0}{p_0} \right| u_1 u_2 u_3 \right| \left. \right\rangle \times \left\langle \frac{\lambda_2'}{p_2^*} \right| \left[ m_1 m_2 m_3 \right] \left\langle \frac{\lambda_2}{p_2} \right| \left. \right\rangle .
\end{align*}
\] (5.5)

One can verify that the reduced AWCs given by (5.5) indeed satisfy the orthogonality relations (5.1) and (5.2). Using the explicit expression of the multiplicity-free RWCs involved in the sum, one can get a closed algebraic expression for the reduced AWCs. Such AWCs may also be useful, especially in two-particle coupling problems. It should be pointed out that though the result (5.6) is the same as required by Brody et al., the labeling scheme is not the same. The multiplicity labels of their AWCs are specified by a U(4) subgroup U(3), while they are now specified by the irrep of SU(3) coupled from the first two irreps.

Using analytic expressions for the multiplicity-free RWCs given in Ref. 15, the following algebraic expression for the reduced AWCs of SU(3)$\supseteq$U(2) can be obtained:

\[
\begin{align*}
\left\langle \frac{\lambda_1\mu_1}{m_1} ; \frac{\lambda_2\mu_2}{m_2} \right| \left[ m_1 m_2 \right] \left[ m_1' m_2' \right] \left[ m_1'' m_2'' \right] \left| u_1 u_2 u_3 \right\rangle &= \delta_{\lambda_2' + \mu_2' + p_0} \delta_{\lambda_2 + \mu_2 + m_2 + m_1 + m_3} \times \frac{(\lambda_2 + 1)!(\lambda_2' + \mu_2' - p_0)!(m_1 - m_2 + 1)!(m_1 - m_2 + 2)!(m_2 - m_1 + 1)!(m_2 - m_1 + 2)!}{(\lambda_2 + \mu_2 - \lambda_2' - \mu_2')!(\lambda_2 + \mu_2 - m_2)!(\lambda_2' + \lambda_2' - m_2')!(\lambda_2 - m_1 + 1)!(\lambda_2' - m_1 + 1)!} \times \frac{(\lambda_2' + \mu_2' - m_2)!(\lambda_2 - m_2 + 1)!(\lambda_2' - m_2 + 1)!(m_1 - m_2 + 1)!(m_1 - m_2 + 2)!}{(\lambda_2 + \mu_2 - \lambda_2' - \mu_2')!(\lambda_2 + \mu_2 - m_2)!(\lambda_2' + \lambda_2' - m_2')!(\lambda_2 - m_1 + 1)!(\lambda_2' - m_1 + 1)!} \times \frac{(\lambda_1 + \mu_1 - \lambda_1' \lambda_1' - m_1')!(\lambda_1 - m_1 + 1)!(\lambda_1' - m_1 + 1)!}{(\lambda_1 + \mu_1 - \lambda_1' \lambda_1' - m_1')!(\lambda_1 - m_1 + 1)!(\lambda_1' - m_1 + 1)!} \times \frac{(\mu_1 - \lambda_1')!(\mu_1' - \lambda_1')!(\lambda_1 + \mu_1 - m_1' + 1)!(\lambda_1' + \mu_1 - m_1' + 1)!}{(\mu_1 - \lambda_1')!(\mu_1' - \lambda_1')!(\lambda_1 + \mu_1 - m_1' + 1)!(\lambda_1' + \mu_1 - m_1' + 1)!}^{1/2} \times \frac{\delta_{\lambda_1' + \mu_1' + p_0} p_0!}{\delta_{\lambda_1 + \mu_1 + m_1 + m_2 + m_3} p_0!} \times \frac{\delta_{\lambda_1' + \mu_1' + p_0} p_0!}{\delta_{\lambda_1 + \mu_1 + m_1 + m_2 + m_3} p_0!} \times \frac{(m_1' + m_2' - p - m_2')!(p - m_2')!(q - m_2')!}{(m_1' + m_2' - p - m_2')!(p - m_2')!(q - m_2')!}^{1/2} \times \frac{(m_1' + m_2' - p - m_2')!(p - m_2')!(q - m_2')!}{(m_1' + m_2' - p - m_2')!(p - m_2')!(q - m_2')!}^{1/2} \times \frac{(m_1' + m_2' - p - m_2')!(p - m_2')!(q - m_2')!}{(m_1' + m_2' - p - m_2')!(p - m_2')!(q - m_2')!}^{1/2} \times \frac{(m_1' + m_2' - p - m_2')!(p - m_2')!(q - m_2')!}{(m_1' + m_2' - p - m_2')!(p - m_2')!(q - m_2')!}^{1/2} \times U \left( \left( \frac{\lambda_1 - k_1, \mu_1 + k_1}{m_1' m_1''} \right) \left( \frac{\lambda_1' - k_1', \mu_1' + k_1'}{m_2' m_2''} \right) \right) \times F_2 \left( \left( \frac{\lambda_1 - k_1, \mu_1 + k_1}{m_1' m_1''} \right) \left( \frac{\lambda_1' - k_1', \mu_1' + k_1'}{m_2' m_2''} \right) \right).
\end{align*}
\]
where the \((\lambda_2, \mu_2)\) irrep is written in coupled form \((\lambda'_2, 0) \times (\mu'_2, 0)\) with \(\lambda'_2 + 2\mu'_2 = \lambda_2 + 2\mu_2\),

\[
\lambda = \lambda_1 + \mu_1 + \lambda'_2 + \mu'_2,
\]

\[
[u_1,u_2,u_3]= [\lambda-k_1-k_2,\mu_1+k_1,k_2],
\]

\[
\min(q) = \max(k_2,m_{12}^n+m_{22}^n-p-\lambda+k_1+k_2),
\]

\[
\max(q) = \min(\mu_1+k_1,m_{12}^n+m_{22}^n-p-\mu_1-k_1),
\]

and

\[
F \left( \frac{\lambda'_2 + \mu'_2}{m_{12}' + m_{22}' - p} ; \frac{\mu'_2}{p} ; \frac{\lambda_2 + \mu_2}{m_{12}^n + m_{22}^n} \right) = \sum_t (-)^t \frac{(m_{12}'-p+t)! (\lambda_2 + \mu_2 - m_{12}' - m_{22}' - p-t)!}{t! (\lambda'_2 + \mu'_2 - m_{12}' - m_{22}' - p+t)! \cdot (m_{12}^n + m_{22}^n - p-\mu_2 + t)!}. \]

(5.7)

\[
(5.8)
\]

\[
(5.9)
\]

VI. DISCUSSION

In this paper, the complementary group technique proposed in (I) for a resolution of outer multiplicity problem of SU(n) is used to obtain a general procedure for the derivation of SU(3) \bigotimes U(2) RWCs with multiplicity. The procedure uses a recoupling formula together with a special transformation within the multiplicity space. The outer multiplicity labeling scheme is shown to be nonunique; one can transform from one scheme to another under SO(m), where m is the multiplicity of [m_1 m_2 m_3] in the decomposition \((\lambda_1 \mu_1) \times (\lambda_2 \mu_2)\). From this perspective, Biedenharn’s resolution for the outer multiplicity of SU(n) can also be regarded as a complementary group resolution. In their work, the complementary group is U(n). Unlike the resolution proposed in (I) where only special Gel’fand basis are considered, here the subirreps of the complementary group U(n) are all considered in order to resolve the multiplicity of SU(n) \times SU(n). This is why the multiplicity formulae cannot be worked out easily. The canonical schemes all belong to an equivalent class.

Using the method proposed, RWCs of SU(3) \bigotimes U(2) with multiplicity can be derived recursively. A computer algorithm based on this procedure can be easily developed, which will make numerical calculation possible in practical applications. Furthermore, one can also obtain closed algebraic expressions for these RWCs for small m values. However, the expression remains cumbersome with summation over many variables. The complexity increases with increasing the multiplicity.

Only RWCs of SU(3) in the canonical chain SU(3) \bigotimes U(2) are discussed. Nonetheless, this method can also be applied to noncanonical basis of SU(3), for example SU(3) \bigotimes SO(3), by using RWCs of SU(3) \bigotimes SO(3) with one symmetric irrep, and therefore multiplicity-free.\(^{50,51}\) In addition, the procedure outlined in this paper can be extended to the general SU(n) case. The SU(4) \bigotimes U(3) RWCs will be the topic of the next paper.

A closed algebraic expression of reduced AWCs proposed in Ref. 11 is obtained using the new approach. These coefficients satisfy different orthogonality relations and therefore may be useful in many-particle coupling problems.
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