Complementary group resolution of the SU($n$) outer multiplicity problem. I. The Littlewood rules and a complementary $\mathcal{U}(2n-2)$ group structure

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A complementary group to SU($n$) is found that realizes all features of the Littlewood rules for Kronecker products of SU($n$) representations. This is accomplished by considering a state of SU($n$) to be a special Gel'fand state of the complementary group $\mathcal{U}(2n-2)$ with labels of the latter used to distinguish multiple occurrences of irreducible representations of SU($n$) (irreps) in the SU($n$) × SU($n$) × SU($n$) decomposition that is obtained from the Littlewood rules. Furthermore, this realization also helps us to determine SU($n$) ⊇ SU($n$-1) × U(1) Reduced Wigner Coefficients (RWCs, frequently called Isoscalar Factors) and Clebsch–Gordan Coefficients (CGCs, or full (nonreduced) Wigner Coefficients) of SU($n$), using algebraic or numeric methods, in either the canonical or a noncanonical basis. New explicit formulas for the SU(3) and SU(4) multiplicities are obtained by using this technique. © 1998 American Institute of Physics.

I. INTRODUCTION

Reduced Wigner Coefficients (RWCs) of SU($n$) ⊇ SU($n-1$) × U(1) are of importance in many physical applications. Except for those of SU(2), which have been discussed extensively and expressed in various forms, RWCs of SU($n$) ⊇ SU($n-1$) × U(1), which can be used to evaluate Clebsch–Gordan Coefficients (CGCs) of SU($n$) in the canonical basis according to the Racah Factorization Lemma, have only been given analytically for some special cases. [Reduced Wigner Coefficients are often called Isoscalar Factors and Clebsch–Gordan Coefficients as used in this article are also known as full (nonreduced) Wigner Coefficients or simply Wigner Coefficients.] The biggest challenge involves the outer multiplicity in the decomposition of Kronecker products of SU($n$) × SU($n$) × SU($n$). The first nontrivial but simplest $n=3$ case was studied as part of the first applications of nonmultiplicity-free CGCs of SU(3) in nuclear and particle physics. There are several distinct approaches to the problem: (i) a tensor operator method; (ii) an infinitesimal generator approach, in which matrix elements of SU($n$) generators are used to determine recursion relations for the RWCs and CGCs; (iii) a polynomial basis and generating invariants, in which a convenient model space is used to realize a basis of irreps; and (iv) use of the Schur–Weyl duality relation between SU($n$) and the symmetric group $S_f$. Among these are various means of addressing the problem; indeed, sometimes a combination of two or more methods is used. There are also different schemes for handling the outer multiplicity, especially for SU(3), and these are usually referred to as either the canonical or a noncanonical labeling scheme.

A thoroughly discussed approach to this problem is the canonical unit tensor operator method developed by Biedenharn and collaborators in a series of publications.\textsuperscript{1–8} The unit tensor operator approach is particularly useful for deriving multiplicity-free CGCs of U($n$). The techniques that are part of this method have also proven to be useful in other approaches, but this methodology has not been used to produce a closed algebraic solution to the general outer multiplicity problem. This method was revisited in the late 1980s in a Bargmann–Hilbert space representation using Vector Coherent State (VCS) theory.\textsuperscript{9–11} Although the results seem no simpler than those found earlier, they do show that there is a relationship between U(3) ⊇ U(2) RWCs and 3$nj$ coefficients.
of SU(2), with some of these being a consequence of the Schur–Weyl duality relation between the unitary and symmetric groups given by Ališauskas et al.\textsuperscript{12–14}

Noncanonical definitions of SU(n) outer multiplicity labels, especially of SU(3), have also been discussed rather extensively, for example, by Moshinsky et al.,\textsuperscript{15,16} Derome and Sharp,\textsuperscript{17,18} Resnikoff,\textsuperscript{19} and Pluhar et al.\textsuperscript{20,21} A wider class of RWCs has been considered by Hecht,\textsuperscript{22} Klimyk and Gavriliuk,\textsuperscript{23} and Le Blanc and Rowe,\textsuperscript{24} who used definitions related to the canonical scheme. Generally, however, these results are for noncanonical labeling schemes. A further example is the extensive work of Ališauskas,\textsuperscript{25–28} who investigated paracanonical coupling relations and symmetries and various pseudocanonical coupling schemes, which lead to biorthogonality among the corresponding coefficients. It should be stated that noncanonical definitions for SU(n) coupling coefficients normally lead to nonorthogonality with respect to the outer multiplicity. In such cases, a Gram–Schmidt process can be adopted to recover orthonormality, but this procedure includes an arbitrary choice in ordering the orthogonalized elements. In addition, except for a few simple cases where analytical expressions are available, only numerical implementations of the corresponding algorithm is possible.\textsuperscript{24–28}

The Schur–Weyl duality relation between SU(n) and $S_f$ is another method that has been used by several authors. This method was studied first by Moshinsky,\textsuperscript{16} Kramer,\textsuperscript{27} and Ališauskas and Jucy,\textsuperscript{12–14} who were able to show that the scheme works for nonmultiplicity-free as well as multiplicity-free cases. For nonmultiplicity-free couplings, however, numerical orthogonalization is again required. This approach is illustrated for some simple cases in the work of Chen et al.,\textsuperscript{30} and by Pan and Chen for the $U_q(n)$ generalization of $U(n)$.\textsuperscript{31}

Based on these methods, several packages have been developed for numerically evaluating CGCs of $U(n)$, especially of SU(3). The earliest one is the well-known Akiyama–Draayer code for SU(3) based on a combination of the tensor operator and infinitesimal generator methods.\textsuperscript{32,33} Another is Chen’s code for various couplings of $U(n)$ based on symmetric group techniques.\textsuperscript{30} Still another is the RWC and CGC code for SU(3) developed by Kaeding and Williams.\textsuperscript{34–36}

Very recently, Parkash and Sharatchandra worked out an algebraic formula for the general CGCs of SU(3).\textsuperscript{37} The method used in their paper is based on a polynomial realization in Bargmann space using generating functions, which was first studied by Shelepin and Karasev for the multiplicity-free case.\textsuperscript{38,39} Up to a normalization factor, the final results are expressed in terms of a restricted sum over 33 variables. To determine the value of a single CGC within this formulation is not easy; neither the algebraic nor numeric results are simple. Nevertheless, it is the first algebraic expression for CGCs of SU(3) with multiplicity. It should be noted, however, that to extend this method to $n \geqslant 4$ cases will be much more complicated. Therefore, another simpler and more direct approach to a resolution of the outer multiplicity problem for SU(n) is desirable.

The present paper is the first (I) in a series which has this as its objective. First of all, a complementary group $U(2n-2)$ realization of the Kronecker product $SU(n) \times SU(n) \supset SU(n)$ is found according to the well-known Littlewood rules. The scheme gives a simple exact resolution of the outer multiplicity. An analysis of the Littlewood rules is also used to derive a new multiplicity formula for SU(n). Examples are given for the SU(3) and SU(4) cases, which can, in principle, be extended to SU(n). A procedure for evaluating CGCs or RWCs of SU(n) is another method that has been used to a normalizability, the final results are expressed in terms of a restricted sum over 33 variables. To determine the value of a single CGC within this formulation is not easy; neither the algebraic nor numeric results are simple. Nevertheless, it is the first algebraic expression for CGCs of SU(3) with multiplicity. It should be noted, however, that to extend this method to $n \geqslant 4$ cases will be much more complicated. Therefore, another simpler and more direct approach to a resolution of the outer multiplicity problem for SU(n) is desirable.

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II. LITTLEWOOD RULES AND THE COMPLEMENTARY GROUP

The Littlewood rules\textsuperscript{40} for determining Kronecker products of SU(n) in SU(n) \times SU(n) \supset SU(n) is a reflection of the Schur–Weyl duality relation between SU(n) and the symmetric group $S_f$. According to the Schur–Weyl duality relation, an irrep $[\lambda]$ of SU(n) can also be regarded as the same irrep of $S_f$ with $\sum_{i=1}^{n}\lambda_i = f$. Therefore, the Kronecker product of two SU(n) irreps $[\lambda] \times [\mu]$ in the decomposition $SU(n) \times SU(n) \supset SU(n)$ can be obtained from the product of two $S$ functions of the corresponding symmetric groups:\textsuperscript{31}

$$[\lambda] \times [\mu] = \sum_{\nu} \{\lambda \mu \nu\}[\nu],$$

(2.1)
where \( \{ \lambda \mu \nu \} \) is the number of occurrences of \([ \nu ]\) in the product. To determine all the irreps that appear on the rhs of (2.1), one can use the well-known Littlewood rules. First fill in the Young diagram \([ \mu ] = [\mu_1, \mu_2, \ldots, \mu_n] \) with \( \mu_1 \) symbols \( a_1 \) in the first row, \( \mu_2 \) symbols \( a_2 \) in the second row, \( \mu_3 \) symbols \( a_3 \) in the third row, and \( \mu_n \) symbols \( a_n \) in the \( n \)th row. Then, the final irrep denoted by a Young diagram \([ \nu ]\) can be obtained by augmenting the Young diagram \([ \lambda ]\) with the \( \mu_1 \) symbols, \( \mu_2 \) symbols, ..., and \( \mu_n \) symbols, respectively, in ways specified by the following three conditions.

(a) No identical symbols should appear in the same column of the diagram.
(b) If the \( a_1, a_2, \ldots, a_n \) symbols are counted from right to left starting at the top, then at each stage the number of \( a_1 \) symbols must not be less than the number of \( a_2 \) symbols, which must not be less than the number of \( a_3 \) symbols, and so on.
(c) The Young diagram \([ \nu ]\) obtained after the addition of each symbol must be standard, that is, \( \nu_1 \geq \nu_2 \geq \cdots \geq \nu_n \).

The Young diagram \([ \nu ]\) filled with symbols \( a_1, a_2, \ldots, a_n \) under restrictions (a)-(c) can be regarded as a special Weyl tableau of a unitary group. Recall some basic definitions for the Weyl tableau: A Weyl tableau is a Young diagram with the boxes filled by a set of ordered indices regarded as a special Weyl tableau of a unitary group. Every irrep \( [ \nu ] \) of \( U(n) \) can be characterized by a partition \( [ \nu ] = [m_{1n}] = [m_{11}, m_{2n}, \ldots, m_{nn}] \) of \( n \) non-negative integers obeying the relation \( m_{1n} \geq m_{i+1n} \); conversely, every such partition denotes a unique irrep. It is well known that any basis vector of \( U(n) \) in its canonical basis, i.e., the basis adapted to \( U(n) \supset U(n-1) \supset \cdots \supset U(2) \supset U(1) \) can be labeled by the Gel'fand symbol,

\[
\begin{pmatrix} \nu \\ m \end{pmatrix} = \begin{pmatrix} m_{11} & m_{22} & \cdots & m_{1n-1} & m_{n-1n} & m_{nn} \\ m_{1n-1} & m_{2n-1} & \cdots & m_{12} & m_{nn-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{12} & m_{22} \end{pmatrix}.
\]

Equivalently, sometimes the Weyl tableau is also convenient in labeling the canonical basis of \( U(n) \). The one-to-one correspondence between the Gel'fand symbol and the Weyl tableau is realized in the following way:

\[
\begin{pmatrix} \nu \\ m \end{pmatrix} \mapsto W^{[\nu]} = \begin{pmatrix} f_{11}a_1 \bar{s} & f_{12}a_2 \bar{s} & \cdots & f_{1n}a_n \bar{s} \\ f_{21}a_1 \bar{s} & f_{22}a_2 \bar{s} & \cdots & f_{2n}a_n \bar{s} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}a_1 \bar{s} & f_{n2}a_2 \bar{s} & \cdots & f_{nn}a_n \bar{s} \end{pmatrix}
\]

(2.2)

where

\[
\begin{align*}
f_{1k} &= m_{1k} - m_{1k-1}, & f_{2k} &= m_{2k} - m_{2k-1}, & \ldots, \\
f_{k-1k} &= m_{k-1k} - m_{k-1k-1}, & f_{kk} &= m_{kk}.
\end{align*}
\]

(2.3)

In other words, a Weyl tableau \( W^{[\nu]} \) filled with \( a_1, a_2, \ldots, a_n \), corresponds to the \( n \) partitions, \( [\nu]([m_{1n}],[m_{1n-1}],\ldots,[m_{12}]) \) and \( [m_{11}] \) of a Gel'fand symbol, where \( [m_{ik}] \) is the Young diagram resulting from deleting all the boxes in the Weyl tableau occupied by the symbols \( a_n, a_{n-1}, \ldots, a_{k+1} \).

It is clear that the definition of a Weyl tableau and the rules for placing symbols in a Young diagram given by Littlewood rules are the same except for some of the restrictions given by \( b) \), that is, \( a_i \)'s can appear in the \( i \)th rows with \( i < k \), and the number of \( a_i \)'s can be greater than the number of \( a_i \)'s with \( i < k \) counted from right to left and from top to bottom, while these cases are forbidden by the restriction \( b) \) of the Littlewood rules. Therefore, it is obvious that the Littlewood
rules for placing symbols in a Young diagram can be regarded as a special Weyl tableau for a unitary group. Hence, under the restrictions of Littlewood rules given by (b), one obtains a special Gel’fand basis of a corresponding unitary group, which is called the complementary group for Kronecker products of SU(n).

Assume the irrep \( [\lambda] \) has \( p_1 \) rows, while \( [\mu] \) has \( p_2 \) rows. Then, the final irrep \( [\nu] \) has, at most, \( p_1 + p_2 \) rows, with \( p_1 + p_2 \leq n \). Therefore, the complementary group corresponding to the Kronecker product of SU(n) is \( \mathcal{U}(p_1 + p_2) \). A general SU(n) irrep has, at most, \( n^2 \) rows because one can always use the equivalence condition \( \lambda_1 \ldots \lambda_{n-1} \leq \mu_1 \ldots \mu_{n-1} \) to remove the \( n \)th row if it exists. From this, it follows that the minimum complementary group is \( \mathcal{U}(2n) \) for general Kronecker products of SU(n).

Using the correspondence between the Weyl tableau and a Gel’fand symbol, one can easily find the following relations among coupled and uncoupled state labels of \( \mathcal{U}(2n) \).

\[
\begin{align*}
\begin{array}{c}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_{n-1} \\
\end{array}
&= \begin{bmatrix}
[\lambda_1 \lambda_2 \cdots \lambda_{n-1} 0] \\
\Gamma_1 \\
\end{bmatrix} = \begin{bmatrix}
\lambda_1 \lambda_2 \cdots \lambda_{n-1} 0 & \mathcal{U}(2n) \\
\cdots \\
\Gamma_1 & \mathcal{U}(n-1) \\
\end{bmatrix} \\
\begin{array}{c}
\mu_1 a_1 b \\
\mu_2 a_2 b \\
\vdots \\
\mu_{n-1} a_n b \\
\end{array}
&= \begin{bmatrix}
[\mu_1 \mu_2 \cdots \mu_{n-1} 0] \\
\Gamma_2 \\
\end{bmatrix} = \begin{bmatrix}
\mu_1 \mu_2 \cdots \mu_{n-1} 0 & \mathcal{U}(2n-2) \\
\cdots \\
[\mu_1 \mu_2 0] & \mathcal{U}(n+1) \\
[\mu_1 0] & \mathcal{U}(n) \\
[0] & \mathcal{U}(n-1) \\
\end{bmatrix}
\end{align*}
\]

while the final coupled \( \mathcal{U}(2n-2) \) basis is

\[
\begin{bmatrix}
[\nu_1 \nu_2 \cdots \nu_{n} 0] \\
\Gamma(\tau) \\
\end{bmatrix} = \begin{bmatrix}
\nu_1 \nu_2 \cdots \nu_{n} 0 & \mathcal{U}(2n-2) \\
(\tau) & \mathcal{U}(n-1) \\
\lambda_1 \lambda_2 \cdots \lambda_{n-1} 0 \\
\rho & \mathcal{U}(n-1) \\
\end{bmatrix},
\]

where \( \tau \) stands for intermediate sublabels between \( \mathcal{U}(2n-2) \) and \( \mathcal{U}(n-1) \) given by the Littlewood rules, which can be regarded simultaneously as the outer multiplicity label of both \( \mathcal{U}(2n-2) \) and SU(n), and \( \rho \) represents sublabels of \( \mathcal{U}(n-1) \), which is unimportant for labeling the multiplicity of SU(n) × SU(n).

In order to make this point clear, let us give an example for SU(3) × SU(3) | SU(3) for the coupling \([21] \times [21] | [321] \). According to our procedure, uncoupled basis vectors of SU(3) × SU(3) for the irreps \([21] \times [21] \) can be labeled as

\[
\begin{bmatrix}
\Gamma_1 \\
[210] \\
(\nu_1) \\
\end{bmatrix} \quad \begin{bmatrix}
\Gamma_2 \\
[210] \\
(\nu_2) \\
\end{bmatrix} = \begin{bmatrix}
\rho \\
[21] \\
[210] \\
(\nu_1) \\
\end{bmatrix} \begin{bmatrix}
0 \\
[210] \\
[210] \\
(\nu_2) \\
\end{bmatrix},
\]

where \( (\nu_1) \), \( (\nu_2) \) are sublabels for two SU(3) irreps \([210] \), respectively,

\[
\Gamma_1 = \begin{bmatrix}
[210] \\
[21] \\
\rho \\
\end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix}
[200] \\
[00] \\
0 \\
\end{bmatrix},
\]
are \( \mathcal{U}(4) \) sublabels for the same irreps [2100] required from the Littlewood rule, and \( \rho \) stands for unimportant sublabels of \( \mathcal{U}(2) \). While the final coupled basis vectors for [321] is denoted by

\[
\begin{pmatrix}
\Gamma(\eta) \\
[321] \\
(\nu)
\end{pmatrix} =
\begin{pmatrix}
\rho \\
[21] \\
[32-\eta\eta] \\
[321] \\
(\nu)
\end{pmatrix},
\]

where \( \eta = 0, \) or 1, which is determined by the Littlewood rule directly, can be understood as the multiplicity label.

From this procedure, it can be seen that the \( \mathcal{U}(2n-2) \) labels are twofold. On the one hand, the final coupled \( \mathcal{U}(2n-2) \) sublabels \( \Gamma(\eta) \) can be used to label the outer multiplicity of \( SU(n) \times SU(n) \) or \( SU(n) \times SU(n) \), while the uncoupled \( \mathcal{U}(2n-2) \) sublabels \( \Gamma_1 \) and \( \Gamma_2 \) just indicate how the final coupled sublabels of \( \mathcal{U}(2n-2) \), which gives the multiplicity labels, can be obtained. General cases for \( SU(3) \) and \( SU(4) \) will be shown in the next section. Therefore, this situation is similar to the upper Gel'fand pattern introduced by Biedenharn et al.,1-8 in which the upper Gel'fand patterns \( \Gamma_2 \) of \( \mathcal{U}(n) \) are used to label the outer multiplicity of \( U(n) \times U(n) \), while other two upper Gel'fand patterns \( \Gamma_1 \) and \( \Gamma \) of \( \mathcal{U}(n) \) are set to the highest weight. While, in our scheme, the sublabels of uncoupled \( \mathcal{U}(2n-2) \) labels are fixed according to the Littlewood rule, the final coupled \( \mathcal{U}(2n-2) \) sublabels \( \Gamma(\eta) \), which can be taken as different values according to the Littlewood rule, are used as outer multiplicity labels for \( SU(n) \times SU(n) \). However, in the \( SU(3) \) case, for example, in their upper pattern scheme, there are some restrictions on the ranges of \( \Gamma_{22} \) in the upper pattern \( \{\Gamma_{12}, \Gamma_{22}\} = \{m_1 + m_2 - \lambda_1 - 2\mu_1 - \Gamma_{22}, \Gamma_{22}\} \) for the coupling \( (\lambda_1, \mu_1) \times (\lambda_2, \mu_2) \), were \( \lambda_1, \lambda_2, \mu_1, \mu_2 \) that come from the Littlewood rules for \( SU(3) \). Actually, the ranges of \( \Gamma_{22} \) should be the same as those of \( \eta \), to be determined in the next section, which cannot be derived directly from restrictions on upper pattern labels. For example, \([422]\) occurs only once in the decomposition \([310] \times [310], \) but there are two sets of upper pattern labels:

\[
\begin{pmatrix}
\Gamma_{11} \\
\Gamma_{12} \\
\Gamma_{22}
\end{pmatrix} =
\begin{pmatrix}
1 & 0 \\
2 & 0 \\
1 & 1
\end{pmatrix},
\]

are both allowed by the upper pattern labeling scheme, but according to the Littlewood rules the

\[
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
\]

result should be eliminated. Therefore, in addition to the usual rules for determining allowed upper Gel'fand patterns, restrictions that follow from an application of Littlewood rules must be applied, a result not noted in the papers of Refs 1-8 and, indeed, one that cannot be deduced directly from their upper Gel'fand pattern methodology. While, in our scheme, the possible values of \( \eta \) gives exactly the outer multiplicity of \( SU(3) \). In this sense, the complementary group is meaningless, but it provides us with the labeling scheme for the outer multiplicity of \( SU(n) \times SU(n) \), and tells us how these labels can be obtained. On the other hand, in order to evaluate the CGCs of \( SU(n) \), one can consider the following \( SU(n) \times U(n) \) \( SU(n-1) \times \cdots \times U(1) \times U(2n-2) \) coupling coefficients, for example, in the canonical chain \( (SU(n) \supset SU(n-1) \supset \cdots \supset U(1)) \times U(2n-2) \supset U(2n-3) \supset \cdots \supset U(1)) \):

\[
\begin{pmatrix}
(\Gamma_1) \\
[\lambda] \\
(\kappa_1)
\end{pmatrix}
\begin{pmatrix}
(\Gamma_2) \\
[\mu] \\
(\kappa_2)
\end{pmatrix}
\begin{pmatrix}
(\Gamma(\eta)) \\
(\nu) \\
(\kappa)
\end{pmatrix},
\]

where \( [\lambda], [\mu], \) and \( [\nu] \) are irreps of \( SU(n) \) and \( U(2n-2) \), simultaneously, \( (\Gamma_1), (\Gamma_2), \) and \( (\Gamma(\eta)) \) are sublabels of \( U(2n-2) \), and \( \kappa_1, \kappa_2, \) and \( \kappa \) are sublabels of \( SU(n) \). Using the recoupling method, which will be discussed in detail in paper II, we have
where the coupling coefficient of SU($n$)$\otimes\mathcal{U}(2n-2)$ on the lhs of (2.7) can be evaluated by using many methods, which will be discussed in (II), while, on the rhs of (2.7), there is a sum over the outer multiplicity labels $\eta'$ of a products of SU($n$) and $\mathcal{U}(2n-2)$ CG coefficients, the symbols $(\Gamma_i)$ with $i=1,2,$ and $(\Gamma(\eta))$ are used to denote sublabels of $\mathcal{U}(2n-2)$, which are the same as those derived from the Littlewood rule discussed above, and the same symbols without parentheses are used to label the outer multiplicities in the coupling. Equation (2.7) can be used to derive CGCs of SU($n$) algebraically or numerically. A general procedure for SU(3) and SU(4) will be outlined in papers (II) and (III), respectively. From this point of view, the complementary group $\mathcal{U}(2n-2)$ is indeed active in obtaining the CGCs of SU($n$).

III. OUTER MULTIPLICITY PROBLEM OF SU(3) AND SU(4)

As noted above, the outer multiplicity in the decomposition of the Kronecker products of SU($n$)$\otimes$SU($n$)$\mid$SU($n$) is the main obstacle in applications of algebraic methods to physical problems. There are a lot of articles devoted to this subject. In order to resolve the problem for SU(3), Hecht\textsuperscript{22} proposed an external labeling operator of third order, an operator that may be related to the one proposed by Moshinsky\textsuperscript{16} in terms of the complementary U(3)$\otimes$U(2)$\times$U(2) chain. Alisauskas and Kulish\textsuperscript{45} have also proposed an external labeling operator, a fourth-order form suggested by Sharp\textsuperscript{43} in a study of Yang–Baxter equations. There are also other articles on this subject. For example, new Casimir operators, the so-called chiral Casimirs, were introduced in Refs. 16, and 44–45. Also, various formulas\textsuperscript{19,46–49} for the multiplicity of SU(3) exist in the literature, however, such expressions are normally not linked to the SU(3) coupling and recoupling coefficients problem. There is still no general formula for the outer multiplicity of SU($n$) with $n \geq 4$. In this article and forthcoming papers, the complementary group $\mathcal{U}(2n-2)$ to the SU($n$)$\times$SU($n$)$\mid$SU($n$) will be shown to be a powerful tool for deriving both multiplicity formulas and coupling and recoupling coefficients of SU($n$). Multiplicity formulas for SU(3) and SU(4) are considered below.

(1) SU(3) case. Consider the general Kronecker product $(\lambda_1\mu_1) \times (\lambda_2\mu_2)$, where the well-known notation for SU(3) in physics is adopted. The irrep $(\lambda\mu)$ can be expressed in terms of a two-rowed Young diagram $[\nu_1\nu_2]$ with $\nu_1 = \lambda + \mu$ and $\nu_2 = \mu$. Using the Littlewood rules, the decomposition of $(\lambda_1\mu_1) \times (\lambda_2\mu_2)$ can be expressed in terms of a quintuple sum,

$$(\lambda_1\mu_1) \times (\lambda_2\mu_2) = \sum_{k_1=0}^{\lambda_2+\mu_2} \sum_{k_2=0}^{\mu_1} \sum_{k_3=0}^{\lambda_2+\mu_2-k_1} \sum_{n_1=0}^{\min(\lambda_1+1-k_1,\mu_1+k_2-k_3-k_1)} \sum_{n_2=0}^{\min(\mu_2-n_1,\mu_1+k_2-k_3,k_1+k_2-n_1)} \times \sum_{n_3=0}^{\min(\mu_2-n_1,\mu_1+k_2-k_3,k_1+k_2-n_1)} \sum_{n_4=0}^{\min(\mu_2-n_1,\mu_1+k_2-k_3,k_1+k_2-n_1)} \sum_{n_5=0}^{\min(\mu_2-n_1,\mu_1+k_2-k_3,k_1+k_2-n_1)} \times \sum_{n_6=0}^{\min(\mu_2-n_1,\mu_1+k_2-k_3,k_1+k_2-n_1)} \sum_{n_7=0}^{\min(\mu_2-n_1,\mu_1+k_2-k_3,k_1+k_2-n_1)} \sum_{n_8=0}^{\min(\mu_2-n_1,\mu_1+k_2-k_3,k_1+k_2-n_1)},$$

where the constraints

$$\sum_{i=1}^{3} k_i = \lambda_2 + \mu_2, \quad \sum_{i=1}^{2} n_i = \mu_2$$

apply in the summation. Expression (3.1) can be further simplified, for example, to O’Reilly’s formula\textsuperscript{48} in which only a triple sum appears. However, (3.1) can be used to help determine a multiplicity formula and determine the multiplicity labels of the complementary group.
Consider a Young diagram of the resultant irrep \([m_1m_2m_3]\) according to (3.1) with conditions given by (3.2):

\[
\begin{array}{c|c|c}
\lambda_1 + \mu_1 & k_1 \alpha \\
\mu_1 & m_2 - \mu_1 - \eta \alpha \\
m_3 - \mu_2 + \eta \alpha & \mu_2 - \eta \alpha
\end{array}
\]

where \(\alpha\) and \(\beta\) are the \(a_i\) symbols of the Littlewood rules for SU(3). The labels in (3.3) have been arranged to accommodate the constraints of (3.2) and to yield a multiplicity formula in a straightforward manner. In this form it is obvious that a diagram with the same number of boxes in each row can only appear repeatedly when \(\eta\) is not a fixed integer. Therefore, \(\eta\) can be regarded as the multiplicity label of SU(3).

According to Littlewood rules (a)–(c), it is easy to derive the following limits on \(\eta\):

\[
\eta_{\text{min}} \leq \eta \leq \eta_{\text{max}},
\]

where

\[
\eta_{\text{max}} = \min(m_1 - \lambda_1 - \mu_1, m_2 - \mu_1, \lambda_2 + \mu_2 - m_3, \mu_1 + \mu_2 - m_3, m_2 - m_3),
\]

\[
\eta_{\text{min}} = \max(0, \mu_2 - m_3, m_2 - \lambda_1 - \mu_1).
\]

Hence, the multiplicity of \([m_1m_2m_3] = (m_1 - m_3, m_2 - m_3)\) occurring in the Kronecker product \((\lambda_1 \mu_1) \times (\lambda_2 \mu_2)\) is given by

\[
\text{Multi}(SU_3) = \eta_{\text{max}} - \eta_{\text{min}} + 1.
\]

This expression is very simple and more transparent than others found in the literature.

In this case, the complementary group is \(U(4)\). The Gel’fand symbol of \(U(4)\) corresponding to the resultant irrep of SU(3) given in (3.3) is

\[
\begin{bmatrix}
m_1 & m_2 - \eta & m_3 - \mu_2 + \eta \\
\lambda_1 + \mu_1 & \mu_1 & \rho
\end{bmatrix},
\]

where \(\rho\) is the intrinsic label for \(U(1)\), which is not important for our purpose. Some conditions in (3.5) can also be easily obtained from the betweenness conditions of the entries in the Gel’fand symbol (3.7). However, the remaining conditions in (3.5) can only be deduced from the Littlewood rules (b), and cannot be obtained from the betweenness conditions. Hence, only one outer multiplicity label is needed in the decomposition of SU(3) \(\times\) SU(3) \(\times\) SU(3). This is why the CGCs of SU(3) can be determined numerically by using only one type of tensor operator.\(^{32-36}\)

In contrast with the so-called canonical labeling scheme proposed by Biedenharn et al., in which three independent shifts determined by an upper pattern are introduced, the complementary \(U(4)\) group provides only one outer multiplicity label in the SU(3) case. The complementary group labeling scheme is therefore a very economical way to label the outer multiplicity of SU(3), and by extension of SU\(n\).

(2) SU(4) case. A general SU(4) irrep has three rows. Using the Littlewood rules, the following formula for the decomposition of SU(4) \(\times\) SU(4) \(\times\) SU(4) can be determined:

\[
[\lambda_1 \lambda_2 \lambda_3] \times [\mu_1 \mu_2 \mu_3] = \sum_{k_1=0}^{\mu_1} \sum_{k_2=0}^{\min(\mu_1 - k_1, \lambda_1 - \lambda_2)} \sum_{k_3=0}^{\min(\lambda_2 - \lambda_3, \mu_1 - k_2)} \sum_{k_4=0}^{\min(\lambda_1 + k_1 - \lambda_2 - k_2, \mu_2 - k_3)} \sum_{l_1=0}^{\min(\lambda_1 + k_1 - \lambda_2 - k_2, \mu_2 - k_3)} \sum_{l_2=0}^{\min(\lambda_2 + k_2 - \lambda_3 - k_3, \mu_2 - l_1 - k_4 - k_2 - l_1)} \min(\lambda_2 + k_2 - \lambda_3 - l_2 - k_4 - l_1 - k_4 - l_1)
\]
it can be shown that only three variables 

\[ \min(\lambda_3 + k_3 - k_4, \mu_2 - l_1 - l_2, k_1 + k_2 + k_3 - l_1 - l_2) \]

\[ \times \sum_{l_j=0}^{\min(\lambda_3 + l_2 - k_4 - l_1, \mu_3 - n_1, l_1 + l_2 - n_1)} \]

\[ \times \sum_{n_2=0}^{\min(\lambda_3 + k_3 + l_2 - k_4 - l_1, \mu_3 - n_1, l_1 + l_2 - n_1)} \]

\[ \times [\lambda_1 + k_1, \lambda_2 + k_2 + l_1, \lambda_3 + k_3 + l_2 + n_1, k_4 + l_3 + n_2], \]

where the following constraints:

\[ \sum_{i=1}^{4} k_i = \mu_1, \quad \sum_{i=1}^{3} l_i = \mu_2, \quad \sum_{i=1}^{2} n_i = \mu_3, \]

apply in the summation. In the resultant irrep \[ [\nu_1 \nu_2 \nu_3 \nu_4] = [\lambda_1 + k_1, \lambda_2 + k_2 + l_1, \lambda_3 + k_3 + l_2 + n_1, k_4 + l_3 + n_2], \] with the restrictions given by (3.9), there may be six ways to relabel the configuration that leave the irrep unchanged:

\[
\begin{array}{c|c|c}
\lambda_1 & k_1 & \alpha \\
\hline
\lambda_2 & k_2 & l_1 & \beta \\
\lambda_3 & k_3 & l_2 & \gamma \\
\lambda_4 & k_4 & n_1 & \delta \\
\end{array}
\]

(3.10a)

where

\[ k_1 = \nu_1 - \lambda_1, \quad k_2 = \nu_2 - \lambda_2 - \xi_1 - \xi_2, \]

\[ k_3 = \nu_3 - \lambda_3 - \mu_2 + \xi_1 - \xi_3 - \xi_4 - \xi_5 - \xi_6, \]

\[ k_4 = \nu_4 - \mu_3 + \xi_2 + \xi_3 + \xi_5 + \xi_6, \]

\[ l_1 = \xi_1 + \xi_2, \quad l_2 = \mu_2 - \xi_1 - \xi_4 + \xi_5, \quad l_3 = \xi_4 - \xi_2 - \xi_5, \]

\[ n_1 = \xi_3 + \xi_4 + \xi_6, \quad n_2 = \mu_3 - \xi_3 - \xi_4 - \xi_6, \]

and \( \alpha, \beta, \gamma \) are the symbols filling in each box according to the Littlewood rules. However, by using the following transformation:

\[ \eta_1 = \xi_1 + \xi_2, \quad \eta_2 = \xi_3 + \xi_4 + \xi_6, \quad \eta_3 = \xi_4 - \xi_2 - \xi_5, \]

(3.11)

it can be shown that only three variables \( \eta_i \) with \( i = 1, 2, \) and 3 are independent. Therefore, (3.10) can be relabeled in terms of these three variables:

\[ k_1 = \nu_1 - \lambda_1, \quad k_2 = \nu_2 - \lambda_2 - \eta_1, \]

\[ k_3 = \nu_3 - \lambda_3 - \mu_2 + \eta_1 - \eta_2 + \eta_3, \]

\[ k_4 = \nu_4 - \mu_3 + \eta_2 - \eta_3, \]

(3.12)

\[ l_1 = \eta_1, \quad l_2 = \mu_2 - \eta_1 - \eta_3, \]

\[ l_3 = \eta_3, \quad n_1 = \eta_2, \quad n_2 = \mu_3 - \eta_2. \]

Applying the Littlewood rules to this result yields the following boundary conditions for the outer multiplicity labels \( \eta_1, \eta_2, \) and \( \eta_3: \)

\[ \eta_1_{\text{min}} \leq \eta_1 \leq \eta_1_{\text{max}}, \quad \eta_2_{\text{min}} \leq \eta_2 \leq \eta_2_{\text{max}}, \quad \eta_3_{\text{min}} \leq \eta_3 \leq \eta_3_{\text{max}}, \]

(3.13)

where
corresponding irreps because of the Schur-Weyl duality relation between SU($n$) and $SU(\infty)$ applied to eliminate spurious upper Gel'fand patterns. Hence, the results for $SU(\infty)$ as that of $SU(n)$, for example, the multiplicity of the Kronecker product for two two-rowed irreps of $SU(n)$ is a special Gel'fand pattern. For that of $SU(3)$, and that for two two-rowed irreps is the same as that of $SU(4)$, and so on. Hence, the results for $SU(3)$ and $SU(4)$ apply for the general $SU(n)$ case as well. As a trivial example, note that the $SU(3)$ multiplicity expression follows from the one for $SU(4)$ in the two-rowed limit ($\eta_1 \rightarrow \eta$, $\eta_2 = 0$, and $\eta_3 = 0$) of the theory, though this fact cannot be clearly seen from (3.14).

\[
\begin{align*}
\eta_{1\min} &= \max(0, \nu_2 - \lambda_1), & \eta_{1\max} &= \min(\nu_2 - \lambda_2, \nu_1 - \lambda_1), \\
\eta_{2\min} &= \max(\nu_3 - \nu_2 + \eta_1, 0), & \eta_{2\max} &= \min(\eta_1, \mu_3, \nu_3 - \nu_4).
\end{align*}
\]

\[
\begin{align*}
\eta_{3\min} &= \max(2 \eta_2 - \eta_1 + \mu_2 - \nu_3 - \nu_3, \eta_2 - \eta_1 + \lambda_3 + \mu_2 - \nu_3, \eta_2 + \nu_4 - \lambda_3 - \mu_2 + \nu_3 - \nu_3, \eta_1 + \lambda_2 + \mu_2 - \nu_1 - \nu_2, 0, \text{Int}(\eta_2 + \lambda_2 + \lambda_3 + 2 \mu_2 - \nu_1 - \nu_2 - \nu_3, \eta_1 + \lambda_1 + \lambda_2 + \mu_2 - \nu_1).
\end{align*}
\]

\[
\eta_{3\max} = \min(\mu_2 - \eta_1, \nu_4 - \mu_3 + \eta_2, \mu_2 - \mu_3, \lambda_2 - \nu_3 + \mu_2 - \eta_1 + \eta_2, \mu_2 - \eta_2).
\]

where $\text{Int}[x]$ is the integer part of $x$. Thus, the multiplicity of $[\nu_1, \nu_2, \nu_3, \nu_4] = [\nu_1 - \nu_4, \nu_2 - \nu_4, \nu_3 - \nu_4]$ appearing in the Kronecker product $[\lambda_1 \lambda_2 \lambda_3] \times [\mu_1 \mu_2 \mu_3]$ can be calculated by

\[
\text{Multi}(SU(4)) = \sum_{\eta_1 = \eta_{1\min}}^{\eta_{1\max}} \sum_{\eta_2 = \eta_{2\min}}^{\eta_{2\max}} \sum_{\eta_3 = \eta_{3\min}}^{\eta_{3\max}} \eta_1 \eta_2 \eta_3.
\]

The complementary group of the Kronecker product $[\lambda_1 \lambda_2 \lambda_3] \times [\mu_1 \mu_2 \mu_3]$ of SU(4) is $\mathcal{U}(6)$, with the following special Gel'fand labels:

\[
\begin{pmatrix}
[\nu_1 \nu_2 \nu_3 \nu_4] \\
[\nu_1, \nu_2, \nu_3 - \eta_2, \nu_4 - \mu_3 + \eta_2, 0] \\
[\nu_1, \nu_2 - \eta_1, \nu_3 - \mu_2 + \eta_1 + \eta_3 - \eta_2, \nu_4 - \mu_3 + \eta_2 - \eta_3] \\
[\lambda_1 \lambda_2 \lambda_3]
\end{pmatrix}
\mathcal{U}(6)
\]

\[
\begin{pmatrix}
[\mu_1 \mu_2 \mu_3] \\
[\mu_1 \mu_2 \mu_3] \\
[\mu_1 \mu_2] \\
[\mu_1] \\
[0]
\end{pmatrix}
\mathcal{U}(3)
\]

\[
\rho
\]

The Gel'fand labels of $[\lambda_1 \lambda_2 \lambda_3]_0$ and $[\mu_1 \mu_2 \mu_3]_0$ for $\mathcal{U}(6)$ are

\[
\begin{pmatrix}
[\lambda_1 \lambda_2 \lambda_3]_0 \\
[\lambda_1 \lambda_2 \lambda_3]_0 \\
[\lambda_1 \lambda_2 \lambda_3]_0 \\
[\lambda_1 \lambda_2 \lambda_3]_0 \\
[\lambda_1 \lambda_2 \lambda_3]_0 \\
[\lambda_1 \lambda_2 \lambda_3]_0 \\
\rho
\end{pmatrix}
\mathcal{U}(6)
\]

\[
\begin{pmatrix}
[\mu_1 \mu_2 \mu_3]_0 \\
[\mu_1 \mu_2 \mu_3]_0 \\
[\mu_1 \mu_2 \mu_3]_0 \\
[\mu_1 \mu_2 \mu_3]_0 \\
[\mu_1 \mu_2 \mu_3]_0 \\
[\mu_1 \mu_2 \mu_3]_0 \\
\rho
\end{pmatrix}
\mathcal{U}(3)
\]

Again, most of the boundary conditions for the multiplicity labels $\eta_1$, $\eta_2$, and $\eta_3$ can be obtained from the betweenness conditions for the Gel'fand symbol shown in (3.16). However, the remaining conditions can only be deduced from the Littlewood rules because (3.16) is a special Gel'fand basis for the canonical chain $\mathcal{U}(6) \supset \mathcal{U}(5) \supset \mathcal{U}(4) \supset \mathcal{U}(1)$. From this development it is clear that there are at most, three quantum numbers needed to label the outer multiplicity for the decomposition SU(4) × SU(4) | SU(4). In the canonical unit tensor approach proposed by Biedenharn et al. for the SU(4) case, there are four shifts out of six upper labels, of which only three labels are independent. As for the SU(3) case, restrictions that follow from the Littlewood rules must be applied to eliminate spurious upper Gel'fand patterns.

It should be noted that any SU($n$) function, for example, CGCs, RWCs, or Racah coefficients, etc., is rank $n$ independent, and only depends on boxes contained in the Young diagrams of the corresponding irreps because of the Schur-Weyl duality relation between SU($n$) and $S_f$. For example, the multiplicity of the Kronecker product for two two-rowed irreps of SU($n$) is the same as that of SU(3), and that for two three-rowed irreps is the same as that of SU(4), and so on. Hence, the results for SU(3) and SU(4) apply for the general SU($n$) case as well. As a trivial example, note that the SU(3) multiplicity expression follows from the one for SU(4) in the two-rowed limit ($\eta_1 \rightarrow \eta$, $\eta_2 = 0$, and $\eta_3 = 0$) of the theory, though this fact cannot be clearly seen from (3.14).
IV. CONCLUSIONS

In this paper, a complementary group $U(2n - 2)$ to $SU(n)$ is found that gives a complete realization of all the features of the Littlewood rules in the Kronecker product decomposition of $SU(n) \times SU(n) \mid SU(n)$. By using this scheme, the outer multiplicity labels for $SU(n)$ can be easily assigned, being nothing other than a set of sublabels of the special Gel’fand basis of the complementary $U(2n - 2)$ group. Furthermore, within this framework, most of the boundary conditions on the multiplicity labels can be easily obtain from the betweenness conditions of the Gel’fand symbols of $U(2n - 2)$, while the remaining conditions must be deduced from the Littlewood rules. The method was used to obtain simple multiplicity formulas for $SU(3)$ and $SU(4)$. In addition, in the coupling of two $SU(n)$ irreps, the basis for $SU(n)$ can further be labeled by the final $U(2n - 2)$ sublabels $\eta_i$ obtained from the coupling of two uncoupled basis vectors of the corresponding special Gel’fand basis of $U(2n - 2)$, which are missing within the $SU(n)$ group. This situation is similar to the canonical unit tensor approach proposed by Biedenharn et al. However, in the canonical unit tensor approach, there are $n$ independent shifts indicated by the upper pattern of $U(n)$ from the $n(n - 1)/2$ upper labels. While these upper indices can be used to label the outer multiplicity of $U(n)$, there may very well be spurious degrees of freedom among the labels, and these have to be eliminated according to restrictions of the Littlewood rules of $SU(n)$.

It should be noted that the same complementary group for a resolution of $SU(n)$ was considered in Ref. 15. However, the method used and the final outcome are all different from those of this article. First, in Ref. 15, the complementary group structure was derived through boson realizations, whereas in the present case it comes directly from the Littlewood rules. Second, according to Ref. 15, the complementary group should be labeled in terms of a noncanonical chain $U(2n - 2) \supset U(n - 1) \times U(n - 1)$. In this way, the RWCs of $SU(n)$ cannot be derived easily because new inner multiplicity occurs in the decomposition $U(2n - 2) \mid U(n - 1) \times U(n - 1)$. In order to overcome this problem, another type of Wigner coefficients, so-called auxiliary Wigner coefficients, was introduced, which is different from the standard definition and which satisfies other orthogonality conditions. These special Wigner coefficients will be considered in the next paper, where it will also be shown that one can derive analytical expressions for some simple cases and algorithms for $SU(n)$ RWCs or CGCs with general multiplicity in canonical as well as noncanonical bases if the multiplicity-free coefficients in these bases are known.

The complementary $U(2n - 2)$ group scheme for labeling outer multiplicities in Kronecker products of $SU(n)$ is canonical because the basis of $SU(n)$ labeled in this way is orthogonal with respect to the outer multiplicity labels. A general procedure for evaluating RWCs or CGCs for $SU(3)$ and $SU(4)$ will be given in forthcoming papers.

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