ANALYTIC EXPRESSIONS FOR ENERGY CENTROIDS
AND WIDTHS IN THE MICROSOPIC COLLECTIVE MODEL

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Received 31 September 1984

Abstract: Analytic formulae are given for the U(3) centroids of the collective Bohr-Mottelson potential
as the microscopic collective model. In particular, formulae are reported for the centroids of the
quadratic $[\xi (Q - Q^*)]$ and cubic $[\xi (Q - Q^*)^3]$ rotational operators in the microscopic
quadrupole operator. Favorable comparisons for ground-state intruder are achieved between
shell-model diagonalizations and statistical predictions based upon the gaussian approximation to
the energy density. These results suggest that statistical methods can be used reliably for
triangulation of the infinite-dimensional representation space of the microscopic symplectic
collective theory.

1. Introduction

The symplectic collective model is a completely microscopic generalization of the
Bohr-Mottelson geometrical model [1-13]. The symplectic theory extends Elliott’s
SU(3) shell model to encompass core-excited basis wave functions as dictated by
the quadrupole and monopole collective degrees of freedom. Thus, the model space
of the symplectic theory is defined to be an irreducible unitary representation space
for the real symplectic Lie algebra Sp(3, R). But, since the symplectic Lie algebra is
noncompact, the symplectic model space is necessarily infinite-dimensional.

Of course, the hamiltonian eigenvalue problem on an infinite-dimensional space
poses, in general, an intractable technical obstacle. Only very special model
hamiltonians with a large symmetry algebra, e.g., the harmonic oscillator $H_0$, may
be solved analytically. Fortunately, because of the nuclear shell structure, the
symplectic-model space can be truncated safely to span only the first few major
oscillator shells. For example, in symplectic-model calculations in the $d$-shell for
$^{20}$Ne and $^{24}$Mg, a truncation including $10\hbar \omega$ oscillator excitations above the $10\hbar \omega$
d shell was found to be adequate [11]. Nevertheless, the dimension (145, 530) of the
$^{24}$Mg $10\hbar \omega$ truncated space is still formidable. For nuclei in the rare-earth region,
a truncation beyond 10hω may be required, and the resulting dimensions are prohibitive. Hence, it would be desirable for light nuclei and an absolute necessity for heavy nuclei to truncate the model space further.

The most extreme truncation is the Sp(1, R) model of Aricks, Brookhove and Devenney. In this approximation, one selects from each \( n \) h\( \omega \) harmonic-oscillator subspace contained in a symplectic-model space a single SU(3) irreducible representation \( (\lambda_n, n, \mu_n) \) for \( n = 0, 2, 4, \ldots \). Here \( (\lambda_n, \mu_n) \) denotes the Elliott \( n \)th SU(3) irreducible representation on which the symplectic core excitations are built.

A less severe truncation is the Sp(2, R) model of Peterson and Liebman. In the Sp(2, R) model space, only the SU(3) irreducible representations \( (\lambda_n + n - 2, \mu_n + 2) \) for \( j = 0, 1, \ldots, \min(n + 1, \lambda_n + 2) \) are chosen from the \( n \)th oscillator eigenstate. Yet another truncation scheme appropriate for prolate nuclei is to select the SU(3) representations \( (\lambda, \mu) \) with \( \lambda \geq \mu \) and discard the oblate representations with \( \lambda < \mu \).

In order to invoke one of these truncation schemes, an a priori justification is required. Thus, a quantitative measure is needed which estimates the contribution of each SU(3) irreducible representation space to the ground-state wave function. Moreover, a useful indicator ought to be simple to compute.

In this paper, a reliable estimate for the intensities of the U(3) representation spaces in the ground state is achieved in terms of the U(3) centroids and widths of the Hamiltonian \( H \) by making the gaussian assumption of statistical spectroscopy. For both \( \text{^{26}Na} \) and \( \text{^{24}Mg} \), we find remarkably good agreement between theoretical predictions for the intensities and shell-model results for the intensities.

The U(3) centroids of \( H \) are determined solely by its U(3) core operator \( H_c \); i.e., the U(3) scalar part of the Hamiltonian. But for a collective Hamiltonian, i.e., a polynomial in the symplectic algebra generators, the U(3) core must be a polynomial in the Sp(3, R) = U(3) integrability basis. Since the integrability basis consists of rather simple operators, the explicit expression for the core of the collective hamiltonian provides analytic formulas for the energy centroids. A similar analysis will also be carried out for the core of an SU(3) effective residual interaction; its U(3) core is a polynomial in the SU(3) Casimir invariants.

The hamiltonian considered in this paper is in the Sp(3, R) enveloping algebra. It is the sum of the harmonic oscillator, the collective quadrupole potential \( V_{QQ} \), and the SU(3) residual interaction \( V_{Res} = H_c + V_{Res} + V_{QQ} \). The collective potential is a rotational scalar in the mass quadrupole tensor \( Q_{ij}^2 = Q_{ij} \cdot Q_{ij} \). Here \( Q_{ij} \) denotes the mass tensor \( Q_{ij} = \sum (x_i - X_i)(x_j - X_j) \), where \( X_i \) is the center of mass. One knows that every rotational quadrupole scalar must be a function of the two independent rotational scalars

\[
\alpha_2 = \frac{1}{6} \sum_i Q_{ii}^2, \quad \alpha_1 = \text{det}(Q^2),
\]

(1)
We restrict the collective potential to a quartic form:

\[ V_{\text{coll}} = b_2 \mu I + b_3 \mu J + b_4 \mu K, \]

where \( \mu = \mu_0^2 \) and \( b_2, b_3, b_4 \) are real constants.

The SU(3) residual interaction is a totall scalar in the SU(3) algebra. We choose it to be at most quartic in the SU(3) generators:

\[ V_{\text{res}} = \kappa_1 \gamma X_3 + \kappa_2 \gamma X_4 + \gamma J^2 + \gamma K^2. \]

Here the \( \gamma \)-dependent scalars are \( X_3 = L \cdot \{ C^{(3)} \times L \}^{\dagger} \) and \( X_4 = [L \times C^{(3)}]^{\dagger} \cdot [C^{(3)} \times L]^{\dagger} \), where \( C^{(3)} \) is the SU(3) quadrupole generator and \( L \) denotes the total orbital angular momentum. The non-collective aspects of the nuclear force, primarily pairing and spin-orbit, break symplectic symmetry. \( V_{\text{res}} \) should be regarded as an effective interaction arising from the truncation to a single symplectic representation space.

Since the model Hamiltonian is built from symplectic generators, it leaves each irreducible representation space of Sp(3, \( \mathbb{R} \)) invariant. Each such space is generated from a starting Elliott U(3) representation \( \mathcal{N}_3(2\mu_0) \) at the 0\( \mu_0 \) major oscillator shell by successively acting on it with the 2\( \mu_0 \) symplectic raising operator

\[ A_\mu = \frac{1}{2} \sum \epsilon_{ijk} \gamma^i \gamma^j e^k - \frac{1}{24} \left( \sum \gamma^i \gamma^j \right) \left( \sum \gamma^k \gamma^l \right), \]

where \( \epsilon_{ijk} = 2 \cdot (\gamma^i \gamma^j \gamma^k) \) and the sums are carried over the \( A \) nucleons. This creates model states as oscillator excitation energies \( \mu = 0, 2, 4, \ldots, \mu_0 \), which are viewed as spanning a vertical slice of collective states contained within the complete antisymmetrized \( A \)-nucleon Hilbert space. Note that the \( 1/4 \) term in the raising operator eliminates spurious center-of-mass excitations.

In addition to being a U(3) irreducible representation space, the starting point \( \mathcal{N}_3(2\mu_0) \) must also annihilated by the 2\( \mu_0 \) symplectic lowering operator \( B_\mu = A_\mu^\dagger \).

These special starting representations appear naturally in the shell model. For example, in the case of \( ^{76}\text{Ge} \), any state vector consisting of a closed \( ^{40}\text{Ca} \) core plus four nucleons distributed in the 2\( \mu_0 \) shell is annihilated by \( B_\mu \) since the 1\( \mu_0 \) shell is occupied completely. The leading U(3) representation for \( ^{76}\text{Ge} \) is the \( \mathcal{N}_3(2\mu_0) = 48.5(8.9) \). For \( ^{24}\text{Mg} \), it is 62.5(8.4).

In order to construct a natural basis for a slice, we exploit the fact that the raising operator is a \( (2,0,0) \) U(3) tensor operator. Then, the product of raising operators may be coupled to good total U(3) symmetry \( (n_1, n_2, n_3) \). But, since the \( A_\mu \) commute among themselves, only totally symmetric couplings are non-zero. Thus, we are restricted to U(3) tensor couplings for which \( n_1, n_2, n_3 \) are even negative integers satisfying \( n_1 \geq n_2 \geq n_3 \) [Ref. 25].
A complete infinite-dimensional basis for a slice is produced by finally coupling symmetric products of raising operators to the starting representation,  
\[ |(r_1, r_2, \ldots, r_p) \rangle = \left[ (A \times A \times \cdots \times A)_{n_1 \times n_2 \times \cdots \times n_p} \right] |(s_1, s_2, \ldots, s_p) \rangle, \]  
(5)  
where \( p \) is the multiplicity of \( |s_p \rangle \) in the SU(3) tensor product \((n_1 + n_2, n_3 + n_4) \otimes |s_1, s_2, \ldots, s_p \rangle \). The basis labels for \(|s_p \rangle \) have been suppressed. Note that these states are eigenstates of the harmonic oscillator \( H_0 \) belonging to the eigenvalue \( N_0 = (n_1 + n_2 + n_3) = N_0^* + n \). This basis is not, in general, an orthogonal one. Two basis states belonging to inequivalent irreducible representations of \( \text{U}(3) \) are necessarily orthogonal. But, in the case of nontrivial \( \text{U}(3) \) multiplicity, the equivalent representation spaces are only asymptotically orthogonal as \( N_0 \to \infty \).

2. U(3) centroids

The contribution of a \( \text{U}(3) \) irreducible representation \( n(|s_0 \rangle \) to the ground-state wave function is determined in large measure by its average energy or centroid \( \langle H \rangle_{n(|s_0 \rangle \).  
\[ \langle H \rangle_{n(|s_0 \rangle = \frac{1}{\dim(|s_0 \rangle \left| \psi_{\text{average}}(H) \right.} \]  
(6)

The trace of the Hamiltonian \( H \) is computed within the \( \text{U}(3) \) representation space \( n(|s_0 \rangle \) and \( \dim(|s_0 \rangle \) denotes the dimension of \( |s_0 \rangle \). One anticipates that the lower the centroid, the greater the contribution to the ground state.

The computation of the centroid is simplified considerably if \( H \) is a component of an irreducible \( \text{U}(3) \) tensor operator. In that case, the trace within an irreducible representation space varies unless \( H \) is a \( \text{U}(3) \) scalar.

To prove this, suppose that \( |H_0 \rangle \) are the components of a \( \text{U}(3) \) reducible tensor operator,  
\[ \left[ \bar{X}, H_0 \right] = \sum \bar{X}(x) \bar{X}_x, \]  
for all \( X \in \text{U}(3) \),

where \( \pi \) is an irreducible \( \text{U}(3) \) representation. Although, in general, the \( H_0 \) do not leave the representation space \( n(|s_0 \rangle \) invariant, the \( \text{U}(3) \) operator \( X \) does. Hence, the trace of the commutator \( [X, H_0] \) computed within the \( n(|s_0 \rangle \) space vanishes. We then have  
\[ \sum \bar{X}(x) \bar{X}_x = 0. \]  
Thus, \( \psi_{\text{average}}(H_0) \) is in the kernel of \( \pi \), an invariant subspace. Since \( \pi \) is irreducible,
there are only two possibilities: either the kernel is the entire carrier space of \( n \), i.e., \( n = 0 \), or the kernel is the null subspace. In the former case, \( H_2 \) is a \( U(3) \) scalar. In the latter, the trace vanishes, \( tr_{u(3)}(H_2) = 0 \).

In the general case of an arbitrary Hamiltonian \( H \), we first expand \( H \) as a sum of components of various irreducible \( U(3) \) tensor operators (as may always be done). Since the \( U(3) \) traces of the nonscalar tensors vanish, the centroid of \( H \) is determined completely by the scalar terms in its tensor expansion. We refer to this scalar part of the expansion as the \( U(3) \) scalar core of \( H \) and denote it by \( H^s \). Clearly, \( \langle H \rangle_{u(3)} = \langle H^s \rangle_{u(3)} \).

When \( H \) is a polynomial in the generators of a Lie algebra \( g \) containing \( U(3) \), then an enormous simplification is permissible. Any \( U(3) \) scalar operator in the enveloping algebra of \( g \) must be a polynomial in the \( g \supset U(3) \) integrity basis. If \( g \) is a finite-dimensional simple Lie algebra, then the integrity basis must also be finite-dimensional \(^{16,17}\).

For the case \( g = Sp(3, R) \), the integrity basis is known \(^{18}\). To fourth degree in the symplectic generators, there are eight independent \( U(3) \) scalar operators:

\[
\begin{align*}
[H_0, C_2, C_3, C_2, G_2, G_0, X, Z_1, Z_2] = \{ &H_0, C_2, C_3, C_2, G_2, G_0, X, Z_1, Z_2\} \quad (7)
\end{align*}
\]

\(C_2\) and \(C_3\) denote the quadratic and cubic \( SU(3) \) Casimir invariants with eigenvalues

\[
\begin{align*}
C_2 &= \frac{1}{2}(x^2 + \mu^2 + 4\lambda + 3\mu), \\
C_3 &= \frac{1}{2}(4x^2 + 3\mu^2 - 3\mu^2 - 2\mu^2 + (x^2 - \mu^2 + \lambda - \mu)).
\end{align*}
\]

\(G_2\) and \(G_3\) are the quadratic and quartic Casimir invariants of \( Sp(3, R) \). Thus, they are constant within each infinite-dimensional irreducible representation of \( Sp(3, R) \).

For the vertical slice starting from the \( N \) \((\mu, \lambda, \mu_0) \) representation of \( U(3) \) we have

\[
\begin{align*}
G_2 &= C_2 + 1/2 N_0^2 - 4 N_0, \\
G_3 &= \frac{1}{2}(C_2 - 8C_1 + 26C_0 + 5C_0 - 8C_1 N_0) \\
&\quad + \frac{1}{2}(C_2 N_0^2 + 41 N_0^2 - 2N_0^2 + 4N_0^2 + 48 N_0),
\end{align*}
\]

where \( C_2 \) and \( C_3 \) are evaluated for the \((\mu, \lambda, \mu_0) \) starting representation \(^{18}\).

\(X, Z_1\), and \(Z_2\) are \( U(3) \) scalars in the \( Sp(3, R) \) enveloping algebra which are neither Casimir invariants of \( U(3) \) nor of \( Sp(3, R) \):

\[
\begin{align*}
Y &= \sqrt{3} \{(A \times C) \otimes (B)^{200}\}, \\
Z_1 &= \sqrt{6} \{(A \times C) \otimes (C \otimes B)^{200}\}, \\
Z_2 &= \sqrt{3} \{(A \times C)^{201} \otimes (C \otimes B)^{100}\}.
\end{align*}
\]
These are SU(3)-coupled scalar tensors formed from the (22) raising operator A, its adjoint (22) lowering operator B, and the (11) SU(3) generator C. There are as yet no analytic expressions for the eigenvalues of Y, Z, and Z.

Let us now explicitly compute the U(3) scalar cores of a2, a3, and a4 as polynomials in the Sp(3, R) U(3) integrality basis. Recall that a2 is in the quadratic rotational scalar in the mass quadrupole operator Q0. Since Q0 is a sum of the L = 2 components of the U(3) tensors A(22), B(22), and C(22), a2 itself may be expressed as a sum of the L = 0 components of all the possible U(3) tensor couplings of A(22), B(22), and C(22) [ref. 17]. The U(3) core of a2 is given by just the U(3) scalar coupled terms in the full tensor expansion. For example, the rotational scalar product of the L = 2 components of A and B which is one of the terms in Q2, must be a sum of the L = 0 components of the U(3) tensor operators [4 × B]000 and [4 × B]022. Note that the coupling [4 × B]011 is not involved since it has no L = 0 component. The core of the rotational scalar product of the L = 2 components of A and B is given by disconnecting the (22) tensor and retaining the U(3) scalar [4 × B]000. After collecting all the U(3) scalars in the expansion of a2 and expressing [4 × B]000 in terms of the quadratic Sp(3, R) Casimir operator C2, the final result for the core of a2 is

\[ a_2^2 = \frac{1}{2} G_4 + \frac{1}{2} H_4 - \frac{1}{2} G_2. \]  

(11)

Similar calculations give the cores of a3 and a4. However, since a2 and a3 are of degree three and four in the Sp(3, R) generators, respectively, their cores are correspondingly extended. The U(3) scalar cores of a3 and a4 are given by

\[ a_3^2 = \sqrt{2} G_3 + \frac{1}{2} H_3 - \frac{1}{2} G_2 + \frac{3}{2} C_3. \]  

(12)

The U(3) scalar cores of the residual interaction terms are polynomials in the two Casimir invariants of SU(3), C3 and C4. Reasoning analogous to the more complicated Sp(3, R) U(3) case yields their cores:

\[ X_3 = -\frac{1}{2} \sqrt{2} C_3, \quad X_4 = -\frac{1}{2} C_3(C_3 + 1), \quad (\frac{1}{2} F') = 2 C_2, \quad (\frac{3}{2} F') = -\frac{3}{2} C_3(C_3 - 1). \]  

(14)
Table 1

The asterisk indicates that the corresponding state contains a $J = 0$ state.

$H = h_0 (\mathbf{r}_0 + \mathbf{r}_p) = \mathbf{\alpha}_0 \mathbf{r}_0^2 + \eta \mathbf{x}_0 \mathbf{r}_0^2 + \eta \mathbf{r}_p \mathbf{r}_0^2 + \eta \mathbf{x}_p \mathbf{r}_0^2$, with $\mathbf{\alpha}_0 = 12.0$ MeV and $\eta = -0.2$. $\mathbf{\alpha}_0 = 0.00975, \eta = 0.0025, x_0 = -0.00044, x_p = -0.00045$ MeV.

In Table 1, the $U(3)$ centroids up to $4\hbar \omega$ are listed for the case of $^{24}$Mg using the same potential parameters as in ref. 7. Although the $n\mathbf{h}\omega$ major oscillator shell is split by the collective potential and, to a lesser extent, by the residual $SU(3)$ interaction, the energy centroids of the $U(3)$ representation spaces selected from the $n\mathbf{h}\omega$ shell all lie approximately $n\mathbf{h}\omega$ above the starting $0\hbar \omega$ representation. Hence, the basic oscillator shell structure is preserved. Note also that among the $U(3)$ representations from the $n\mathbf{h}\omega$ shell, the centroid of the stretched state lies low in energy. Thus, we expect the stretched approximation to be good here. Identical conclusions were drawn for $^{20}$Ne previously.7)
3. Ground-state intensities

Although the locations of the U(3) centroids indicate the relative contributions of the U(3) representation spaces to the ground-state wave function, a reliable quantitative measure demands a satisfactory approximation to the energy densities $\rho_{\alpha\beta\gamma}(E)$. Here, following French and collaborators,[14][15], we make the gaussian approximation which, in addition to the centroids, incorporates knowledge of the widths into the densities. Numerical estimates for the intensities of the U(3) spaces in the ground state predicted with the approximate gaussian densities will be found to favorably compare with shell-model diagonalization.

The exact densities are distributions in the energy. Thus, if $f$ is any continuous function of the energy $E$, then the U(3) centroid of $f(E)$ is given by

$$\int_{-\infty}^{\infty} f(E)\rho_{\alpha\beta\gamma}(E)\,dE = \dim (\langle \mu \rangle; f(H)_{\alpha\beta\gamma})$$

In particular, the area under the density is the dimension

$$\int_{-\infty}^{\infty} \rho_{\alpha\beta\gamma}(E)\,dE = \dim (\langle \mu \rangle)$$

and the centroid is

$$\int_{-\infty}^{\infty} E\rho_{\alpha\beta\gamma}(E)\,dE = \dim (\langle \mu \rangle; H)_{\alpha\beta\gamma}$$

In general, the $n$th moment is given by

$$\int_{-\infty}^{\infty} E^n\rho_{\alpha\beta\gamma}(E)\,dE = \dim (\langle \mu \rangle; H^n)_{\alpha\beta\gamma}$$

A formula for the exact density can be expressed in terms of the projection operator $P_{\alpha\beta\gamma}$ onto the U(3) representation space $\rho(\mu)$ and the spectral decomposition of the hamiltonian

$$H = \sum_{\mu} E_{\mu} \rho_{\mu}$$

where $x$ is the projector onto the eigenspace belonging to the eigenvalue $E_{\mu}$. Then the density is the distribution

$$\rho_{\alpha\beta\gamma}(E) = \sum_{\mu} \text{tr}(P_{\alpha\beta\gamma} \rho(\mu) \delta(\epsilon - E_{\mu}))$$

We shall make the gaussian approximation to the density

$$\rho_{\alpha\beta\gamma}(E) \approx \frac{\dim (\langle \mu \rangle)}{\sigma_{\beta\gamma}^2 \sqrt{2\pi}} \exp\left(-\frac{(E - \langle H \rangle)^2}{2\sigma_{\beta\gamma}^2}\right)$$
where \( \langle H \rangle = \langle H \rangle_{\text{cent}} \) is the centroid and \( \sigma \) denotes the width

\[
\sigma_{\text{peak}} = \left[ (\langle H \rangle)_{\text{peak}} - \langle H \rangle_{\text{cent}} \right]^2.
\]

(21)

The Gaussian approximation correctly gives the area (16), centroid (15) and the \( k = 2 \) moment (18), but, in general, fails to reproduce the higher moments \( k \gg 3 \).

Analytic formulae for the widths are given by the \( U(3) \) scalar core of the square of the potential. For the residual interaction, the core has been determined as follows:

\[
\begin{align*}
\langle X^2 \rangle &= \frac{1}{2} C_2^2, \\
\langle X Y \rangle &= \frac{1}{2} C_2^2 + \frac{1}{2} \left( C_2^2 + 1 \right), \\
\langle X^4 \rangle &= \frac{1}{2} C_2^2 + \frac{1}{2} \left( C_2^2 + 1 \right), \\
\langle X^3 \rangle &= \frac{1}{2} C_2^2 + \frac{1}{2} \left( C_2^2 + 1 \right), \\
\langle X^2 Y \rangle &= \frac{1}{2} C_2^2 + \frac{1}{2} \left( C_2^2 + 1 \right).
\end{align*}
\]

(22)

Because of their complexity, the cores of \( V_{\text{G}} \) and \( V_{\text{res}} \) have not been determined analytically. Both are \( U(3) \) scalars of degree eight in the symplectic generators. Their widths have been evaluated by explicitly computing the relevant traces.

In Table 1, the widths for the case of \( ^{14}Mg \) are given. Note that the widths are approximately \( \hbar \omega \). Hence, we expect non-negligible admixtures of higher shells into the ground-state domain. Nevertheless, since the widths are not equal to or greater than \( 2\hbar \omega \), the shell structure is still the dominant underlying feature. Since the widths of the stretched states are the largest in any given shell, these states will correspondingly contribute the most to the ground state.

In order to numerically estimate the ground-state intensities, consider the cumulative distribution

\[
F_{\text{rad}}(E) = \int_{-\infty}^{E} \rho_{\text{rad}}(E') \, dE',
\]

(23)

and the distribution

\[
F(E) = \sum_{E_{i}} F_{\text{rad}}(E).
\]

(24)
For the exact density, the cumulative distribution $F_{nsl}(E)$ is zero for $E$ less than the ground-state energy $E_1$. At $E_1$, it jumps discontinuously to

$$F_{nsl}(E_1) = \text{tr}(\rho_{nsl} \delta(E - E_1)) = |\langle \psi_{nsl} | \delta(E - E_1) | \psi_{nsl} \rangle|^2.$$  \hspace{1cm} (25)

Thus, $F_{nsl}(E_1)$ is the intensity of the U(3) representation space $n(l)\mu$ in the ground state. $F(E)$ jumps discontinuously at $E_1$ to $\text{tr}(\rho_1) = 1$, since the ground-state eigenspace is always one-dimensional if $J = 0$.

In the gaussian approximation, the cumulative densities are continuous functions. In the neighborhood of the ground state, the smoothed distribution function $F(E)$ rises continuously from zero to one. We define the gaussian approximation $F_1$ to the ground-state energy by $F_1(E) = \frac{1}{2}$. This definition is somewhat arbitrary. However, observe that a cut-off at one would have the disadvantage of including non-negligible contributions to the first excited state from the smoothed continuous distribution function. The approximate intensities of the ground state are given now by $2F_{nsl}(E_1)$.

Knowing that the ground state is an $L = 0$ state allows us to make two improvements to the general prescription. Firstly, we omit from the sum for $F$ all U(3) representations which do not contain an $L = 0$ state. Secondly, the areas under the densities $\rho_{n\lambda\mu}$ are normalized to unity instead of $\text{dim} \rho_{n\lambda\mu}$, since there is at most one $L = 0$ state in $l(\lambda)$.

In fig. 1, the $^{24}\text{Mg}$ densities $\rho_{n\lambda\mu}$ are plotted for the $0$ and $2\hbar \omega$ representations and for two selected $4\hbar \omega$ representations. Although the $0\hbar \omega$ representation clearly dominates, there is significant intrusion of $2$ and $4\hbar \omega$ excited states into the ground-state domain. The precise extent of this is shown by the cumulative densities in fig. 2.

In table 2, the computed statistical predictions for the ground-state amplitudes are compared with the results of shell-model diagonalization from ref. (1). The statistical predictions are particularly good for the larger components of the ground-state wave function. Indeed, the stretched states dominate and the comparison for their components is excellent. One expects the smaller components to be given less accurately, since it is only the tails of their distributions which penetrate the ground-state domain. But, in order to reproduce the tail accurately, higher moments are usually required. The third and fourth moments supply the skewness and excess corrections \(^{19-14}\). In view of the relative crudeness of the gaussian approximation and the fact that one anticipates that skewness and excess corrections ought to be especially important for collective states, the qualitative agreement between the statistical predictions and the shell-model diagonalization is remarkable.
Fig. 1. The densities $n_{\text{K} \cdot \text{J}}$ are plotted versus the energy for the $\text{U} \cdot \text{J}$ representations in $^{24}\text{Mg}$ that contribute most to the ground state. The Hamiltonian is from ref. 1.

<table>
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<th>$\text{U} \cdot \text{J}$</th>
<th>$(x,y)$</th>
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<th>Shell model</th>
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<td>(0 3 0)</td>
<td>0.6</td>
<td>0.17</td>
<td>0.15</td>
</tr>
<tr>
<td>(2 0 0)</td>
<td>0.6</td>
<td>0.17</td>
<td>0.15</td>
</tr>
</tbody>
</table>
Fig. 2. The cumulative densities \( f_{\text{cum}} \) and their sum \( F \) are plotted versus the energy for \(^{24}\text{Mg}\).

4. Conclusions

It has been demonstrated that statistical methods, based upon the gaussian approximation to the energy density, provide the means to identify the most important \( U(3) \) basis vectors to include in a truncated symplectic-mode space. Statistical estimates for the \( U(3) \) intensities of the ground-state wave function by and large track the results of shell-model diagonalization. Furthermore, in no case does the statistical prediction fail to recognize a significant \( U(3) \) representation space, cf. table 2. Similar conclusions were reached previously for \(^{20}\text{Ne}\) as well \(^{21}\).

An improvement to the gaussian statistical estimates can be achieved by
incorporating skewness and excess corrections. This requires the evaluation of third- and fourth-order energy moments. With their inclusion, one would expect quantitatively better estimates for the smaller components. However, for the purpose of achieving a reliable truncation scheme, the higher moments are largely irrelevant.

In order to analytically evaluate the widths of the collective potential, the core of its square must be determined. But, since the square of $X_{pl}$ is an eighth-degree polynomial in the symplectic generators, this presents a formidable computational problem. The only conceivable practical avenue is to use computer algebra.

This work has been supported by the Tulane Committee on Research and the US National Science Foundation. Discussions with colleagues attending the 1984 Queen’s University Summer Institute in Theoretical Physics are gratefully acknowledged.

References

29) S. Okubo, Modified 4th order Casimir invariants and indices for simple Lie algebras, Univ. of Rochester report E8-75.