Transitional Nuclear Spectra within the Framework of Interacting Vector Boson Model with 6-Dimensional Davidson Potential

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Abstract.
A new dynamical symmetry of the Interacting Vector Bosons Model, that starts with the $Sp(12,R) \supset SU(1,1) \times SO(6)$ reduction corresponds to the consideration of a model Hamiltonian which contains 6-dimensional Davidson potential. Exact analytic solutions for its eigenstates are obtained in the basis defined by the subgroup chain of $SO(6) \supset SU(3) \otimes O(2) \supset SO(3)$. This algebraic approach leads to simple and direct applications of the theory to nuclear spectra transitional between the rotational and vibrational cases. Something more, energy spectra of nuclei at the critical point of phase/shape transition are also well reproduced, which is illustrated on the example of $^{157}$Sm with proved $X(5)$ symmetry.

1 Introduction
The interaction between competing collective modes in atomic nuclei is very important in determining their structure [1]. The collective modes that manifest themselves most strongly [2] are rotations and vibrations. These modes are characterized by very specific energy level spacings and electromagnetic transition strengths. The need for a description of nuclei in which rotational-vibrational interactions dominate has led to a search for algebraically solvable potentials and a meaningful set of basis states that make the transitional nature of these systems more transparent. With this aim nuclear physicists turned their attention to the Davidson potential [3]

$$ V(r) = \chi(r^2 + \frac{e}{r^2}) $$

(1)
which has known algebraic solutions when applied to diatomic molecules. In an algebraic approach for either the nuclear many-body problem or the Bohr-Mottelson collective model, the addition of the Davidson potential to the Hamiltonian requires the consideration of a dynamical subgroup chain that contains the direct product $Sp(2, R) \otimes SO(n) \subset Sp(2n, R)$, with $n = 3$ and 5, respectively.

In the first case ($n = 3$) the Hamiltonian (including its kinetic part) for a system with a strong rotational-vibrational interaction in harmonic oscillator units takes the following form

$$H_\varepsilon = \frac{1}{2} \hbar \omega^2 \left( -\nabla^2 + \frac{\varepsilon}{r^2} \right).$$

(2)

Both $\nabla^2$ and $r^2$ are $SO(3)$ scalars, and hence $H$ itself is a $SO(3)$ invariant. On the other hand, $H$ can be expressed in terms of the $SU(1, 1)$ generators defined in the following way [4]

$$Z_1 = -\nabla^2 + \frac{\varepsilon}{r^2}, \quad Z_2 = r^2, \quad Z_3 = \frac{1}{2}(r \cdot \nabla + \nabla \cdot r).$$

This means the eigenstates of the system can be classified according to the direct product $SU(1, 1) \times SO(3)$. Using the latter, algebraic solutions (eigenvalues and wavefunctions) can be obtained.

Notwithstanding these considerations, a microscopic description of the collective behavior of a many-particle system in 3-dimensional space requires higher symmetries. Specifically, such a complicated system is usually characterized by an irreducible representation (irrep) of $Sp(6n, R)$ and its $Sp(6, R) \otimes O(n)$ subgroup, where $n = A - 1$ and $A$ is equal to the total number of nucleons in the system. Collective effects emerge within this structure when the system is constrained to a specific irrep of $O(n)$ which in turn determines the $Sp(6, R)$ irrep [5]. When this is done, the Hamiltonian falls within the enveloping algebra of $Sp(6, R)$ rather than $Sp(6n, R)$.

The challenge is to define a basis in terms of irrep labels of groups in a physically meaningful chain of subgroups of $Sp(6, R)$. The basis characterized by the chain $Sp(6, R) \supset U(3) \supset SO(3)$, where $U(3)$ is the group of the quadrupole momentum introduced by Elliott [6], is well known and obtained in both an abstract way [7, 8] as well as in terms of shell-model states [9]. Many successful applications of this theory have been made for deformed nuclear systems.

Another relevant chain for developing collective basis states is $Sp(6, R) \supset Sp(2, R) \otimes SO(3)$ that was considered by Moshinsky and his collaborators in an effort to obtain a simple description of vibrational collective nuclear motion [5]. Indeed if the local isomorphism of the $sp(2, R) \approx su(1, 1)$ algebras is taken into account its relation to the spectrum generating algebra of the many body nuclear system with the Davidson interaction becomes explicit. This provides motivation for considering this reduction in seeking a description of a more complex modes that includes rotational-vibrational interactions.
6-Dimensional Davidson Potential

In the present work the more general case of a 6-dimensional Davidson potential is considered within the framework of the phenomenological interacting vector boson model (IVBM) [10], which has $Sp(12, R)$ as its dynamical symmetry group. Its new reduction chain through the direct product subgroup $Sp(2, R) \otimes SO(6)$ extends the applicability of the theory to include rotational-vibrational interactions. The application of this dynamical symmetry to nuclei confirms the ability of the Davidson potential to reproduce nuclear transitional behavior between the two main collective modes, found in nature.

2 Group Theoretical Background of the New Dynamical Symmetry in the IVBM

A further elaboration in the use of a symplectic geometry in investigating nuclear collective motion, can be achieved if we consider the nuclear many-body system as consisting of interacting proton and neutron subsystems. This leads to the phenomenological IVBM [10], where $Sp(12, R)$ - the group of linear canonical transformation in a 12-dimensional phase space [11] – appears as the dynamical symmetry of the model. The $sp(12, R)$ algebra is realized in terms of creation (annihilation) operators $u_{m}^{\pm}(\alpha)(u_{m}(\alpha))$, in a 3-dimensional oscillator potential $m = 0, \pm 1$ of two types of bosons differing by the value of the "pseudo-spin" projection $\alpha = \mp \frac{1}{2}$ and $\alpha = m = -\frac{1}{2}$. These are related with the cyclic coordinates $x_{\pm 1}(\alpha) = \frac{1}{\sqrt{2}}(x_{1}(\alpha) \pm ix_{2}(\alpha)), x_{0}(\alpha) = x_{3}(\alpha)$ and their associated momenta $q_{m}(\alpha) = -i\partial /\partial x^{m}(\alpha)$ in the standard way

$$u_{m}^{+}(\alpha) = \frac{1}{\sqrt{2}}(x_{m}(\alpha) - iq_{m}(\alpha)),$$

$$u_{m}(\alpha) = (u_{m}^{+}(\alpha))^{\dagger},$$

where $x_{i}(\alpha)$ $i = 1, 2, 3$ are the Cartesian coordinates of a quasi-particle vectors with an additional index, namely the projection of the "pseudo-spin" $\alpha = \pm \frac{1}{2}$.

The bilinear products of the creation and annihilation operators of the two vector bosons (3) generate the boson representations of the non-compact symplectic group $Sp(12, R)$. The commutation relations between the pair creation and annihilation operators and the number preserving operators are calculated in [10]. The linear invariant of $U(6) \subset Sp(12, R)$ is the number operator,

$$N = \sqrt{3}(A^{0}(p, p) + A^{0}(n, n)) = N_{+} + N_{-},$$

that counts the total number of bosons. The action space of the generators of the $Sp(12, R)$ is in general reducible and the invariant operator $(-1)^{N}$ decomposes it into even $H_{+}$ with $N = 0, 2, 4, ..., \text{and odd } H_{-}$ with $N = 1, 3, 5, ...$, subspaces, so in the reduction $Sp(12, R) \supset U(6)$ both the even and odd irreps of the $Sp(12, R)$ decompose into an infinite sum of finite fully symmetric irreps of $U(6)$, $[N]_{6} = [N, 0^{\alpha}]_{6}$, where $N$ is the eigenvalue of the operator (4).
In order to relate the IVBM to the 6-dimensional Davidson potential, we introduce another reduction of the \( Sp(12, R) \) group through its non-compact subgroup [5, 11, 12]:

\[
Sp(12, R) \supset Sp(2, R) \otimes SO(6). \tag{5}
\]

As can be deduced from the considerations given above, this construction obviously survives the addition of Davidson potential. The infinitesimal generators of the \( Sp(2, R) \) algebra

\[
F = \sum_{k,m,\alpha} C_{k1m,\alpha}^{00} u_k^+(\alpha) u_m^+(\alpha) = 2S^+, \tag{6}
\]

\[
G = \sum_{k,m,\alpha} C_{k1m,\alpha}^{00} u_k^-(\alpha) u_m^-(\alpha) = 2S^-, \tag{6}
\]

\[
A = \sum_{k,m,\alpha} C_{k1m,\alpha}^{00} u_k^0(\alpha) u_m(\alpha) = \frac{1}{\sqrt{3}} N = 2S^0 - 1,
\]

are obtained from the \( Sp(12, R) \) generators, by means of contraction with respect to both the spatial \( m = \pm 1 \) and the "pseudospin" \( \alpha = p = 1/2, \alpha = n = -1/2 \) indices. It is straightforward to show that the operators \( S^\tau, \tau = 0, \pm \) commute in a standard way for the \( SU(1, 1) \) algebra generators [13]

\[
[S^0, S^\pm] = S^\pm, \quad [S^+, S^-] = -2S^0,
\]

so the \( sp(2, R) \) and the \( su(1, 1) \) algebras are locally isomorphic with a Casimir operator written as \( C_4(SU(1, 1)) = S^0(S^0 - 1) - S^+S^- \).

By construction, the operators (6) are scalars with respect to 6-dimensional rotations and they commute with the components of the 6-dimensional momentum operators [10],

\[
\Lambda^L_M(\alpha, \beta) = A^L_M(\alpha, \beta) - (-1)^L A^L_M(\beta, \alpha), \tag{7}
\]

which obey the property:

\[
\Lambda^L_M(\alpha, \beta) = (-1)^L \Lambda^L_M(\beta, \alpha)
\]

and generate the \( SO(6) \) algebra. In this way, the direct product of the two groups in (5) is realized. The second order invariant for the \( SO(6) \) group is

\[
\Lambda^2 = \sum_{L,\alpha,\beta} (-1)^M \Lambda^L_L(\alpha, \beta) \Lambda^{-L}_M(\beta, \alpha). \tag{8}
\]

(8) is related to the second order invariant of the \( Sp(2, R) \), as in the direct product (5) the two groups are complementary [11], which means that the irreps of the group \( SO(6) \) determine those of \( Sp(2, R) \approx SU(1, 1) \) and vice versa.
In order to define the basis of the system with (5) as a dynamical symmetry that allows one to include the 6-dimensional Davidson potential, we consider the reduction of the SO(6) algebra to the $SO(3)$ algebra of the angular momentum through the following chain [10, 14]

$$SO(6) \supset SU(3) \otimes O(2) \supset SO(3)$$

which defines the $\gamma$-unstable limit of the IVBM. The single infinitesimal operator of $O(2)$ is proportional to the scalar operator $A^0(\alpha, \beta)$ from the $SO(6)$ generators (7),

$$M_{\alpha\beta} = -\sqrt{3}A_0(\alpha, \beta) = -\sqrt{3}[A^0(\alpha, \beta) - A^0(\beta, \alpha)],$$

and the generators of $SU(3)$ [10] are

$$X_M^2 = \frac{i}{2}(A_M^2(p, n) - A_M^2(n, p)), M = 0, \pm 1,$$

$$Y_M^1 = A_M^1(p, n) + A_M^1(n, p) = -\frac{1}{\sqrt{2}}L_M, M = 0, \pm 1.$$ (11)

Note, that in this case the quadrupole moment $X$ (11) is the proton-neutron interaction. The second-order Casimir invariants of the two groups in the direct product in (9) can be written as

$$2C_2(O_2) = M^2 = \sum_{\alpha, \beta} M_{\alpha\beta}M_{\beta\alpha},$$

$$C_2(SU(3)) = \sum_M (-1)^M(X_M X_{-M} + Y_M Y_{-M}).$$

For $SO(6) \subset U(6)$, the symmetric representation $[N]_6$ of $U(6)$ decomposes into fully symmetric $(\omega, 0, 0)_6 \equiv (\omega)_6$ irreps of $SO(6)$ [15] according to the rule

$$[N]_6 = \bigoplus_{\omega=N, N-2, \ldots, 0} \binom{N}{\omega} = \bigoplus_{\omega=0}^{\frac{N}{2}} \binom{N}{N - 2\omega} \leq \binom{N}{\frac{N}{2}},$$ (12)

where $\leq \frac{N}{2}$ if $N$ is even and $\leq \frac{N-1}{2}$ if $N$ is odd. Furthermore, the following relation between the quadratic Casimir operators $C_2(SU(3))$, $M^2$ of $O(2)$ and $\Lambda^2$ (8) of $SO(6)$ holds [16]:

$$\Lambda^2 = 2C_2(SU(3)) - \frac{1}{3}M^2,$$ (13)

which means that the reduction from $SO(6)$ to the rotational group $SO(3)$ is carried out through the complementary groups $O(2)$ and $SU(3)$ [11]. As a consequence, the irrep labels $[f_1, f_2, 0]_3$ of $SU(3)$ are determined by $(\omega)_6$. 


of $SO(6)$ and by the integer label $(\nu)_2$ of the associated irrep of $O(2)$ where
\[ \nu = \omega, \omega - 2, ..., 0 \]
i.e.
\[ (\omega)_6 = \bigoplus \left( f_1, f_2, 0 \right)_3 \otimes (\nu)_2. \] (14)

Using the relation (13) of the Casimir operators, for their respective eigenvalues in terms of the Elliott’s notation $(\lambda, \mu)$ one obtains:
\[ (\omega)_6 = \bigoplus_{\nu=\omega, \omega-2, ..., 0(1)} \left( \frac{\omega + \nu}{2}, \frac{\omega - \nu}{2} \right) \otimes (\nu)_2. \] (15)

Finally, the convenience of this reduction can be further enhanced through the use of the standard rules for the reduction of the $SU(3) \supset SO(3)$ chain:
\[ K = \min(\lambda, \mu), \min(\lambda, \mu) - 2, ..., 0 (1) \]
\[ L = \max(\lambda, \mu), \max(\lambda, \mu) - 2, ..., 0 (1); K = 0 \] (16)
\[ L = K, K + 1, ..., K + \max(\lambda, \mu); K \neq 0. \]

The latter is the usual reduction of an irrep $(\lambda, \mu)$ of $SU(3)$ into irreps $L$ of $SO(3)$ where the multiplicity number $K$ is used to label the collective bands in the energy spectra of the system.

The basis, labeled by the quantum numbers classified by the group-subgroup chain (9), can be written as
\[ |N\omega; (\lambda, \mu)\nu; K, L\rangle \] (17)
where the reduction rules for obtaining specific values for each state are given earlier. By means of these labels, the basis states can be classified in each of the two irreducible even $H_0$, with $N = 0, 2, 4, ...,$ and odd $H_\pm$ with $N = 1, 3, 5, ...$ representations of $Sp(12, R)$. We illustrate this in Table 1 for the even $H_0$ irreducible representation, where $N$ with the set of $\omega$ contained in it (12) label the rows and the values of the quantum number $\nu$ label the columns. The $SU(3)$ quantum numbers $(\lambda, \mu)$ define the cells of the Table 1 as they are obtained with the help of $\omega$ and $\nu$ (15).

The Hamiltonian with the considered dynamical symmetry (5) is expressed in terms of the first and second order Casimir operators of the different subgroups in its corresponding chain (9):
\[ H = aN + bN^2 + \alpha_2 \Lambda^2 + \alpha_2 M^2 + \beta_3 L^2. \] (18)

and it is obviously diagonal in the basis (17) labeled by the quantum numbers of their representations. The second order invariant of $SU(3)$ is dropped in (18), because of its linear dependence on the Casimir operators of the $SO(6)$ and $O(2)$ (13). Then the eigenvalues of the Hamiltonian (18) that yield the spectrum of a system interacting with 6–dimensional Davidson potential are:
\[ E(N, \omega, \nu, L) = aN + bN^2 + \alpha_5 \omega (\omega + 4) + \alpha_2 \nu^2 + \beta_3 L (L + 1). \] (19)
3 Application to Real Nuclei

In applications of this new dynamical symmetry of the IVBM to real nuclear systems that follows, we exploit the "algebraic" definition of yrast states as introduced in [17]. This means that we consider those states with maximal value of the angular momentum \( L \) for a given number of bosons \( N \) to be yrast. With this definition, the states of the ground band, which are the yrast states of the nucleus, are basis states with \( N = \omega = 2L, \lambda = \mu = L \) where \( N = 0, 4, 8, \ldots \) (\( \Delta N = 4 \)) from the \( \nu = 0 \) column of Table 1. Hence using the above assignment of the basis states to the experimentally observed states of the ground band their energies are presented in the following way:

\[
E_{gr}(L) = \alpha L + \beta L(L + 1)
\]

where \( \alpha = 2a + 4a_6 - 4b \) and \( \beta = 4b + 4a_6 + \beta_3 \) reflect the mixing of the vibrational collectivity with equidistant spectra depending on \( L \) and the one with rotational character \( L(L + 1) \).

The states of excited bands are not yrast states. The correct placement of the bands in the spectrum strongly depends on their band-head configurations and in particular, on the number of bosons, \( N = N_{\text{min}} + L \), from which they are built [18]. Hence we choose first a corresponding \( N_{\text{min}} \) for the band-head state and then the bands are developed along the "diagonals" of Table 1 by changing simultaneously \( \omega \) and \( \nu \) depending on the respective values of \( L \) and defining the corresponding values of (\( \lambda, \mu \))

For the \( \beta \)-bands with \( K^+_i = 0^+_i \), where \( i \) enumerates the \( 0^+ \) excited states in the order of increasing energy, we choose \( \omega = \nu = L \), that leads to a sequence of \( su(3) \) multiplets (\( \lambda = L, \mu = 0 \)) for the states in the bands with \( \Delta L = 2 \). Obviously, these bands \( \Delta N = 2 \) and \( N_{\text{min}} \) is included in the energy expression

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \omega )</th>
<th>( \nu )</th>
<th>6</th>
<th>4</th>
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<th>0</th>
<th>-2</th>
<th>-4</th>
<th>-6</th>
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<td>(5,1)</td>
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...
\[ E_{2\Lambda}(N_{\text{min}}, L) = aN_{\text{min}} + bN_{\text{min}}^2 + 2bLN_{\text{min}} + \alpha_{2\Lambda} L + \beta_{2\Lambda} L(L+1), \quad (21) \]

where \( a \) and \( b \) are the first two parameters from (19) that are already evaluated from the ground state band. Hence from (21) the dependence of the bandhead position on the number of bosons \( N = N_{\text{min}} \) that build the initial state with \( L = 0 \) is made explicit. The other two parameters \( \alpha_{2\Lambda} = a - b + 3\alpha_2 - \alpha_2 \) and \( \beta_{2\Lambda} = b + \alpha_6 + \alpha_3 + \alpha_2 \) that reflect the mixing of the rotational and vibrational modes include the parameter \( \alpha_2 \) of the Hamiltonian, which is obtained by a fitting procedure to the experimental energies of the states belonging to the \( K_F^+ = 0^+ \) bands.

For the excited \( K^+ = L^+ (L \neq 0, L = 2, 4, \ldots) \) bands, the band-head configurations are again obtained by the evaluation of the respective \( N = N_{\text{min}} + L \).

Since in these bands \( \Delta L = 1 \) we take from each \( N \) two values of \( L - L \) and \( L + 1 \) with the initial \( L \) even. Then the bands are developed again along the diagonals, but there is a difference in the values of \( \omega \) and \( \nu \), since as prescribed by the reduction rules \( \omega \geq \nu \). In the given examples we consider the low lying \( \gamma \)-bands as belonging to diagonals of \( \text{su}(3) \) multiplets \( (\lambda = L + s, \mu = 2) \) with fixed value of \( \mu = 2 \) and \( \lambda = L + s \), where \( L \geq 2 \) and \( s \geq 2 \). This leads to the following values of \( \omega = L + s + 2, \nu = L + s - 2 \), for \( L = 2, 4, 6, \ldots \). In this case \( s \) should be convenient even number evaluated in the fitting procedure. As a result the energy expression for the states of the \( \gamma \)-bands is more complicated:

\[ E_{\gamma}(N_{\text{min}}, L) = aN_{\text{min}} + bN_{\text{min}}^2 + 2bLN_{\text{min}} + \alpha_{\gamma} L + \beta_{\gamma} L(L+1) + \delta_1 c + \delta_2 s(s+2L) + \epsilon, \quad (22) \]

where the so introduced parameters \( \alpha_{\gamma} = a - b + 7\alpha_6 - 5\alpha_2, \beta_{\gamma} = \beta_{\Lambda} = b + \alpha_6 + \alpha_3 + \alpha_2, \delta_1 = 8\alpha_6 - 4\alpha_2, \delta_2 = \alpha_6 + \alpha_2 \) and \( \epsilon = 16\alpha_6 + 4\alpha_2 \) are different linear combinations of the interaction strengthes of the Hamiltonian (18), which are already evaluated. Note that there is an additive constant term \( \epsilon \). The important band characteristics that have to be evaluated in this case are the \( N_{\text{min}} \), that fixes the position of the band-head and also the value of \( s \) that shifts these kind of bands in respect to the ground and \( \beta_4 \)-bands.

This variety of possible choices for the excited bands allows us to reproduce correctly the behavior of these bands with respect to one another, which can change a lot even in neighboring nuclei because of the mixing of the vibrational and rotational collective modes [19].

From (19) it is obvious that there are 5 free parameters, which we determine by fitting the theoretical predictions for the energies of the ground and few excited bands to the experimental data [1], using a \( \chi^2 \)-procedure. The parameters that were obtained, the number \( k \) of experimental states, \( \chi^2 \), and \( N_{\text{min}} \) are all given in Table 2 for four different nuclei.

We have chosen the nuclei in the presented examples for their long spin sequences in particular in their ground bands, from where most of the parameters are obtained. The first one \(^{168}\text{Yb}\) is a well deformed nuclei from the rare
Figure 1. (Color online) Comparison of the theoretical and experimental energies for ground and excited bands of \(^{168}\text{Yb}\).

Figure 2. (Color online) The same as in Figure 1, but for the nucleus \(^{152}\text{Gd}\).
earth region with experimental ratios \( R_4 = E_4 / E_2 \) of the ground band energies

\( R_4 = 3.26 \)

One can see in Figure 1, the good agreement between theory and experiment for the ground, first two \( \beta \) and \( \gamma \) bands in the rotational \( ^{168}\text{Yb} \) nucleus.

The values of \( N_{\text{min}} \) that determine the start of the band-heads for the considered excited bands are rather high and equal in the first \( \beta \) and \( \gamma \) bands, which are almost degenerated. Although this is the dynamical symmetry for the \( \gamma \)-soft nuclei the well deformed ones are well described as well for a convenient choice of the parameters, given in Table 2.

Additionally, in Table 2 we give the results for the \( ^{148}\text{Nd} \) and \( ^{152}\text{Gd} \), which are considered to be \( \gamma \)-soft nuclei nuclei, both with \( R_4 = 2.49 \), which corresponds to the \( SO(6) \)-limit of the IBM (see Figures 3 and 2). The parameters of the Hamiltonian in both cases are equal within the given accuracy. The only difference coming from the shifting of the band head configurations of the excited \( \beta \) and \( \gamma \) bands by means of the different values of \( N_{\text{min}} \) evaluated for them. This reveals the importance of the consideration of the position of the excited bands in the interpretation of the character of the collectivity exhibited by nuclear spectra.

The most interesting example is the last nucleus \( ^{152}\text{Sm} \) (\( R_4 = 3.1 \)), which is transitional between the \( \gamma \)-soft (\( R_4 = 2.5 \)) and rotational nuclei with (\( R_4 = 3.33 \)) and is a nuclei at the critical point of phase/shape transition [20] with so-called \( X(5) \) symmetry. As shown in Figure 4, the experimental data is reproduced remarkably well, especially for the \( \beta \) and \( \gamma \) bands, that are well distin-

<table>
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<th>Nucleus</th>
<th>( k )</th>
<th>( N_{\text{min}} )</th>
<th>bands</th>
<th>( \chi^2 )</th>
<th>parameters</th>
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<td>ground</td>
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<td>( a = 0.3493 )</td>
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<td>20</td>
<td>( \gamma )</td>
<td>0.0006</td>
<td>( b = 0.00059 )</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>20</td>
<td>( \beta_1 )</td>
<td>0.0073</td>
<td>( a_8 = 0.00055 )</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>24</td>
<td>( \beta_2 )</td>
<td>0.0125</td>
<td>( a_2 = -0.00245 )</td>
</tr>
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<td>( ^{152}\text{Gd} )</td>
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<td>0</td>
<td>ground</td>
<td>0.0031</td>
<td>( a = 0.07571 )</td>
</tr>
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<td>( \gamma )</td>
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<td>( b = 0.00049 )</td>
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<td>8</td>
<td>( \beta_1 )</td>
<td>0.0078</td>
<td>( a_9 = 0.00087 )</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>12</td>
<td>( \beta_2 )</td>
<td>0.0053</td>
<td>( a_2 = 0.00777 )</td>
</tr>
<tr>
<td>( ^{148}\text{Nd} )</td>
<td>7</td>
<td>0</td>
<td>ground</td>
<td>0.0031</td>
<td>( a = 0.07571 )</td>
</tr>
<tr>
<td>( R_4 = 2.49 )</td>
<td>5</td>
<td>12</td>
<td>( \gamma_1 )</td>
<td>0.0029</td>
<td>( b = 0.00049 )</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>12</td>
<td>( \beta_1 )</td>
<td>0.0018</td>
<td>( a_9 = 0.00087 )</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>16</td>
<td>( \gamma_2 )</td>
<td>0.0006</td>
<td>( a_2 = 0.01124 )</td>
</tr>
<tr>
<td>( ^{152}\text{Sm} )</td>
<td>9</td>
<td>0</td>
<td>ground</td>
<td>0.0022</td>
<td>( a = 0.02868 )</td>
</tr>
<tr>
<td>( R_4 = 3.01 )</td>
<td>7</td>
<td>20</td>
<td>( \gamma )</td>
<td>0.0028</td>
<td>( b = 0.00072 )</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>16</td>
<td>( \beta_1 )</td>
<td>0.0002</td>
<td>( a_9 = 0.00073 )</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>30</td>
<td>( \beta_2 )</td>
<td>0.0051</td>
<td>( a_3 = 0.00344 )</td>
</tr>
</tbody>
</table>
Figure 3. (Color online) Comparison of the theoretical and experimental energies for $^{148}\text{Nd}$.

Figure 4. (Color online) The same as on Figure 1 for $^{152}\text{Sm}$.
guished. This is reflected by the values of $N_{\text{min}}$, which vary quite a lot, but less than the parameters of the Hamiltonian.

The values of $\chi^2$ are rather good (small) for all of the examples considered. This suggests that the model is appropriate for the description of a rather broad range of nuclei, and most importantly nuclei that display different degree of mixing of the rotational and vibrational degrees of freedom.

4 Summary and Conclusions

In this work we introduce a new reduction of the dynamical group $Sp(12, R)$ of the algebraic Interacting Vector Boson Model, which contains the 6-dimensional Davidson potential by means of the spectrum generating algebra $SU(1, 1) \otimes SO(6)$. It is naturally contained in the group of dynamical symmetry $Sp(12, R)$ of the IVBM. Further, the reduction of the boson representations of $SU(1, 1) \otimes SO(6)$ to the angular momentum group $SO(3)$ is obtained in order to provide for a complete labeling of the basis states of the system and the model Hamiltonian is written in terms of the first and second order invariants of the groups from the corresponding reduction chain. Hence the problem is exactly solvable within the framework of the IVBM which, in turn, yields a simple and straightforward application to real nuclear systems.

We present results that were obtained through a phenomenological fit of the models' predictions for the spectra of collective states to the experimental data for nuclei from the rare-earth major shell exhibiting rotational spectra, $\gamma$-soft and transitional between $\gamma$-soft and vibrational spectra. The good agreement between the theoretical predictions and the experiment results confirms the applicability of the IVBM to a broad range of nuclei with quite different collective properties. This is possible not only because of the changes in the values of the variational parameters of the Hamiltonian, that mix the collective modes, but also because of the symplectic extension of the model that yields as an important consequence a possible change of the values of $N_{\text{min}}$ that determine the band head configurations of the excited bands. These features could be further developed to study the phase/shape transitions in nuclear systems [21], which of late has been a subject of high interest from a theoretical [22] as well as from experimental [20] point of view.

The most important feature of the model, from a physical point of view, is that it leads to a successful description of different types of nuclear collective spectra as well as mixed-mode results with the proton and neutron substructures and interactions between them taken into account explicitly. This is accomplished within the framework of a symplectic symmetry that allows for a change in the number of bosons of each type.

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References

[1] Mitsuo Sakai, Atomic Data and Nuclear Data Tables 31 (1984), 399; Level Retrieval Parameters, http://icnrd.isca.or.at/nudat/levform.html