SYMMETRIES AND STATISTICAL BEHAVIOR IN FERMION SYSTEMS

J. B. French

Department of Physics and Astronomy, University of Rochester
Rochester, New York 14627

and

J. P. Draayer

Department of Physics and Astronomy, Louisiana State University
Baton Rouge, Louisiana 70803

1. Introduction

The subject is the interplay between statistical behavior and symmetries in
nuclei, as revealed for example by spectra and by distributions for various kinds
of excitations. Much of the argument and general procedure may be applied also to
other many-fermion spaces which have (as do nuclear shell-model spaces) a direct-
product structure, being representable by distributing a certain number, m, of
fermions over N single-particle states \(\Omega_N\mathcal{H}_N\). I shall talk about methods and
general results because that seems appropriate here, rather than about specific
applications, of which however there have been many.

Most people think of statistical behavior as being a relevant way to describe
things only at high excitation, whereas the natural domain of symmetries, with the
obvious exception of those which are exactly (or almost exactly) conserved\(^*\), is
near the nuclear ground state. But even rather badly broken symmetries are useful
in roughly partitioning an energy spectrum and play thereby an important role in
determining locally averaged strengths, etc.; thus symmetry effects extend toward
in energy away from the ground-state region. On the other hand, if we introduce
Hamiltonian ensembles to describe the statistical behavior (which is necessary for
fluctuations, though not for locally averaged eigenvalue densities, etc.), it
turns out that appropriate ensembles gives results which are analogous and, when

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\(^\ast\)We speak of a symmetry as conserved if each wave function belong to a single
irreducible representation
expressed in an energy scale supplied by the local eigenvalue density, are stationary in energy.\(^1\) Thus statistical behavior extends downward into the ground-state domain so that the interplay which we discuss exists in all parts of the spectrum.

As I have indicated, statistical behavior can be discussed in terms of secular averages and of fluctuations about the average.* A finer decomposition is in terms of the sequence of \(\ell\)-point correlation functions (\(\ell = 1, 2, \ldots\)), of which \(\ell = 1\) gives the density, while \(\ell = 2\) dominates all of the measures which have been introduced (and perhaps all which can be experimentally measured). There are interesting connections between fluctuations and symmetries, for, while fluctuations are "small" and carry little information, the little which they do carry is closely connected with the existence or non-existence of a conserved or almost conserved symmetry; this is because the existence of such a symmetry moderates the "level repulsion" which dominates the fluctuation behavior. Methods for studying these connections have only recently been developed. I shall therefore say nothing more about fluctuations but focus instead on the (secular) behavior of locally averaged quantities.

2. Elementary Principles

The \(m\)-fermion wave functions for fixed \(m = 0, 1, \ldots, N\) form an irreducible (anti-symmetric) tensor, of dimensionality \(d(m) = \binom{N}{m}\), with respect to the group \(U(N)\) of unitary transformations in the single-fermion space. To each state \(\psi_a(m)\) there exists a complementary one \(\tilde{\psi}_{ac}(N-m)\) which forms a contragradient tensor. The system has two vacuum states \(|0\>\) and \(|N\>\) and any state \(\psi_a(m)\) can be described in terms of either:

\[
\psi_a(m) = \psi_a(m)|0\> = \tilde{\psi}_{ac}(N-m)|N\>
\]

(1)

in which the operation \((-)\) replaces every creation and destruction operator by its adjoint so that \(\tilde{\psi}_a(m) = \psi_{a^\dagger}(m=a^+_1\cdots a^+_N)\). Since holes are also fermions (the fundamental commutation rules being invariant under \((-)\)) we have that for any real

*In nuclear model systems, at least, there is a remarkable (well-understood) separation between the two phenomena which allows us to discuss them separately.
\[(\text{Trace } F \text{ in m-particle space}) = \langle\langle F \rangle\rangle^m = d(m)\langle\langle F \rangle\rangle^m = \langle\langle F \rangle\rangle^{N-m} \]  

(2)

where, as indicated, \(\langle\langle F \rangle\rangle^m\) is the trace and \(\langle F \rangle^m\) the average eigenvalue (or expectation value).

A k-body operator, \(F(k) = \Sigma k\eta^\dagger(k)k\eta\phi^\dagger(k)\phi^\dagger(k)\), transforms according to \((k+1)\) irreps of \(U(N)\); for example, with \(n\) the number operator, 

\[F(2) = \left(\begin{array}{c}
\binom{n}{2}F(0) + \binom{n-1}{1}F(1) + F(2)
\end{array}\right) = \sum_{v=0}^{2} F(v)\]  

(3)

where \(\binom{n}{2}\) measures the spectrally-averaged eigenvalue (which then "propagates" with particle number as \(\binom{n}{2}\)). \(\binom{n-1}{1}\) \((N-1)\) \((F(1))\) (in which \(F(1)\) is an \(SU(N)\) generator) gives the modification of the single-particle energies, and \(F(2)\) may be regarded, in the context of the \(U(N)\) decomposition, as the "true" residual interaction.

There is, for each particle number \(m\), a natural geometry for traceless operators in which 

\[(\text{norm } F)^2 = d^{-1}(m)\langle F^*F \rangle^m = \langle F^*F \rangle^m\]  

(4)

The unitary decomposition of \(F\) is orthogonal in this geometry (true also for operators which do not conserve particle number). With this geometry we can consider such questions as "how much Q-Q is contained in a given \(H\), acting in an \(m\)-particle space?" (answer: a large amount; the scalar product between the unit vectors for \(H\) and \(Q-Q\) (in statistical terms the "correlation coefficient" between \(H\) and \(Q-Q\)) is negative and, for light nuclei at least, \(-0.5\). Note that \(Q-Q\) is a sum of squares of traceless 1-body operators; it appears that realistic \(H\)'s are rather well representable in this way.

There are central-limit theorems (CLT's) which ensure that, if \(N\) is large enough the smoothed spectrum converges, as we increase the particle number \(m\), to a characteristic density describable by a few parameters. For example with non-interacting particles \((H=H(k=1))\) the single-particle spectrum convolutes and rapidly becomes Gaussian as we increase the particle number. The real function of the CLT is to make the unitary geometry an effective one. There are two aspects of
this; the unitary norm for an operator might well be criticized on physical
because it treats all states on the same footing and therefore concerns itself
mostly with states which lie high in energy (where the density is highest), even
though most of these are not of direct physical significance and our interest might
well be in the lowest ones. This difficulty disappears when the energy variation
of the eigenvalue density is closely specified by a CLT, for what happens then at
all energies is determined by the trace of $H$ and the norms of low powers of $H$. At
the same time we avoid the hopeless technical problem of evaluating high-order
moments.

3. A Simple Example: Single-state Occupancies

The Strutinsky explanation of shell effects in fission may be taken as demonstrat-
ing the "survival" of single-particle information in the presence of strong
interactions. As a simpler example of this let us ask whether the ground-state sum
rule for the single-nucleon-transfer strength (which measures single-particle in-
formation) can tell us anything about the effective interaction.

The sum rule for transfer in an orbit $j$ determines the ground-state occupancy
$<N_j | n_j | N_j>$. In general, for any $H$-eigenstate $|\Psi\rangle$ of eigenenergy $W$, we have, with
$\rho(W)$ the eigenvalue density function, $P_j(W)$ the orthonormal polynomials defined
with $\rho(W)$ as weight function, and $<x^m>$ the $m$-particle (or model-space) average with
$E(x)=<H|x>^m$ the eigenvalue centroid, that

$$<W|K|\Psi>=<W|\delta(H-W)|\Psi>=\sum_j P_j(W)^m P_j(W)=<E-x^m+<K(H-E)>^m\frac{1}{E-x^m}(H-E)+...$$

(5)

It is easy to see that the $v$-th order term would arise from the $v'$th order deforma-
tions of $\rho(W)$ under the infinitesimal change of the Hamiltonian, $H+\delta x K$. But
then, in the CLT limit, the terms of order larger than 1 must vanish (except for
very high order, involving "wavelengths" comparable with the level spacing, for
which the CLT smoothing is ineffective). Thus we have a linear form for the
smoothed $<W|K|\Psi>$, the slope of which is proportional to the scalar product
$<K(H-E)>^m$. For any one-body operator, $K=N_j$ for example, only the $[2,1]^{N-2}$
part of $H$ contributes (by orthogonality); the part of this deriving from the two-
body $H$ measures the Hartree-Fock-like contribution to the effective single-particle
...proportional to \( \sum_s \sum_j N^{-1} \sum_j W_{jsj} \). Observe how the CLT in (5) and the unitary geometry here have combined to modify a single-particle property of a complicated system. The modification has been measured in the \( ds \) shell, and used in determining which model interactions are acceptable.\(^2\) Occupancies can of course be used in many other ways also.

4. Subgroup Structures, Spectra and Characteristic Forms

If, in the example above, we focus our attention on an \( N_1 \)-dimensional subset of the single-particle states we are led naturally to think about the \( U(N_1) \) subgroup of \( U(N) \), and more generally about "additive" decompositions of the \( U(N) \) algebra. If we consider neutron and proton transfer separately we are led to \( U(N/2) \otimes U(2) \) and a direct-product decomposition. A great variety of other subgroups are encountered in other circumstances and so we must consider the subgroup problem more generally.

A given representation of a subgroup \( G \), as with \( SU_2(T) \) or \( O_3(J) \), may occur very many times in a large space. We then naturally ask what representations of \( G \) are found and how often. This question is answered to some extent by considering the spectrum, in the \( m \)-particle space, of the bilinear Casimir operator \( C \), which is, of course discrete with degeneracies equal to the number of times that the appropriate\(^*\) representation occurs, multiplied by its dimensionality.

For any \( U(N) \) subgroup, \( C \) may be written as a sum of squares of (non-commuting) traceless one-body operators (\( SU(N) \) generators) which have the same norm in all \( m \)-particle spaces, and which may be taken as orthogonal in the \( 1 \)-particle space (and hence via (3) in \( m \)-particle spaces). Thus \( C = \sum_{\alpha} h_{\alpha}^2 \) with \( <h_{\alpha}^2> = 0 \) and \( <h_{\alpha} h_{\beta}^\dagger> = \delta_{\alpha\beta} \). Let us consider the moments of \( C \) in the limit of a large dilute multi-particle space \( (N=m, m=m, \frac{m}{N}=0) \). For a traceless \( h \) we can decompose \( <h_{2\nu}^2> \) in terms of correlation patterns defined by the partition of \( 2\nu \) into integers \( \nu \); then for example the partition \( [2\nu] \) gives a term proportional to \( \left( <h_{2\nu}^2> \nu \right) = (m(N-m)(N-1))^{-1} \). From the \( m \)-dependence it is easy to see that only binary correlations survive in the asymptotic limit.\(^3\) Extending this to the sum of squares and observing that no cross-terms contribute we see that, in the present limit, \( C \) is equivalent to a sum

\(^*\)\( C \) does not by itself distinguish all irreducible representations. We may say that it "classes together all "\( C \)-equivalent" irreps.
of commuting squares so that its distribution is an $\ell$-fold convolution; in other
words only the dimensionality, $\ell$, of the space of generators matters, nothing
further about the nature of the group or about the way in which it is realized in
the space. Thus the asymptotic spectrum of $C_2$ has a $\chi^2_\ell$ form,

$$D_{C_2}(x) = \left(2^{\ell/2} (\ell^{\ell/2} \Gamma(\ell/2))^{-1} x^{\ell/2 - 1} \exp(-x/2a^2)\right)$$

(6)

in which, as above, $a^2$ is the one-particle variance of a generator. An SU(3) example
($\ell=6$) is shown in Fig. 1. For $\ell=3$ (and thus for angular momentum) the
distribution is Maxwellian, a result given in 1936 by Bethe$^4$ for non-interacting
particles and used ever since, with minor corrections, for the J-dependence of
the level density. For a quadrupole interaction ($\ell=5$) Nomura$^5$ evaluated the moments
to order four, and stressed that the asymptotic spectrum is non-Gaussian.

From the argument above we see that the distribution is $\chi^2_5$.

This distribution (6) can be ex-
tended to finite particle number by explicit evaluation of the low-order moments (and, much less easily, to finite $N$ as well). In general terms we can expect that an approximate $\chi^2_\ell$ distribution will ob-
tain as long as we have a half-dozen particles or more. For large $\ell$, by the
standard CLT, the $\chi^2_\ell$ distribution is close to Gaussian. Since, as we have said,
the Hamiltonian itself is often reasonably representable as a sum of squares we see
that its asymptotic spectrum likewise should be approximately $\chi^2_\ell$ ($\ell$ being now the
number of squares of comparable magnitude) and in many cases then close to Gaussian,
as is observed in detailed calculations.

Spectral distributions, as used in practice, involve the distribution in
energy of states of a given symmetry, the distribution being a Hamiltonian-eigen-
distribution if the symmetry is a good one, but otherwise an "intensity" distribution (giving for each Hamiltonian eigenvalue, the contribution to \( \langle \psi | \phi \rangle = 1 \) which arises from basis states of the specified symmetry). The moments of such a distribution follow from the simplest moments, \( \langle H^n \rangle \), by the model-space decomposition, * \( \rho \rightarrow \rho(m, \vec{r}) \) where \( \vec{r} \) defines the symmetry irrep; we come to that in Section 5.

For now observe that if the \( C_2 \)-\( H \) correlation coefficient, \( \zeta \), is small it follows from (5) that a given \( G \) irrep will distribute more or less uniformly over the whole energy spectrum, the decomposition (6) being then locally valid (this goes far towards justifying the use of Bethe's J-decomposition in level density studies).

At the other extreme (\( |\zeta| \rightarrow 1 \)), the \( H \) spectrum will be close to that of \( C_2 \). In many cases the Hamiltonian centroids are determined by a single group invariant, \( C_2 \); in these cases, if we start with \( H = C_2 \) (to within a sign) and allow it to move away from \( C_2 \) (\(|\zeta| \) going gradually to 0) the overall \( x_k^2 \) distribution will be maintained for at least part of the range of \( \zeta \) even while the degenerate \( C_2 \) "spikes" are broadened by the movement of the individual levels and their admixing with levels of other symmetries. The line shape (which one would guess should be Gaussian, except perhaps near the extremes of the spectrum) and more generally the energy distribution of the fixed-symmetry states can be studied by the polynomial methods referred to in Section 3.

5. Propagation of Information

The asymptotic distribution (in energy) for a fixed symmetry is defined by three quantities only, the dimensionality, centroid and variance. For distributions which are not asymptotic the same characteristic form will obtain, but with \( m^{-1} \) and \( m/N \) corrections for which we need a few more pieces of information. Since the Hamiltonian and the other interesting operators of the system are defined in few-particle subspaces "embedded" in the \( m \)-particle space the general problem of understanding what goes on in a many-particle subspace may be thought of in terms of propagation of information from the simple subspaces to the more complicated ones. Alternatively if we measure things in a many-particle space we are "probing" the

*We use \( m \) to denote the \( m \)-particle space.
structure of the physical few-particle spaces. Because of the filtering action of
the CLT much of the information is lost* and conversely the many-particle probe is
sensitive to only a few properties of the system.

Intuitively we might guess for a \( k \)-body operator, averaged over \( m \), that

\[
<F(k)>^m = <F(k)>^k \quad \text{(weight with which } k \text{ is found in } m)
\]

(7)

in which the natural definition of the weight is

\[
<\delta \sum_a \psi_a(k)\psi^*_a(k)>^m = <\delta(k)>^m = \binom{m}{k}
\]

(8)

This result is easily seen to be correct; schematically (with particle number in-
creasing upward so that \( \psi^*_{\alpha}(k) \equiv \beta^* \) and \( \psi_{\beta}(k) \psi^*_{\beta}(k) \equiv 3 \) \( \beta \)) \( F(k) = \sum_{\beta} F_{\beta} \beta(\beta^* \beta) \)
and then

\[
<F(k)>^m = \sum_{\beta} F_{\beta} \left\{ \sum_a \left( \begin{array}{c} 0 \\ a \end{array} \right) \left( \begin{array}{c} a_m \end{array} \right) \right\}
\]

(9)

with \( \rho \) the eigenvalue density operator defined in (8). In going from the first to
the second forms we have used the contragradient representation of the \( m \)-particle
states, this followed by a commutation of state operators (which would not be valid
for the first form) and a use of (2). Note that \( \rho_{\alpha}(N-m) = \rho(N-m) \); \( \rho \) is of course
diagonal in \( \beta \) in the present case but not always so for its symmetry extensions.

Carrying out the sums we have

\[
<\langle F(k) \rangle>^m = <\langle \rho(N-m)F(k) \rangle>^k = <\langle \rho(N-m)F(k) \rangle>^{N-k}
\]

(10)

which, with (8), reproduces the elementary result (7). The intuitive interpreta-
tion of (10) follows from the fact that \( \rho(N-m) \) measures the number of holes in the
complementary space \( (N-m) \). It is easy enough to see that (10) extends to any sub-
space \( \Gamma \) of \( m \) (even to single states) so that, with \( \langle \langle F \rangle \rangle^m \) the trace over all \( m \)-

*The CLT does not smooth out the fluctuations but the "lost information is not to be found there. Fluctuations, as we have said, carry almost no information.
particle states belonging to all irreducible representations \( \Gamma \) of a group \( G \), a subgroup of \( U(N) \), (involving therefore a sum over all irreps \( \Gamma \))

\[
\langle \langle F(k) \rangle \rangle^m_{\Gamma} = \langle \langle \tilde{\rho}_\Gamma (N-m) F(k) \rangle \rangle^k_{C} = \sum_{\Gamma'} \langle \langle \tilde{\rho}_\Gamma (N-m) F(k) \rangle \rangle^{k\Gamma'}_{\Gamma'}
\]

\[
(s) \sum_{\Gamma'} \langle \langle \tilde{\rho}_\Gamma (N-m) \rangle \rangle^{k\Gamma'}_{\Gamma'} \langle F(k) \rangle^{k\Gamma'}_{\Gamma'} = \sum_{\Gamma'} \langle \langle \tilde{\rho}_\Gamma (k) \rangle \rangle^m_{\Gamma} \langle F(k) \rangle^{k\Gamma'}_{\Gamma'}.
\]

(11)

the last two forms of which (the second of these following from the first via the same operations used in (9)) are valid when \( \rho_{\Gamma} \) (N-m) behaves as a multiple of unity in each subspace, in which case we have simple (s) propagation, from the space in which \( F \) is defined, to the more complicated spaces of interest to us. This form displays beautifully how the trace information propagates throughout all the subspaces, the propagation coefficient being dependent only on the pair of representations involved, in fact on the weight with which one representation space is found in the other. The nature of simple propagation (in this case for "non-scalar" information) will become clearer on inspection of Fig. 2, for which the density operators are also available.

By construction the operators \( \rho_{\Gamma} (k) \) are k-body \( G \) scalars which satisfy \( \rho_{\Gamma} (k) \psi_{\alpha} (k) = \delta_{\Gamma \Gamma'} \psi_{\alpha} (k) \). These properties completely define the operators and show that the set of them for fixed \( k \) forms a Green's Function, on the \( m=k \) subspace, by means of which we can calculate traces away from the defining "surface", i.e. for \( m \neq k \). We can obviously construct one operator for each set of \( G \) irreps \( \Gamma \). But there are also off-diagonal \( G \) scalars that can be constructed by coupling states from two equivalent irreps. Suppose that \( G \) is a subgroup of a factor \( U(k) \) where \( U(N) \supset U(N/k) \oplus U(k) \) and we specify also the

Fig. 2 Propagation on the isospin lattice for an operator with isospin-rank 2 and maximum particle rank 8. The triangle shows the effective defining lattice and the cross-hatching the "shadow zone" into which no propagation occurs for the part of the operator defined in the \((m=8,T=3)\) subspace.
(single) irrep \( \Lambda \) of \( U(k) \); if now in the reduction \( \Lambda \rightarrow \Gamma \) there is a multiplicity \( k \), we have \( k \) diagonal \( G \) scalars and \( \frac{1}{2}k(k-1) \) off-diagonal ones, the total being \( \frac{1}{2}k(k+1) \). Similarly if \( G \) is the last member of a chain, as in \( U(N) \supseteq G \), and we also specify a single irrep \( \Lambda \) of \( \kappa \). In these circumstances for each \((\Lambda, \Gamma)\) pair there is a matrix of density operators which should enter into the propagation of information; we shall see that they do.

For \( U(N) \) itself \((G=U(N))\), \( \Gamma \) is of course unique and \( \rho_{\kappa}(k)=\binom{n}{k} \) as we have seen.

For \( G=U(k) \) a factor of \( U(N) \supseteq U(N/k) \supseteq U(k) \) (and similarly for an additive decomposition) \( \Gamma \) is not unique but one sees easily that \( \rho_{\kappa} \) is a function only of the generators of \( G \) and hence of its Casimir invariants. The total number of \( G \) irreps for \( k \) spin particles then equals the number of terms in a polynomial, of maximum particle rank \( m \), involving the \( U(N) \) invariant, \( n \), and the \( G \) Casimir operators, this being true as long as \( N \) is large enough so that all the irreps can in fact be generated. In this case, which includes the nuclear physicist’s isospin and spin-isospin \( SU(4) \), the propagation is simple in the sense of (11). We should mention here that the first systematic discussion in group theoretical terms of simple propagation and some of its extensions is given in an important paper by Quesne.

Going beyond simple propagation, consider \( U(3) \supseteq O(3) \). The density operators are \( O(3) \) scalars constructed from the generators of \( U(3) \). By a simple counting of polynomial terms and representations one finds immediately that the polynomials are not constructible from the Casimir invariants of \( U(3) \) and \( O(3) \) alone; one new invariant is needed \( ^* \) for \( m=3 \) and one for \( m=4 \). We make contact here with the cataloging of invariants by means of the “integrity basis”, that basis in the present case consisting of \( n \), \( C_{2}(O_{3}) \), \( C_{2}(U(3)) \), \( C_{3}(U(3)) \), \( X_{3} \) \( X_{4} \). As a demonstration that there are no higher ones we have for \( m=50 \) (and \( N \) large enough to generate the irreps) a polynomial of 35,993 terms and the same number of density operators (including the \( \kappa(k-1)/2 \) off-diagonal ones which enter when there is a \( \kappa \)-fold \( U(3) \supseteq O(3) \) multiplicity). It should be noted that \( X_{4} \) is redundant for the state-

\footnote{The first four operators give 5 polynomial terms for \( m=2 \) and 9 for \( m=3 \); there are 5 irreps \( (\Lambda_{1}; \Gamma_{1}) \) for \( m=2 \) and 10 for \( m=3 \). In constructing the polynomials all of the scalars including the new ones can be taken as effectively commuting since commutation reduces the particle rank by unity giving scalars which have already been counted.}
The problem, and thus the propagation of information, for which \( X_4 \) is necessary, makes more demands on the group structure than does the simpler problem.

For the decomposition \( U(N) \supset U(N/4) \otimes U(4) \) with \( U(4) \supset SU(2) \otimes SU(2) \), which is of considerable interest to nuclear physicists, we find as the integrity basis (by counting the \( U(4) \supset SU(2) \otimes SU(2) \) irreps) \( n, G_2, G_3, G_4, S^2, T^2 \) and non-Casimir invariants \( X_3, X_4, Y_4, Z_4, X_5, X_6, Y_6 \), explicit forms for which have in fact been given by Miller.\(^\text{11}\) But here the integrity-basis counting gives numbers which grow slightly more quickly than the number of density operators, for \( m=7,9,25,50 \) the numbers are \((73,201,57125,11766281)\) for the former, and \((73,200,48653,6541506)\) for the latter. The source of the difference cannot be that we have too many basic invariants; it must therefore be that not all products and powers of the basic invariants are independent. The redundancies here could be resolved once again by counting irreps. Alternatively these redundancies can be anticipated from generating-function forms \(^\text{9,12}\) for the number of scalars of given polynomial order.

In this case we find that \( Z_4 \) (\( S \cdot T \) where \( S \) is the Gamow-Teller operator) and \( X_5 \) (\( S \cdot S^3 \cdot T \)) can only occur linearly while no products of \( X_6 \) (\( (S \cdot S)^2 \cdot (S^4)^2 \)) and \( Y_6 \) (\( (S^4)^2 \cdot (T \cdot T)^2 \)) can occur at all. With these restrictions the polynomial counting precisely reproduces the irrep counting above.

As another example we mention Elliott's \( SU(3) \) which appears in the decomposition \( U(N) \supset U(4) \otimes U(N/4) \); \( U(N/4) \supset SU(3) \). Here one finds easily that when \( N=12,24 \) the invariants up to \( m=2 \) have a Casimir basis so that \( SU(3) \) centroids propagate simply in the \( p \) and \( ds \)-shell spaces; not so however for \( N=40 \) (\( fp \) shell), for which another invariant of particle rank 2 is needed.

We shall not discuss the explicit construction of the density operators for the non-Casimir case. We remark however that, in general the final forms of the propagation equation \((11)\) will not apply, while, for the other forms we might have to resort to an explicit evaluation for each irrep (not just for the sum over all equivalent ones).

Finally we remark that we should be interested also in "non-scalar" information, as in Fig. 2. As a trivial example the quadrupole moment operator \( Q_2 \) has a zero average expectation value in the substates of a \( J \)-level but nonetheless carries
information (about the intrinsic quadrupole moment) via the "double-barred" or
tensor-coupled matrix element. Then there enter naturally the non-scalar density
operators \( \rho^{\alpha}_A(m) \approx \sum_{\alpha \beta} \langle \varphi^{\alpha}_A(m) \times \varphi^{\beta}_A(m) \rangle^\nu \) (in which \((x)\) denotes tensor coupling and
\( \varphi^{\alpha}_A \) the tensor adjoint) and the corresponding extension of the integrity basis.

This is particularly important for strength distributions. 12

6. Information Propagation in Dilute Spaces

Looking at things from the standpoint of many-body physics we might well ask
for the source of the great complications which show up in constructing the
integrity basis. We recall that one type of unpleasantness arises in many-body-
theory from the Pauli Principle which forbids two fermions to be in the same state.
The (blocking) corrections involved disappear in dilute systems, those in which the
single-state occupancies are much less than one. Thus \( m < N \); the more detailed
requirement arises from the need to avoid "condensations" in subsets of single-
particle states whose occupancies then would be comparable with unity. There's no
trouble calculating the occupancies (just go back to section 3) so that whether or
not a system is dilute can in fact be determined.

If we first use completeness to write \( \langle c_{q}(k) c_{p}^{\dagger}(k) \rangle^m = \langle c_{p}^{\dagger}(k) c_{q}(k) \rangle^{m-k} \),
and then use (10) we find different propagation forms,

\[
\langle F(k) \rangle^m = \langle F(k) \rangle^{m-k} = \langle F(k) \rangle^{m-n+k} = \langle \rho(m-k) F(k) \rangle^k
\]

\[
\sum_{p=0}^{k} \langle \bar{F}(p) \rangle_{p}^{m-k} \sim \sum_{p=0}^{k} \langle \rho(m-k-p) F(k) \rangle_{p}^{d}
\]

In the second branch of this equation \( \bar{F}(k) \) is the \( p \)-body part of the \( k \)-hole
operator \( F(k) \) which of course has particle ranks \( 0,1, \ldots k \). The \( p \)-decomposition
given here is important for dilute systems \( (N \rightarrow 0, m/N \rightarrow 0) \) because in such systems
\( \langle \bar{F}(k) \rangle^{m-k} \) is dominated by its \( p=0 \) part; this comes about because the system has a
great many holes, which are essentially counted by \( \bar{F}(k) \), but very few particles to
inhibit the counting. The \( p \)'th term in fact is proportional to \( (m/N)^d \), as one sees
by noting that for the prototype operator \( \begin{bmatrix} n \end{bmatrix}_k \) we have \( \begin{bmatrix} n \end{bmatrix}_p = (-1)^p \begin{bmatrix} N-p \end{bmatrix}_k \begin{bmatrix} n \end{bmatrix}_p \).

Equation (12) is trivial as it stands; we need its decomposition according to a group \( G \), producing thereby the equivalent of (11). But things are quite different now because any decomposition \( m^p \varepsilon(m,\Gamma) \) will, for most operators, destroy the completeness used in the first step of (12); we get around this**, just as one does with multipole sum rules, by making use of the non-scalar density operators \( \rho_\Gamma^\lambda(m) \) introduced above and using recoupling algebra to isolate single intermediate \( \Gamma \) spaces. We end up then with a tensorial extension of (12) which we can use for propagating scalar or non-scalar information, and which is particularly appropriate for dilute systems.

In \( p=0 \) approximation we find for a \( k \)-body operator, \( F^0(k;\lambda,\lambda) \), whose tensorial structure is \( (\psi(k)\otimes \delta(k))^0 \), and which therefore is defined in the \( (k,\lambda) \) subspace, that

\[
\langle \langle \mathcal{F}^0 \rangle \rangle^\Gamma = d(\Gamma)d^{-1}(\lambda) \langle \langle \mathcal{F}^0 \rangle \rangle^k_{\Gamma_1} \delta_{\Gamma_1} \sum_{\Gamma_2} \delta_{\Gamma_2} \delta_{\Gamma_1} \delta_{\Gamma_1}
\]

(13)

where \( d_\lambda \) is a multiplicity, \( d(\Gamma) \) a dimensionality, and \( \delta_{\Gamma_1 \Gamma_2} \) the triangle function.

For the coupled trace of a non-scalar operator we would find not a sum over multiplicities but instead a Racah transform of the multiplicity function. The structure revealed here is as simple as could be; \( F^0 \) is measured in terms of its defining trace and the contribution to any \( (m,\Gamma) \) irrep is simply proportional to the number of parents which \( (m,\Gamma) \) has in the space \( (m-k) \). By taking \( F^0 \) as the \( (k,\lambda) \) density operator we evaluate the propagation coefficient precisely in terms of the weight of one space in the other. The complexities of propagation then disappear in the dilute-system limit. They gradually re-enter of course as we take into account terms of higher \( p \).

**For scalar traces \( F(k) = \langle \mathcal{F}(k) \rangle^k \) so that the result is general. Note that the higher-\( p \) terms in (12) can also be expanded similarly, so that we have an expansion in \( (m/N) \). The error made in truncating at any stage is of course calculable in this simple case, but is insignificant because it can be applied to group decompositions as well.

***In defining \( \rho_\Gamma^\lambda \) above we have assumed that the tensor coupling is multiplicity free, but this restriction is in fact not necessary.
7. Conclusion

Our subject has been spectral distribution and group symmetries though we have been able to refer to only a few papers in each domain. We have stressed that group theory constructions enter in a particularly elegant way when many-body problems are treated via spectral distributions; and by the same token, many-body results shed some new light on problems in group theory.

References

11) W. Miller, Jr., as quoted in Ref. 11.