STATISTICAL SPECTROSCOPY OF THE Sp(3,R) COLLECTIVE MODEL

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ABSTRACT

Analytic expressions are reported for the U(3) centroids of the Bohr-Mottelson collective Hamiltonian within irreducible representations of the symplectic algebra Sp(3,R). A favorable comparison is achieved between shell model diagonalizations and statistical predictions for the U(3) content of states in the ground state domain. These results imply that statistical measures can be used reliably to truncate infinite-dimensional representation spaces of Sp(3,R) so that microscopic calculations become feasible. Analytic results are also given for the major oscillator shell centroids. These provide constraints on the parameters of the collective potential in order for the underlying oscillator shell structure to be preserved.

1. Introduction

The symplectic collective model is a fully microscopic generalization of the Bohr-Mottelson liquid drop model which is compatible with the harmonic oscillator shell model. Basis states of the model span an infinite dimensional discrete series irrep of the non-compact real symplectic Lie algebra Sp(3,R). A representation space is generated from an Elliott SU(3) irrep \( \lambda_0^{0|0} \) at the \( 0\hbar \omega \) oscillator shell by successively acting on it with the Sp(3,R) raising operators. This creates model states with harmonic oscillator energies \( (N+n)\hbar \omega \), \( n = 0, 2, 4, \ldots \). The basis is chosen to be symmetry-adapted to the maximal compact subalgebra U(3) whose irreps are labelled by \( n(\Lambda_0) \). This basis labelling, however, is not multiplicity free.

The generators of Sp(3,R) include the angular and vibrational momenta which span GL(3,R), the kinetic energy operator, and the harmonic oscillator Hamiltonian \( H_0 \). Sp(3,R) also contains the mass quad-
The tensor $Q_{ij}^{(2)} \equiv \sum_{\alpha i \alpha j} (x_i x_j - 1/3 \delta_{ij} \delta_{\alpha}^{\alpha})$, where the sum is over the particle index $\alpha$ and $i, j$ denote Cartesian components. Since the Bohr- Mottelson potential $V$ is a polynomial in the quadrupole tensor, $V$ is in the symplectic enveloping algebra. Indeed, because $V$ is a rotational scalar, it is a polynomial in the fundamental scalars

$$a_2 \equiv b_2 \sum_{ij} Q_{ij}^{(2)}$$

and

$$a_3 \equiv \text{det } Q^{(2)}.$$  \hspace{1cm} (1)

Usually $V$ is restricted to quartic forms

$$V = b_2 a_2 + b_3 a_3 + b_4 a_4.$$  \hspace{1cm} (2)

where $a_4 \equiv (a_2)^2$ and $b_2, b_3, b_4$ are real constants. The Hamiltonian $H = H_0 + V$.

The eigenvalue problem for $H$ can be solved analytically in only very special cases. Thus, the infinite dimensional representation space poses a serious technical impediment to a computationally tractable theory. Because of the shell model, any physically sensible $H$ must yield a ground state containing small contributions from high-lying major shells ($n \geq 10$ in the $ds$ shell). However, even with a truncation at $10\%$, the dimension of the model space can be prohibitive, e.g. for $^{24}$Mg the dimension of the truncated space is 145,530.

2. $U(3)$ Centroids

Statistical spectroscopy methods, as adapted to the symplectic model, provide the means to identify the most important $U(3)$ subspaces to include in a truncated model space. A significant factor governing the contribution of a $U(3)$ irrep $\lambda$ to the ground state is its average energy or centroid, which is defined as the trace of $H$ in the $U(3)$ irrep space divided by its dimension

$$<H> = \text{tr } (H)/\text{dim } (\lambda).$$  \hspace{1cm} (3)

If $a_2$, $a_3$ and $a_4$ are expressed as a sum of $U(3)$ tensor operators, then non-scalar terms give a vanishing trace and, hence, do not contribute to the centroid. The trace-equivalent operators are polynomials in the $U(3)$ scalar operators in the $Sp(3,R)$ enveloping algebra. An integrity basis for the $U(3)$ scalars up to quartics is given by $H_0$, the quadratic and cubic $SU(3)$ Casimir operators $C_2$ and $C_3$, the quadratic and quartic $Sp(3,R)$ Casimir operators $C_2$ and $G_4$, the cubic $U(3)$ scalar $Y$ and two quartic $U(3)$ scalars $Z_1$ and $Z_2$. Thus, the trace-equivalents of $a_2$ and $a_3$ are
\[(a_2)_{TE} = \frac{25}{28} C_2 + \frac{5}{72} H_0^2 - \frac{5}{24} C_2 \]  

(4)

\[(a_3)_{TE} = \frac{7}{3} (a_2)_{TE} + \frac{7}{48} C_3 + \frac{7}{24} \sqrt{3} \gamma - \frac{35}{96} C_2 - \frac{35}{27} H_0. \]  

(5)

The \(a_4\) trace-equivalent is given in 3).

By making a Gaussian assumption, ground-state intensities can be predicted statistically 4). The agreement between the statistical prediction and explicit shell model diagonalization was found to be remarkably good for \(^{20}\text{Ne}\). To about a 10% level and for \(n \approx 4\), the statistical results correctly anticipate which \(U(3)\) representations dominate. See 3) for further details.

3. Major Shell Centroids

Next consider the centroid \(\langle H \rangle_n\), the average over the entire \(n\) major oscillator shell. To preserve the fundamental shell structure,

\[
\langle H \rangle_2 \approx \langle H \rangle_0 + 2n_\omega. \]  

(6)

This provides a constraint on the parameters of the collective potential (2).

The major shell centroid of \(a_k\) is a kth degree polynomial in \(n\) whose coefficients depend on the \(Sp(3,R)\) labels \(N_0(\lambda_0, \mu_0)\). For example,

\[
\langle a_2 \rangle_n = \frac{65}{336} n^2 + \left( \frac{125}{84} + \frac{5}{36} N_0 \right) n + \langle a_2 \rangle_0, \]  

(7)

where the \(0\)\(\omega\) centroid depends upon the quadratic \(SU(3)\) Casimir value for \(\lambda_0, \mu_0\) and \(N_0\),

\[
\langle a_2 \rangle_0 = \frac{5}{16} C_2 + \frac{5}{6} N_0. \]  

(8)

In earlier work on \(^{20}\text{Ne}\) with \(b_2 = 0\), the parameters \(b_3\) and \(b_4\) were adjusted to give a best fit to the B(E2) rates in the ground band 1). The ratio \(b_3/b_4\) of the best fit values was found to be -80.

Using condition (6), we predict the ratio -75. Clearly, the agreement is excellent. It gives direct evidence of the compatibility of the shell model and the collective model. Further results are given in 5).

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References


