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Spectroscopy from the Bottom Up

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Abstract

We contrast standard "top-down" shell-model theories with the popular "bottom-up" approach implicit in algebraic models. The latter, whether representing a boson (e.g., IBM = Interacting Boson Model) or fermion (e.g., MCM = Microscopic Collective Model) dynamics, are shown to share a common underlying group structure. And we present an algorithm for constructing the most general hamiltonian of given order in the generators of a dynamical group $G$ which is invariant under the operation of a symmetry subgroup $H \leq G$. In this the concept of an integrity basis of $H$ scalars in $G$ plays a central role. When an integrity basis exists in one-to-one correspondence with the invariants of a group lattice, $G \cong G_1 \cong H$, the theory embodies natural symmetry limits and analytic results can be given for observables. The theory is illustrated for the $G = SU(3) = SO(3) \times H$ geometry which is common to both the IBM and MCM and forms the backbone of the Elliott model. Calculations are presented for $^{24}$Mg and $^{164}$Er.

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1. Introduction

The popularity of the Interacting Boson Model\(^1\) (IBM) is well founded. Conceptually it is a very simple model, one built around the elementary notion of the existence of fundamental s and d boson degrees of freedom in nuclei. It is elegant in that it exploits group theory and, as a consequence, embodies certain identifiable symmetry limits: SU(3) = rotations, U(5) = vibrations, and O(6) = gamma unstable configurations. And it is a tractable theory, simple enough to yield a qualitative picture of many diverse types of data yet subtle enough to challenge the most sophisticated of competing theories.

The IBM theory has come under sharp attack from both collective-model theorists,\(^2\) who question the physical interpretation offered for the bosonic-degrees of freedom, and shell-model theorists,\(^3\) who offer a closely related challenge concerned with apparent algebraic inconsistencies associated with an interpretation of the bosons as correlated fermion pairs. There have been serious attempts towards resolving these conflicts and offering a better understanding of both the strengths and weaknesses of the model.\(^4,5\)

Our purpose here is neither to applaud nor critique the IBM industry; we remain committed to providing a microscopic fermion interpretation of the structure of nuclei. Nevertheless, in what follows we shall frequently use the IBM geometry to illustrate matters for the algebraic properties of interest to us transcend, for the most part, questions regarding the microscopic structure of the nucleon degrees of freedom. And, as suggested above, the mathematical elegance of the IBM is unsurpassed. Turning things around, one obviously has that our results for fermions have their boson counterparts.

We have set about to ask and seek answers to the following questions:

1. What are the essential differences between standard and algebraic shell-model theories?

2. Is there a new technology encountered in exploiting dynamical symmetries?

3. Can traditional theories be simplified and enhanced by incorporating and adopting algebraic methods?

The first of these questions is rather easy to offer a response to. This is dealt with in Section II, ahead. The second has a positive answer and our present understanding of these matters is discussed in Section III. Some results are known regarding the final question. These are presented by way of examples given in Section IV. In the last section we put forward some observations and speculations with the hope of stimulating input!
II. General Considerations

In Figure 1 we compare in a highly schematic way "standard" and "algebraic" shell-model theories. The traditional approach can be viewed as a "top-down" theory. One begins with an infinite Hilbert space and true physical operators. Truncation is then invoked to achieve a tractable model space and renormalization procedures applied to deduce the corresponding set of effective model-space operators. It is quietly assumed that if the model space is chosen large enough so that the required renormalizations need only compensate for high-lying, distant-orbit effects they will be mild and a relatively smooth function of particle number. Furthermore, it is customary to incorporate this dependence on particle number into the mean field while keeping the effective two-body interaction A independent. And operators associated with observables like E2 transitions are usually only rescaled.

In contrast with this, algebraic models may be viewed as "bottom-up" theories. One begins with a set of operators which represent physical observables and form a Lie algebra. The model space is then defined in terms of irreducible representations (irreps) of that algebra. The associated group, say G, is usually referred to as a dynamical symmetry. In the IBM case the thirty-six bilinear combinations of s and d boson creation and annihilation operators generate the dynamical group U(6). One then proceeds by looking for subgroups of G, say G₁, which contain the exact symmetry or invariance group H of the hamiltonian as a subgroup. For an IBM theory H is the rotation group SO(3) and the G₁ are SU(3), U(5), and O(6). Such a model is deemed to be successful if an H invariant hamiltonian of low degree in the generators of G can be found which yields agreement with experimental results. The G₁ define various symmetry limits of the G \rightarrow H model.
Interaction parameters are usually fit to the available data with additional liberties sometimes taken in defining transition operators. This is portrayed in more detail in Figure 2.

It is tempting to follow the naming practices of our high energy colleagues and attach a "bottomless"/"topless" tag to these two contrasting theories. But that would be quite inappropriate. By moving upstream from even the simplest of algebraic models one can usually achieve a traditional type geometry and, of course, the opposite is true as well. The best illustration of this is the microscopic collective model (MCM) of Rosensteel and Rowe. The basic structure is the Elliott SU(3) \rightarrow SO(3) geometry. By adding to this the Zn\omega raising and lowering operators associated with collective quadrupole degrees of freedom one gains the Lie algebra of the noncompact Sp(3,R) group. Though the representations of Sp(3,R) are themselves infinite dimensional, they form a subspace of the full shell-model space. Indeed, as Rosensteel and Rowe have shown, the full space can be partitioned into vertical slices, with each slice being an irreducible representation of Sp(3,R). In this case G = Sp(3,R), H = SO(3) and there is one G_l = SU(3), see Figure 3. By augmenting the Sp(3,R) model interaction with horizontal mixing of a type that breaks the SU(3) symmetry and couples Sp(3,R) representations, one can move over into a traditional shell-model regime.

The shell-model truncation exercise is usually carried out using a configuration geometry with single-particle orbitals defining the relevant unitary spaces. Hence ds-shell calculations done in JJ[LS]-coupling have a U(24) \times U(12) \times U(6) \times U(4) \times SU_3(2) group structure. A truncation of this type automatically excludes couplings to the 2\Omega configurations included in the Sp(3,R) model. Clearly the two models emphasize different features --- the standard approach pays closest attention to single-particle features while the Sp(3,R) model focuses on the collective E2 properties. Elliott was the first to recognize and appreciate the significance of the harmonic oscillator SU(3) \rightarrow SO(3) substructure of the LS-coupled shell model. Indeed, if one truncates the shell model beyond one major shell right down to a single representation of the oscillator one has the core of Sp(3,R) model referred to above. These two models share the same parent dynamical symmetry, SU(3) \rightarrow SO(3). This situation clearly offers many opportunities for comparisons. In particular, a very natural but as yet unexplored question, concerns the ability and/or effectiveness of a renormalization theory to compensate for ingredients missing from one model but present in the other.

From this brief discussion the strengths and weaknesses of each approach should be apparent. Again, at the risk of over simplifying matters, we offer in Table 1 a summary of this. Philosophically the traditional or standard approach has the greatest appeal. It champions the ideal of maintaining a direct connection with nucleon-nucleon forces. But its weakness lies in the qualitative nature of the truncation decision and, perhaps even more importantly, in uncertainties associated with the necessary renormalization procedures. Symmetries are certainly involved but compared with algebraic theories they play a recessive role. The question, "How good is a symmetry?" normally does not have a simple answer. Indeed, an adequate response can usually only be given upon completion of a calculation.

The use of an algebraic theory, on the other hand, implies "biting-the-bullet" from the start. Though operator selection is guided by the physics, no claim is made regarding completeness of the set selected nor is a direct connection with nucleon-nucleon forces espoused. The need for parameter...
fitting is acknowledged from the onset. Justification for this is claimed when the structure of a whole series of nuclei can be reproduce without the need for major changes in parameter sets. And clearly symmetries play a dominant role. Whether or not they are good has a simple yes/no answer, depending simply upon the success of the model.

In what follows we hope to demonstrate, by a closer look at the structures encountered and via some detailed examples, that in some sense both theories overlook a very important aspect of the whole shell-model problem. By relaxing constraints inherent in the renormalization procedures accompanying a standard approach and by expanding upward in degree the allowed operator forms encountered in algebraic theories one can enjoy the best of both. At the risk of impropriety, we suggest christening any such theory a “High-bred” Algebraic Model. As will become clear, the obvious choice for an acronym is IBM, for Integrity Basis Model, but impropriety is a far lesser charge than heresy so HAM it will be!

III. Integrity Basis and Dynamical Symmetries

The group structure common to all algebraic models is indicated in Figure 4. There is a largest group $G$ which is a "conserved" symmetry. This designation derives from the fact that by definition all operators are expressible in terms of the generators, $X_i$, of $G$,

$$O(X) = A^{(0)} I + A^{(1)} X_1 + \frac{1}{2!} A^{(2)} X_1 X_j + \frac{1}{3!} A^{(3)} X_1 X_j X_k + \cdots$$  \(1\)

As a consequence, no operator has matrix elements coupling different $G$ irreps. In particular, every eigenstate of the system Hamiltonian $H$ belongs to a single $G$ irrep. And there is an "exact" symmetry subgroup $\mathcal{H} \subset G$. This is the invariance group of $H$. If the system is rotationally invariant $H$ is $SO(3)$, or $SU(2)$ for theories which explicitly include spin degrees of freedom. Every eigenstate of $H$ belongs to a degenerate representation of $\mathcal{H}$.

Between $G$ and $\mathcal{H}$ there may be other groups, $G \supset G_1 \supset \mathcal{H}$. They are neither conserved nor exact symmetries for a generic $H$ of the model has matrix elements coupling different $G_1$ irreps. We refer to these intermediate groups as "imposed" symmetries. Mathematically they form part of a $G \supset \mathcal{H}$ group lattice. They define special symmetry limits of the model. For example, in the $O(6)$ limit of the IBM the hamiltonian has no matrix elements coupling different $O(6)$ irreps. The $O(6)$ structure above $O(6)$ is then excess baggage which serves no role in the dynamics. Therefore, in the $O(6)$ limit of the IBM, $O(6)$ replaces $U(6)$ as the conserved symmetry. So associated with each $G_1$ is a special symmetry limit of the $G \supset \mathcal{H}$ model. Normally, however, $\mathcal{H}$ couples different $G_1$ irreps. The $G_1$ are then broken symmetries and it is meaningful to ask how "good" or how "bad" each is or, equivalently, how "close" or "far" $\mathcal{H}$ is from a $G_1$ symmetry limit. Though such "information" is necessarily
A summary of characteristics of the "conserved", "imposed", and "exact" symmetry types is given in Table 2. Note that there is no literature standard regarding naming conventions. The frequently used designation "noninvariance group" includes both conserved and imposed symmetries while "invariance group" is reserved for the exact symmetry subgroup. Dynamical symmetries also usually refer to all symmetries other than the exact one, though some authors' usage seems to suggest a more restrictive definition. For our purposes it is convenient to have a simple descriptive label attached to each. We have chosen "imposed" over "broken", for example, because to some a conserved symmetry is broken because it is not exact. As indicated in the Table, the three types are also distinguishable in terms of spectral features; that is, by the action of H on a group irrep. By definition every group lattice has a conserved and an exact symmetry, but may or may not have imposed ones. For example, the O(6) limit of IBM does [G=O(5)] but the SU(3) limit does not.

Now rather than asking the standard shell model question, "How good is an embedded symmetry?" which can usually only be answered upon completion of a calculation, or the question, "Is (are) the symmetry (symmetries) good?" which is appropriate for an algebraic theory, one can ask the more probing question, "How much symmetry breaking is necessary?" The latter implies starting at the bottom with a particular G,H pair and only expanding the lattice upward after exploiting that geometry to its fullest. This is what we have christened a HAM (Highbred Algebraic Model). Clearly we have in mind a theory that is free not only of the constraints of a standard shell model theory; namely, that each model space operator be the actual renormalized version of the physical operator it represents, but also of the quietly accepted practice of algebraic modeling enthusiasts which limits the set of allowed operators to those which have a structure similar to the ones used in a standard theory. For example, in algebraic theories it is customary to restrict H to (0+1+2)-body forms. So even though the coefficients multiplying the various factors in such forms are left as free parameters to be fit to experiment, there is a constraint and as we shall see, it is a severe one.

There is a precedent for this modus operandi. It is well-known that rotational bands in collective nuclei can be fit with a low-order power series expansion in L^2. Here L^2 has eigenvalue L(L+1), L being the angular momentum. The coefficient of L^2 is taken to be a parameter, usually fit to the 0^- - 2^+ energy difference, and given the physical interpretation of an inertial parameter 1/2I, I being the principal moment of inertia of the system. An L^4 term (4-body?) is introduced to look after centrifugal stretching or antistretching effects, etc. Indeed, the rotational model can be viewed as exploiting, in the sense suggested above, a G = SU(3) × H geometry. Clearly one should insist, as has been done in the case of the rotational model, that the parameters of such a theory display consistency when the model is applied to neighboring species.

An essential step in a HAM theory is seeking an answer to the question, "What is the structure of the most general interaction of the G × H geometry?" It is this question that we now turn to. Consider again expansion (1). Terms of degree d in the generators may be taken as symmetric under permutation of the X_i's. If they are not symmetric, they may be written as a sum of a symmetric and an antisymmetric part. The degree of the antisymmetric part can then be reduced by one by using the commutation relations of the Lie
algebra of G. There are therefore \((n+d+1)/(n-1)!(d)\) operators which are homogeneous of degree d and symmetric under permutation of the n generators of G. These operators span a linear vector space of operators, \(u(d)(G)\). The direct sum of these operator spaces is called the universal enveloping algebra of G.12)

\[ U(G) = \bigoplus_d u(d)(G) \]  

A simple answer to the question posed is then that it is the H-scalars in \(U(G)\), and hence in \(u(d)(G)\), that qualify as operator forms for H.

But this simple answer is grossly inadequate for it provides no real directive for constructing the required scalar operators. For that the following three results, which we give without proof, are essential:13)

1. If H is a k-body interaction, then \(H(X)\) is of degree \(d = k\).

2. An algorithm can be given for determining the number of independent H-scalars of degree d in \(U(G)\):

   A. Determine \(I^H(G)\), the representation of G carried by \(u(d)(G)\).

   B. Extract the number of times the identity representation of H, \(\Lambda^H(1)\), occurs in \(I^H(G)\).

   C. The basis operators for each \(\Lambda^H(1)\) in \(I^d(G)\) are the H scalars in \(u(d)(G)\).

3. Noether's Theorem: If \(H \subseteq G\), then the set of H-scalars in G is finitely generated.

The third result means that there exists a finite-dimensional integrity basis (IB) of H scalars in G in terms of which any H scalar can be expressed in multinomial form.14) It should also be clear, for example from (2) above, that the integrity basis operators can be graded by degree.

\[ IB(G = H) : \{I^H(k_j), j=1,2,\ldots,s\} \]  

In (3), \(k_j\) specifies the degree in generators of G of the j-th H-scalar operator, \(I^H\). Then, because of the first result,

\[ H(k) = \sum_{0 \leq a \leq k} h(a) \prod_{i=1}^{s} (Z_i)^{a_i} \]  

This is a formal result but, as we shall see, it is also a very useful and practical response to our question. In (4), k is the particle rank of H (k-body interaction) and \(a\) and \(\bar{a}\) are s-dimensional vectors whose components specify, respectively, the power and degree of the \(I^H\) terms in the multinomial expansion of H.

From this development it is very easy to see that the number of parameters in a HAM theory of maximum particle rank k is just the number of solutions of the constraint condition \(0 \leq a \leq k\bar{a}\) with the components of \(a\) restricted to be nonnegative integers. (Actually, there may be additional constraints on the \(a_i\) which come about because of polynomial relationships, called syzygies, among operators of the integrity basis set.15) For example, for \(G = SU(3) \supset SO(3) = H\) which we consider in detail ahead, there is a sixth order scalar which is independent of all others, but its square, and hence any higher power, is not.) Clearly, if the integrity basis for the G,H pair is known, the problem of constructing the most general \(H(k)\) for the \(G \supset H\) geometry is resolved. Indeed, it suffices to know \(\bar{a}\) (plus syzygies) to determine the number of parameters in such a theory.

To illustrate this, note that if there are no syzygies the number of parameters associated with terms of degree n is just given by the coefficient of \(D^n\) in an expansion of the generating function.
If there are syzygies, the affected terms must be removed from the denominator factor and entered appropriately in the numerator. For example, since the
integrity basis for SU(3) \rightarrow SO(3) has two scalars of degree two, two of degree three, one of degree four, and one of degree six with powers greater than one of the latter not independent of the former, \( L = (2,2,3,3,4,6) \), and we have that

\[
N[\text{SU}(3) \rightarrow \text{SO}(3)] = \frac{1+6^6}{(1-0^2)^2 (1-0^3)^2 (1-0^4)}
\]

(6)

The algorithm, listed second above, gives one a direct method for determining the structure of the integrity basis. One must simply carry out steps A-C for \( d = 1,2,\ldots \). The number of new scalars of degree \( d \) is just the multiplicity of \( X_0^d(H) \) in \( r^d(G) \) less the number of independent polynomial scalars of degree \( d \) can be formed using those previously found for \( d' < d \). This can also be shown to be equivalent to counting the number of H-scalar density operators one can build using \( d \)-particle states in the \( G \) geometry.\(^{17}\)

Though we clearly leave much unsaid about all this, we are now in a position to move forward and see how it can be used to further simplify algebraic theories. The key feature in this is that the invariants of \( G \), the invariants of \( H \), and the invariants of any \( G_i \)'s that may exist are all H-scalar operators. They can therefore be substituted for some, and in some cases, all of the \( Z_i \) operators of the integrity basis. To see how this works, consider again the SU(3) \rightarrow SO(3) case. An integrity basis which explicitly includes the invariants of SU(3) and SO(3) is \( ^{16}\)

\[
\text{Id}[\text{SU}(3)] = \text{SU}(3) \downarrow i \downarrow 2 \downarrow 3 \downarrow 4 \downarrow 5 \downarrow 6.
\]

(7)

Here \( C_2 \) and \( C_3 \) are the second and third order Casimir invariants of SU(3), \( L^2 \) is the second order Casimir invariant of SO(3), and \( Z_3, Z_4 \) and \( Z_6 \) are \( 3^{rd}, 4^{th} \) and \( 6^{th} \) order noninvariant H-scalars. There are no intermediate \( G_i \) symmetries and hence no other invariants that can be substituted for the three noninvariant H-scalars (\( Z \)'s). There is one syzygy: powers of \( Z_6 \) greater than one do not yield operators independent of polynomials in the other five H-scalars. In terms of the three angular momentum operators \( L_\mu \) and the five quadrupole operators \( Q_\mu \), which together generate SU(3),

\[
\begin{align*}
Z_3 & \sim (L \times Q_3)^0 \\
Z_4 & \sim [(L \times Q_0)^1 \times (Q_4)^0]^0 \\
Z_6 & \sim [Z_3, Z_4]
\end{align*}
\]

(8)

Note, if \( Z_6 \) is included in an interaction the corresponding parameter must be pure imaginary for \( H \) to be hermitian. In (7) the couplings are SO(3) couplings and [ , ] denotes a commutator.

Now consider the corresponding SU(3) \rightarrow SO(3) dynamics. If we restrict \( H \) to be of degree \( (2,3,4) \) in the generators of SU(3), from (6) we see that we have a \( (3,5,9) \) parameter theory,

\[
H(2) = h_1 + h_2 C_2 + h_3 L^2
\]
\begin{equation}
H(3) = H(2) + d_4 C_3 + h_2 Z_3
\end{equation}
\begin{equation}
H(4) = H(3) + d_4 C_2 + h_2 L^2 + h_8 L^4 + h_2 Z_4
\end{equation}

But within a given irrep of SU(3), C2 and C3 are multiples of the identity. So in reality this reduces to a (1,2,4) parameter theory, respectively. The eigenvalue of $L^2 - L(L+1)$ is trivial. Only $Z_3$ and $Z_4$ have nonsimple matrix elements. But since these have a simple representation (8) in terms of the generators of SU(3), this is not really a serious problem either.

Furthermore, note that powers or products of integrity basis operators are, because there are no nonzero couplings involved, as easy to calculate as linear forms. So by exploiting an integrity basis technology, we see that a G $\rightarrow$ H dynamics can be reduced to its simplest and most elementary form.

In the IBM case it can be shown that the invariants of U(6), SO(3) and the three imposed symmetries O(6), U(5), SU(3) suffice for a k<2 theory if one restricts oneself to degenerate U(6) representation. But for an IBM theory which distinguishes between neutron and proton degrees of freedom, nondegenerate representations of U(6) enter and, accordingly, additional SO(3) scalar operators are required.\(^{18}\) For a complete analysis of this case, see Ref. 13. Let it suffice for the present to emphasize that HAM physics, when coupled with an integrity basis technology, removes much of the mystic that seems to surround the algebraic modeling game. Next we see, by means of examples, that in addition it carries real physical significance.

IV. Some Examples

To illustrate both the simplicity and utility of a HAM theory, we have undertaken a full study of the G = SU(3) $\rightarrow$ SO(3) = H geometry. This structure is a common substructure of both the IBM and MCM models and forms the foundation of the Elliott model. It seems fair to ask, "How much can be achieved without breaking SU(3)?" Though here we restrict ourselves to fermion model applications, it should be clear that similar results can be generated, for example, for the SU(3) limit of the IBM using the same technologies. We will give results for $^{24}$Mg and $^{164}$Er,\(^{19,20}\) Results for other rare earth nuclei ($^{160}$Yb, $^{162}$Yb, $^{164}$Yb, $^{166}$Er, $^{168}$Er, $^{169}$Yb, $^{168}$Yb) and some actinide species ($^{232}$Th, $^{234}$U, $^{236}$U, $^{238}$U) are also available in the literature.\(^{21}\)

In Section III we saw that the SU(3) $\rightarrow$ SO(3) integrity basis consists of six operators, two of which are SU(3) invariants and, as a consequence, contribute nothing to the dynamics within a single representation of SU(3). The most general SU(3) $\rightarrow$ SO(3) inter-irrep $k<4$ interaction has the simple form

\begin{equation}
H(\text{SU(3)$\rightarrow$SO(3);k<4}) = aL^2 + bZ_3 + cZ_4 + dL^4
\end{equation}

In terms of spectrum characteristics the effect of each term is easily understood. Of course, a = 1/2I where I is the moment of inertia. And d is a parameter that looks after centrifugal distortions (stretching or antistretching) from the rigid rotor $L(L+1)$ spectrum of $L^2$. The $Z_3$ and $Z_4$ operators also look after K-band effects. They have a long history starting with Racah\(^{22}\) and including, to be sure, Moshinsky\(^{23}\) (the $Q$ operator) as well as others.\(^{24}\) One of the two or a linear combination serves to resolve the
SU(3) \to SU(3) state labeling problem. Here we find a purpose for both; both are required for a full SU(3) \to SU(3) dynamics.

We performed some theoretical "experiments" to test the ability of such a simple interaction to accommodate special K-band features in a spectrum. We found that the spectrum of \( H = L^2 \pm \alpha K^2 \) can be fit with \( H^{SU(3)\to SO(3);k\leq4} \) to high accuracy even with the parameter \( \alpha \) close to unity. (With \( \alpha \leq 1 \) the differences for levels of the \( \Lambda \mu \) = (84) irrep of \( SU(3) \) gave, independent of whether the + or - sign was used in \( H \), a \( \chi^2 \approx 2\text{MeV}^2 \).) Also, we found that a linear dependence on \( K \) of the type \( H = L^2 \pm \alpha K \) can be followed if \( \alpha \) is small but with \( \alpha \leq 1 \) deviations become large and significant irregularities in the calculated results emerge. Here, as above, the \( \chi^2 \) measure was found to be the same for equal positive and negative K-band shifts. We concluded that at a minimum \( H^{SU(3)\to SO(3);k\leq4} \) can accommodate small (large) linear (quadratic) dependences on the bandhead label \( K \).

The result of a best fit \( H^{SU(3)\to SO(3);k\leq4} \) integrity basis theory for \( ^{24}\text{Mg} \) is shown in Figure 5. A weight of five was assigned to the two \( L=2 \) states composed to one for all other levels. This was done to emphasize the moment of inertia and K-band splitting features of the spectrum. The best fit values for the parameters in \text{MeV} are \( (a,c,d) = (0.252, -0.00234, -0.00166, -0.000438) \). Note how accurately theory reproduces the \( -3\text{MeV} \) K-band splitting of the \( ^2J \) states. It does not, however, reproduce the deviation of the \( K=2 \) band \( 3^+ \) and \( 4^+ \) states from a rotational \( L(L+1) \) sequence. To reproduce such detail would require higher degree polynomial terms in the model \( H \). Such features lie beyond that available using a simple \( K < 4 \), \( SU(3) \to SU(3) \) integrity basis interaction. Our results for \( ^{24}\text{Mg} \) can be shown to compare favorably with a much more sophisticated multi-\( \alpha \) MCHM \([Sp(3,R)]\) calculation which yields, incidentally, excellent agreement between theory and experiment for \( B(E2) \) values without introducing an effective charge.\(^{25}\)

As a further test of the \( SU(3) \to SU(3) \) theory, we fit the ground and gamma band spectra of \(^{164}\text{Er} \). Now the fermion \( SU(3) \to SU(3) \) algebra appropriate to the rare earth and actinide regions also has a long history.\(^{26}\) It is realized by recognizing that the relevant sets of valence spherical shell-model orbitals, which differ in many ways from those of an isotopic harmonic oscillator due primarily to single particle spin-orbit effects, can be split into normal and abnormal parity parts and that the normal parity parts can be mapped onto pseudo spin-orbit pairs \( (\Xi = \sum_{\Xi} \Xi = \sum_{\Xi} \) which when taken together form a "pseudo" oscillator shell of one less quantum than that of the "real" oscillator shell from which they derive (\( N=N-1 \)). This separation and mapping is shown for the proton space of \(^{164}\text{Er} \) in Figure 6.

The \( SU(3) \) is therefore a "pseudo" \( SU(3) \) of the "pseudo" oscillator just as the Elliott \( SU(3) \) is an \( SU(3) \) of the "real" oscillator.

What makes this scheme so effective is that the pseudo spin-orbit pairs are less strongly split than for the usual shell model. For the coefficients \( a(\Xi) \) of \( \sum_{\Xi} \sum_{\Xi} \Xi \) in \( H(\Xi) \) one has that \( a \leq a/10 \). And furthermore, the counterpart of the single-particle \( x^2 \) term, which in the regular shell model also serves to destroy the oscillator structure, is again weaker. Hence the pseudo oscillator structure for rare earth and actinide species is a much better first-order symmetry than the real oscillator is for the ds shell. As \( SU(3) \) is the symmetry of the oscillator, one expects, based on successes of the Elliott model in ds-shell applications, the pseudo \( SU(3) \) scheme to be a good symmetry for deformed rare earth and actinide species. And there is good evidence supporting this.\(^{27}\) It is important to note that \( Q=Q; \) that is, to a high degree the "real" quadrupole operator maps unto the "pseudo" quadrupole operator with only small (known) correction terms.\(^{28}\) Hence the pseudo \( SU(3) \)
scheme accommodates both the major shell rearrangement effects of the single-particle $\pi$-$\sigma$ interaction while maintaining a simple picture of quadrupole collectivity.

The abnormal parity level that penetrates into the domain of the normal parity levels is treated using a quasispin geometry. This seems appropriate since even a quadrupole-quadrupole interaction in a single j shell correlates strongly with pairing. For this reason, in this first theory, we assumed a seniority (v) zero contribution to the dynamics from the abnormal parity space. If one allows for v=2 excitations one has a built-in alignment/backbending mechanism. The result of all this for the case of $^{164}$Er is shown in Figure 7. One uses the Nilsson model, with orbits appropriately relabelled with pseudo quantum numbers, to determine the distribution of neutrons and protons among the normal and abnormal parity levels. For a deformation $\epsilon = 0.2$, the dominant configuration should have $10(6)[8(8)]$ protons(neutrons) in the normal(abnormal) parity spaces. It has been shown that even a small neutron-proton interaction will drive the system towards an SU(3) strong coupled limit. State vectors are then built by weak coupling to strong coupled SU(3) states the quasispin configurations of the abnormal parity spaces ($J_A = 0$ for $v = 0$, so $J = J_N + J_A = J_N$).

Results for $^{164}$Er are given in Figures 8 and 9. The experimental spectrum for ground and gamma band states are compared with a three and a four parameter theory. The three parameter results are with d, the coefficient of $L^4$, set at one-half its optimum (four parameter) value. This was done because at approximately 2MeV pair-alignment (backbending) effects enter. The $L^4$ term can, in part, mock-up that effect but only at the expense of inverting higher spins ($J \geq 20$). The real mechanism is probably dominated, as the name suggests, by a mixing of $v=0$ abnormal parity configurations with aligned $v=2$ ones. As the latter are not included in this first version of the theory, it seemed appropriate to constrain the parameter multiplying $L^4$ to be smaller than its "best-fit" value. The B(E2) values, which are a more sensitive test of the calculated wavefunctions, are also well-reproduced. A single parameter, the effective charge (1.5e), was set by requiring the calculated $2^+ \rightarrow 0^+$ ground band transition to match experiment.
V. Conclusion

A very general comparison of standard "top-down" and algebraic "bottom-up" shell-model theories was given to emphasize both similarities and differences in the two approaches. That led us to a careful and detailed analysis of the common features encountered in algebraic theories. We discovered that there is a structure which transcends, for the most part, even questions regarding the microscopic structure of the nucleon degrees of freedom. And if one is willing to free oneself of the constraint that H be two-body, what might a first attempt to be a real "can of worms" leads to a whole new technology which yields a spectroscopy that is both simpler to understand and more elegant in form than that encountered in traditional theories. We suggested christening such a theory a Hybrid Algebraic Model (HAM). By affixing a \( (G=H;k\leq 1) \) identifier to the HAM abbreviation, the specific geometry and effective particle rank of the theory can be specified. For example, IBM = HAM\( (U(6)\to SO(3);k<2) \). Some detailed results were presented for \( 24\text{Mg} \) and \( 16\text{Sn} \) using a HAM\( (SU(3)\to SO(3);k<4) \) theory.

The concept of an integrity basis plays a central role in all algebraic shell-model theories. (Indeed, though it may be presumptuous and certainly unnecessary, it is tempting to suggest that some significance be added to the IBM "logo" by claiming it as an acronym for Integrity Basis Model, replacing the HAM abbreviation which carries it peculiar connotations all its own!) In particular, Noether's theorem, which says that for H \( \triangleright G \) the set of all H scalars in G is finitely generated, means that any H(GoH) has a multinomial representation in terms of a finite number of H scalars. (For the \( G=SU(3)\to SO(3)\to H \) case the number of operators is six and this yields for \( k<4 \) a four parameter theory.) By replacing the integrity basis scalars by invariants of \( G, \) of \( H \) and of intermediate "imposed" symmetries \( G_1, \) a problem
can be reduced to a form which includes natural symmetry limits. In the case of IBM-1, this procedure is equivalent to writing $H$ for $k < 2$ in terms of the invariants of $\text{U}(6), \text{O}(6), \text{U}(5), \text{SU}(3),$ and $\text{SO}(3)$. For a theory which distinguishes between neutrons and protons, IBM-2, there is a corresponding result which includes additional lattice symmetries.

Concerning the relaxing of constraints on the effective particle rank of an interaction in a HAM theory, we now simply wish to say that our results strongly suggest that it is a practical modus operandi which seems to work well. Justification is most certainly to be found in a (as yet unavailable) symmetry adapted effective interaction theory.

Table Captions

1. A comparison of some positive and negative features of standard and algebraic shell-model theories.

2. Classification of group symmetries.
Figure Captions

1. A schematic comparison of standard ("top-down") and algebraic ("bottom-up") shell model theories. Either implicit or explicit in all algebraic theories is a postulate regarding the existence of a set of group operators which is sufficient for representing the dynamical degrees of freedom of a system. This is in contrast with a standard shell-model approach which employs a truncation/renormalization prescription and champions the ideal of maintaining a connection to real nucleon-nucleon forces. The latter has a "space" focus and the former an "operator" focus. A single-particle configuration geometry is normally used in standard theories with special symmetries, like the Elliott $SU(3)$ or quasispin, emerging after the fact as being important as a result of the dominance of special interactions (quadrupole-quadrupole or pairing, respectively) in the Hamiltonian. Hence special symmetry play a recessive role in such theories as compare to the dominant place they have in algebraic models.

2. A schematic representation of the space and spectrum of an algebraic theory. The group $G$ is a "conserved" symmetry, $H$ an "exact" one, and if there exists intermediate "imposed" symmetries $G_1$, $G \supset G_1 \supset H$, they can be used to resolve, at least in part, the $G \times H$ multiplicity. The Hamiltonian can always be written as a function of the invariants of $G$, the invariants of $H$, and other noninvariant $H$-scalars, $Z_p$. Some, and in some cases all, of the noninvariant scalars may be replaced by invariants of the imposed symmetries. It is the noninvariant scalars in the Hamiltonian that split otherwise degenerate $H$ multiplets.
3. A boson (IBM + Interacting Boson Model) and fermion (MCM + Microscopic Collective Model) example of the general group structure $G \rightarrow G_3 \rightarrow G$ found in all algebraic models. The structures encountered transcend, for the most part, questions regarding the microscopic character of the nuclear degrees of freedom. In the IBM case all the groups are compact and hence the model spaces encountered are finite dimensional. In the MCM case the compact $SU(3)$ is embedded in the noncompact $Sp(3, \mathbb{R})$. The $Sp(3, \mathbb{R}) \supset SU(3)$ part of the model augments the compact Elliott $SU(3) \supset SO(3)$ core by explicitly including the $2\hbar_\omega$ collective excitation generated by the quadrupole degrees of freedom. For two-body hamiltonians the invariants of the IBM (MCM) lattice are sufficient (insufficient) for a complete dynamics.

4. All algebraic models have a largest symmetry group $G$. It is a "conserved" symmetry for all operators of the system are assumed to be representable in term of the generators of $G$ and hence have no matrix elements coupling different $G$ irreps. And there is an "exact" symmetry subgroup of $G, H$, which is the invariance group of the hamiltonian. Every eigenstate of the hamiltonian belongs to a degenerate representation of $H$. Between $G$ and $H$ there may be "imposed" symmetries, which are neither exact nor conserved for a generic hamiltonian has matrix elements coupling different $G_3$ irreps. The invariants of the $G_3$ can be used as multiplicity labels and to replace some, and in some cases all, of the noninvariant $H$-scalars that enter into the construction of a model hamiltonian.

5. The $^{24}\text{Mg}$ experimental spectrum is given on the left together with results of a best fit $SU(3) \supset SO(3)$ calculation using the $k < 4$ integrity basis interaction $H = aL^2 + bJ_3 + cZ_4 + dL^4$ on the right. The model calculation includes only the $(\lambda\mu) = (84)$ ds-shell representation. The integrity basis interaction parameters in MeV are $(a, b, c, d) = (0.2519, 0.02342, -0.001664, -0.0004380)$. To emphasize the $K$-band splitting and moment of inertia features of the structure, the two $L=2$ states were assigned a weight five times greater than the others in the least-squares fitting procedure.

6. For rare earth and actinide nuclei, valence protons and neutrons occupy single-particle spherical orbitals $(\tau \pi \nu \lambda)$ that can be mapped onto a set of pseudo-spin orbitals $(\tau + \lambda \pi + \nu)$ which together form a pseudo oscillator $(N)$ shell:

$$
(9/2^2 \ell_5/2 \ell_3/2 \ell_1/2) \rightarrow N=3
$$

$$
(5/2^2 \ell_5/2 \ell_3/2 \ell_1/2) \rightarrow N=4
$$

$$
(11/2^2 \ell_5/2 \ell_3/2 \ell_1/2) \rightarrow N=5
$$

In addition to these "normal" parity states there is in each case an "abnormal" parity level which is pushed down from the next shell up by the spin-orbit interaction, $h11/2, i13/2$, and $j15/2$ for the $N=3, 4$, and 5 cases, respectively. For an interaction with significant quadrupole plus pairing parts, an $SU(3) \supset SO(3)$ geometry dominates within normal parity configurations and quasispin in the abnormal parity levels. The results shown are for the $N=3$ case; that is, for proton configurations of rare earth species.

7. A Nilsson level scheme is used as a guide in selecting the distribution of
particles among the normal and abnormal parity levels. Results shown are for $^{164}\text{Er}$. We expect the dominant configuration to have $10(8)[8(6)]$ protons(neutrons) in normal(abnormal) parity orbitals. The proton-neutron part of the interaction drives the normal parity part of the system towards $SU(3)$ strong coupling and pairing dominates in the abnormal parity parts. The final $J$ is obtained by weak coupling the two subspaces.

8. The experimental ground and gamma band spectra of $^{164}\text{Er}$ are compared with three and four parameter best-fit integrity-basis calculations using a $k < 4$ interaction. The three parameter results are for a theory with the coefficient of $L^4$ fixed at half its best-fit (four parameter) value. This was done because pair alignment effects, which are not included in the current version of the theory, are expected to be a significant part of the backbending phenomena that occurs around 2 MeV (solid horizontal line) and a strong $L^4$ term in the hamiltonian can, in part, mock-up such an effect.

9. Experimental (circles with error bars, see Refs. 29,30) and theoretical (dots connected by lines) $B(E2)$ values for $^{164}\text{Er}$. The effective charge $(1.5e)$ was set by fitting to the ground band $2^+ \rightarrow 0^+$ transition. Note the scale changes by a factor of ten for the inter-band transitions.

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Table 1. A comparison of some positive and negative features of standard and
algebraic shell-model theories.

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>Shell-Model Theory</th>
<th>Algebraic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interaction derived from nucleon-nucleon forces</td>
<td>YES</td>
<td>NO</td>
</tr>
<tr>
<td>Model space truncation and renormalization required</td>
<td>YES</td>
<td>NO</td>
</tr>
<tr>
<td>Extension/enhancement</td>
<td>YES</td>
<td>NO</td>
</tr>
<tr>
<td>Role of symmetries in the theory</td>
<td>RECESIVE</td>
<td>DOMINANT</td>
</tr>
<tr>
<td>Simple model-space algebra/geometry</td>
<td>NO</td>
<td>YES</td>
</tr>
<tr>
<td>Tractable technology, simple algebraic limits</td>
<td>NO</td>
<td>YES</td>
</tr>
<tr>
<td>Simple relationships between input and output</td>
<td>NO</td>
<td>YES</td>
</tr>
</tbody>
</table>
Table 2. Classification of group symmetries.

<table>
<thead>
<tr>
<th>Type</th>
<th>Algebra 1</th>
<th>Representations 2</th>
<th>Examples 3</th>
</tr>
</thead>
</table>
| Conserved  | \([H, X_1] \neq 0 \) but \( H = H(X_1) \) | Internal Spreading | IBM: \( U(6) \)  
                   |            |                   | MCM: \( Sp(3, R) \) |
| Imposed    | \([H, X_1] = 0 \) | External Mixing   | IBM: \( O(6), U(5), SU(3) \)  
                   |            |                   | MCM: \( SU(3) \) |
| Exact      | \([H, X_1] = 0 \) | Degenerate Multiplets | \( SO(3) \)  
                   |            |                   | Parity       |

1 There is no literature standard - these three categories are descriptive and useful for describing the properties of algebraic models.
2 \( X_1 \) refers to a generic group element so \( [H, X_1] = 0 \) only if so for all the elements of a group.
3 Reference is to the action of \( H \) on a typical state belonging to an irreducible representation of the symmetry group.
4 Group chains and structures encountered in spectroscopy usually contain all three symmetry types in various combinations, see Figure 3 for examples of this.
**ALGEBRAIC MODELS**

**SPACE**

$G$ = UNIVERSE OF ALL OPERATORS
(NONINVARiANCE GROUP——CONSERVED SYMMETRY)

$H$ = SYMMETRY GROUP OF HAMILTONIAN
(INVARiANCE GROUP——"EXACT" SYMMETRY)

**SPECTRUM**

$\Lambda \cdot \Lambda_2$ GOOD BUT SPLIT BY NONINVARiANT $H$-SCALARS

$\Lambda_1$, $\Lambda_2$, $\Lambda_3$, $\Lambda_4$, $\Lambda_5$, $\Lambda_6$

$C_\rho (G) \rightarrow C_\rho (H) \rightarrow Z_\rho (H)[C_\rho (G)]$

$\mathcal{H} \times \mathcal{H} [C_\rho (G), C_\rho (H), Z_\rho (H)/C_\rho (G)]$

$G$ IRREP

$\Psi_{\Gamma\Lambda}$ = $\Gamma\Lambda\psi$

$\text{MULTIPlicITY (EIGENVALUES OF } \mathcal{H} \text{ IMPOSED SYMMETRIES } G\rightarrow H \text{ AND/OR OTHER SCALARS) }$

$G$ IRREP

**EXAMPLES**

IBM (BOSON)

$G = U(6) \& H = SO(3)$

$G \rightarrow \Gamma \rightarrow S_p(3, R)$

$U(6) \rightarrow \text{"CONSERVED"}$

$U(5)$, $O(6)$, $SU(3)$

$G_i \rightarrow \alpha \text{ \"IMPOSED\"}$

$O(5)$

$\text{SU(3)}$

$\text{SO(3)} \rightarrow \text{"EXACT"}$

MCM (FERMION)

$G = S_p(3, R) \& H = SO(3)$

$\text{SU(3)}$

$\text{SO(3)}$
GROUP GEOMETRY

\[ G \quad \text{"CONSERVED"} \]
\[ G_i \quad \text{"IMPOSED"} \]
\[ H \quad \text{"EXACT"} \]

NONINVARIANCE GROUPS (DYNAMICAL SYMMETRIES)

\[ \mathcal{H} = \mathcal{H} \quad (G \text{ GENERATORS; } H \text{ SCALARS}) \]

INVARIANTS OF \( G \)

\[ \alpha \]

INVARIANTS OF \( H \)

\[ \alpha \]

NONINVARIANT \( H \) SCALARS (INVARIANTS OF \( G_i \) + OTHERS)

\[ \Psi [ \frac{1}{16} \text{E}^{-} \]

\[ \frac{5}{2} - \frac{1}{2} \text{ CORE} \]

\[ \frac{1}{16} \text{VALENCE} \]

SAMPLE COUPLING SCHEME

\[ \frac{2}{3} \text{Mg} \]

---

ENERGY (M.e.V.)

14
12
10
8
6
4
2
0

EXPERIMENT

THEORY
SAMPLE MODEL SPACE
(RARE EARTH/PROTONS)

"REAL"  "NORMAL"  "ABNORMAL"
gds-shell  parity part  parity part

\[ s_{\frac{1}{2}} \rightarrow \tilde{p}_{\frac{1}{2}} \]

\[ d_{\frac{3}{2}} \rightarrow \tilde{p}_{\frac{3}{2}} + h_{\frac{1}{2}} \]

\[ h_{\frac{1}{2}} \rightarrow \tilde{f}_{\frac{3}{2}} \]

\[ d_{\frac{5}{2}} \rightarrow \tilde{f}_{\frac{5}{2}} \]

CORE

\[ \gamma_{\frac{1}{2}} \rightarrow \tilde{\gamma}_{\frac{1}{2}} \]

"PSEUDO"  "INTRUDER"

\[ g_{\frac{7}{2}} \rightarrow \tilde{\gamma}_{\frac{7}{2}} \]

\[ g_{\frac{9}{2}} \rightarrow \tilde{\gamma}_{\frac{9}{2}} \]

\[ s_{\frac{1}{2}} \rightarrow \tilde{p}_{\frac{1}{2}} \]

SAMPLE COUPLING SCHEME

\[ \Psi \left[ \begin{array}{c}
164 \\
68 \\
50
\end{array} \right] _{96}
\left. \begin{array}{c}
- 82 \\
18 \\
14
\end{array} \right] _{\text{CORE}}
\left. \begin{array}{c}
\text{VALENCE}
\end{array} \right] 
\]

\[ = \Psi \left\{ \left[ (\tilde{r}^\dagger s)^{10} (\tilde{g}^\dagger d)^{10} \right] \times \left[ (h_{\frac{1}{2}})^{10} (i_{\frac{3}{2}})^{10} \right] \right\} \]

NORMAL

PARITY

\[ \downarrow \]

SU(3)

QUASISPIN

\[ J_A \times J_A \rightarrow J_A \]

(\( \ldots \), \( V \leq 2 \))

COUPLING

\[ (\lambda \mu) J_M \]

\[ x \]

\[ v J_A \]

\[ J \quad (\text{FINAL}) \]
### Table 1: Energy Levels in $^{164}$Er

<table>
<thead>
<tr>
<th>Energy (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
</tr>
<tr>
<td>E1</td>
</tr>
</tbody>
</table>

### Diagram 1: Beta Emission Probabilities

- **Ground Band**
  - $B(E2; I_i \rightarrow I_i - 2)$
- **Gamma Band**
  - $B(E2; I_i \rightarrow I_i - 2)$
- **Ground-Gamma**
  - $B(E2; I_g \rightarrow I_y)$

The diagrams illustrate the beta emission probabilities for different bands in $^{164}$Er, with the $B(E2)$ values plotted against the angular momentum $I_i$. The data points are compared to theoretical predictions (TH(3), TH(4)) and experimental results (EXP).
Instituciones patrocinadoras de este evento.
Instituto de Física, U.N.A.M.
Consejo nacional de ciencia y tecnología.

Editores de estas notas:
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