Shell Model Scheme for Quadrupole Phenomena in Heavy Deformed Nuclei

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ABSTRACT: The microscopic collective model, which extends the seminal work of Elliott on the SU(3) model for describing the structure of light deformed nuclei to include multiple 2ℏω shell-model excitations of the monopole (l=0) and quadrupole (l=2) type, is shown to be a logical extension of the pseudo SU(3) scheme for heavy deformed nuclei. The main advantage of this extended theory is that E2 transition strengths, intraband as well as interband, can be reproduced without introducing effective charges. Results for both 24Mg and 168Er are presented. The symmetry algebra of this extended theory is that of the symplectic group Sp(3,R) which contains SU(3) as a subgroup. It is therefore also called the symplectic model and in the case of the pseudo SU(3) extension, the pseudo-symplectic model. Basic premises of the theory are reviewed.

1. INTRODUCTION

The many-particle shell model, which works well for describing the properties of light nuclei, does not enjoy the same degree of success for heavier systems. The problems are obvious and many: huge model spaces, unknown interactions, inadequate computer power, etc. Solutions offered range from applications of the Cranked Strutinsky Method when rotational correlations dominate1, to Hartree-Fock-Bogoliubov studies to accommodate pairing correlations2, to microscopic calculations using the MONSTER and VAMPIR codes3, to use of the more intuitive but less fundamental Dynamic Deformation Model4. An alternative to these approaches is the algebraic models that capitalize on approximate symmetries in nuclei and use this to reduce the shell-model space to manageable size. Here the list extends from the Generalized Collective Model that describes the dynamics in terms of a many (l=2) boson theory [U(5) → SO(3)]5, to wholly algebraic theories like the popular Interacting Boson Model that develops the dynamics of low-lying states in terms of "s" and "d" boson pair creation and annihilation operators [U(6) → G → SO(3)] with the intermediate group G being either U(5), O(6) or SU(3) for the vibrational, gamma unstable, or rotational limit of the theory, respectively6, the Fermion Dynamical Symmetry Model that

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takes the Pauli Exclusion Principle in account correctly while attempting to
duplicate the simplicity inherent in the "s" and "d" boson pair approach7, and the pseudo SU(3) model and its symplectic extension, the Pseudo-
Symplectic Model, that is considered in this paper8.

In recalling the history of the pseudo coupling schemes [SU(2) as well as
SU(3)] it is important to go back to the seminal work of Elliott on the SU(3)
Model9. This theory steps away from the single-particle shell-model picture
by recognizing and reasserting the importance of the quadrupole-quadrupole
interaction (QQ) to the development of rotational collectivity in nuclei. The
main results can be appreciated by noting that the three angular momentum
operators (Lα, α = -1, 0, +1) and five quadrupole operators (Qμ, μ = -2, -1, 0,
+1, +2), when the latter set is constrained to act within a single major shell,
generate SU(3) and that QQ = 4C2 - 3L2 where C2 is the second order SU(3)
invariant which has the eigenvalue λ2+μ2+3(λ+μ) in the (λ,μ) irreducible
representation (irrep) and L2 is the square of the angular momentum with
eigenvalue L(L+1). This means that whenever QQ dominates both the one-
body and residual two-body interactions, as it most certainly does in strongly
deformed nuclei, calculated low-lying states are well represented by leading
irreps of SU(3). Because the expectation value of QQ is proportional to the
deformation squared, this organization of the shell-model space into SU(3)
irreps allows the basis to be ordered so the most deformed configurations
lie lowest and the least deformed highest.

The pseudo-spin concept was introduced into nuclear theory by two
groups working independently: Hecht and his student Adler10 and Arima,
Harvey and Shimizu11. The physics of the scheme is simply a realization
that in heavy nuclei the spherical shell-model orbitals with j = l+1/2 and j
= (l+2)-1/2 lie very close in energy and can therefore be organized into
pseudo spin-orbit doublets with \( \tilde{l} = l+1 \) or \( \tilde{l} = l \) and \( \tilde{s} = 1/2 \) where \( \tilde{j} = \tilde{l} + \tilde{s} = l + s = j \). A
less obvious but even more important result is that because of the large spin-
orbit (ls) splitting in heavy nuclei the shell-model space can be reorganized
in pseudo shells of an oscillator with one less quanta per shell, \( \tilde{n} = n - 1 \). This
leads to the pseudo SU(3) model for heavy deformed nuclei much like the
SU(3) of Elliott for light nuclei. The logic of the pseudo coupling scheme is
reviewed below. As for the Elliott SU(3) Model, the full shell-model space
can again be organized so the most deformed configurations lie lowest and
the least deformed highest, a feature that is essential to understanding the
microscopic underpinnings of rotations in heavy deformed nuclei.

The microscopic collective model, which is also known as the symplectic
shell model, extends the SU(3) scheme to include multiple 2ℏω shell-model
excitations of the monopole \((l=0)\) and quadrupole \((l=2)\) type\(^{12}\). This is a necessary extension if E2 transition strengths, intraband and interband, are to be reproduced without introducing effective charges. An essential point of the theory is that the action of the deformation inducing \(Q\cdot Q\) interaction is not restricted to a single shell of the oscillator. Specifically, \(Q \sim r^2 Y_2(\theta, \phi)\) has nonvanishing matrix elements between major oscillator shells that differ by two quanta, \(n' = n, n \geq 2\). So to realize the full effect of the \(Q\cdot Q\) interaction it is necessary to include vertical \((n\hbar \omega, n = 2, 4, \ldots)\) excitations in addition to the traditional horizontal \((0\hbar \omega)\) ones. The underlying symmetry of the Hamiltonian \(H = H_0 + Q\cdot Q\) is the noncompact group (infinite dimensional irreps) \(Sp(3, \mathbb{R})\) that has \(SU(3)\) as its maximal compact (finite dimensional irreps) subgroup. This symplectic enhancement of the \(SU(3)\) model applies equally well to the pseudo \(SU(3)\) model and, as is shown below, allows for a realistic treatment of quadrupole collectivity in heavy deformed nuclei.

2. PSEUDO \(SU(3)\) SCHEME

For heavy nuclei the \(SU(3)\) symmetry of the isotropic oscillator is badly broken by the spin-orbit interaction. In particular, the highest spin state, \(J_{\text{max}} = n\), of each major shell, \([J_{\text{max}} - 1/2, \ldots, 1/2] = [J]\), is pushed down among orbitals of the next lower shell. Whereas this precludes the use of \(SU(3)\) for a description of these nuclei, and therefore a symplectic scheme for incorporating quadrupole coherence through mixing with higher-lying configurations, the assignment of pseudo orbital and pseudo spin labels to the single-particle states has led to an even better realization of the \(SU(3)\) symmetry, the pseudo \(SU(3)\) scheme. This model capitalizes on the fact that among the set of remaining orbitals, \([J_{\text{max}} - 1, J_{\text{max}} - 2, \ldots, 1/2] = [\tilde{J}]\), those with \(J = l + l/2\) and \(J = (l+2) - 1/2\) lie close in energy forming pseudo spin-orbit doublets with \(\tilde{J} = \tilde{l} \pm \tilde{s}\) where \(\tilde{l} = l + 1\) and \(\tilde{s} = 1/2\) with \(\tilde{J} = j\). (A tilde \(\tilde{\cdot}\) is used throughout to denote pseudo quantities.) Moreover, the full set \([\tilde{J}]\), when interpreted in terms of the \(\tilde{l}\) and \(\tilde{s}\), can be identified as members of a pseudo harmonic oscillator shell of one less quanta, \(\tilde{n} = n - 1\). The corresponding pseudo \(SU(3)\) symmetry of this pseudo oscillator has been used in a number of different applications: determining decoupling parameters for certain rare earth and actinide nuclei\(^6\), predicting alpha-particle transfer strengths for certain Ni and Zn isotopes\(^{13}\), studying backbending in \(^{126}\)Ba\(^{14}\), examining forking in the \(^{68}\)Ge nucleus\(^{15}\), looking into the role of \(Q_x\cdot Q_y\) and the strong coupling limit of the model\(^{16}\), explaining systematics of unique parity spin sequences in deformed odd-A
nuclei\textsuperscript{17}, as well as determining E2, M1 & M3 strengths for certain rare earth and actinide nuclei\textsuperscript{18}.

The logic behind the pseudo SU(3) scheme can be understood in another more fundamental way. Specifically, the "normal" many-particle hamiltonian has the form

\[ H = H_0 + C \sum_i l_i^2 s_i + D \sum_i l_i^2 + V_r, \]
where \( V_r = \frac{1}{2} \Omega Q + 2P + \ldots \) \hspace{1cm} (1)

whereas for its "pseudo" counterpart the terms are rearranged as

\[ H = \tilde{H}_0 - \frac{1}{2} \bar{Q} \bar{Q} + \bar{V}_r, \]
where \( \bar{V}_r = \bar{C} \sum_i l_i^2 s_i + \bar{D} \sum_i l_i^2 + 2\bar{P} + \ldots \). \hspace{1cm} (2)

This rewriting of the hamiltonian would be a meaningless exercise if it were not for the fact that for form (2), \( \bar{C} = 0 \). Furthermore, it can be shown that \( D = \bar{D}, \chi = \bar{\chi} \) and \( d = \bar{d} \) and for the operators, \( \bar{Q} = Q \) and \( \bar{P} = P \). This has far-reaching consequences because in (1) it is mainly the \( l s \) interaction that breaks the SU(3) symmetry of the oscillator and in (2) this \( l s \) interaction has, for the most part, been pulled into the \( \tilde{H}_0 \) which preserves the \( \tilde{S}U(3) \) symmetry. Furthermore, the \( \bar{Q} \bar{Q} \) interaction splits but does not break the \( \tilde{S}U(3) \) symmetry. The principal symmetry breaking interaction in (2) is the residual \( \tilde{l}^2 \) interaction and even this is relatively mild because in general, unlike the \( l s \) term which mixes space and spin degrees of freedom, \( \tilde{l}^2 \) conserves spatial symmetry. So the residual interaction \( \bar{V}_r \) in (2) is truly small whereas the \( V_r \) in (1) is not. Pairing and the other terms that are represented by the \( (+ \ldots) \) factor in both (1) and (2), though important for providing a microscopic interpretation of such phenomena as backbending, tend not to dominate the low-lying dynamics.

As an example of the interplay between the \( l^2 \) and \( \tilde{l}^2 \) interactions, the intensity of \( \tilde{S}U(3) \) representations in calculated \( J^\pi = 0^+ \) states for the model hamiltonian \( \bar{H}_0 = \tilde{H}_0 - (1/2)\bar{Q} (4\bar{C}_2 - 3\bar{L}^2) + \bar{D} \sum_i \tilde{l}_i^2 \) in the four-particle space \( (\tilde{d}s)^4[I=\frac{1}{2}] \) with \( \tilde{L}^2 = [8,0], [4,2], [0,4], \) and \( [2,0] \) is shown in Figure 1. The parameter \( \tilde{D} \) was set at 0.2 (Mev), which reproduces the real \( ds \)-shell splitting generated by \( \bar{D} \), while \( \bar{X} \) was given values between 0 and 0.1 (Mev). These results for \( ^{208}\text{Ne} \) with no spin-orbit splitting simulates an application of the pseudo SU(3) theory to deformed nuclei of the rare earth and actinide regions. A full breakdown of the intensity is given for the \( 0^+ \) yrast state.
Calculated intensities of O\(^+\) eigenstates in the configuration \([ds]_4^f f = 4\) with \((\lambda, \mu)'s = (8, 0), (4, 2), (0, 4) & (2, 0)\) for the reduced many-body hamiltonian \(H = -2\gamma C_2 + D\Sigma \tilde{n}_1^2\). The parameter D was set at 0.2 (Mev), and \(\chi\) allowed to range between 0.0 and 0.1 (Mev). Values for \(\chi\) in the range 0.05 to 0.07 are representative of normal ds-shell and pseudo-shell applications.

while results for the main \(\tilde{S}\tilde{U}(3)\) component only are shown for each of the remaining states, \(O^\alpha_s, \alpha = 2, 3, & 4\). Note that the \(\tilde{S}\tilde{U}(3)\) symmetry breaking decreases very sharply, particularly for the yrast state, as the strength of the deformation inducing \(\tilde{O}\tilde{O}\) interaction increases. A realistic value for \(\tilde{X}\) in this case is about 0.06. Even at half this value, the yrast state is more than 80% pure \((\lambda, \mu) = (8, 0)\). Based on these results, yrast states for normally deformed configurations of heavy nuclei are expected to be dominated by the leading pseudo SU(3) symmetry. Since the expectation value of \(\tilde{C}_2\) is proportional to the square of the deformation, the dominance of the \(\tilde{O}\tilde{O}\)
interaction over the $T^2$ term should be even greater for superdeformed configurations and as a consequence they are expected to be even better $\tilde{SU}(3)$ nuclei than their normally deformed counterparts.

For heavy deformed nuclei of the rare earth and actinide regions there are two open shells, one for protons ($\pi$) and one for neutrons ($\nu$) with each comprised of a set ($J$) of normal parity orbitals and the companion unique or abnormal parity intruder level with $J = J_{\text{max}} + 1$ from the shell directly above. The Hamiltonian for this direct-product system has the form,

$$\mathbf{H} = \mathbf{H}^\pi + \mathbf{H}^\nu + \mathbf{V}_{n},$$

where the $\mathbf{H}^\pi$'s are given by (2) and the $\mathbf{H}^\nu$'s by an appropriate form for the unique parity space, for example, a surface-delta interaction. Here the "n" and "u" superscripts are used to distinguish the "normal" and "unique" parity subspaces, respectively. For applications of the theory, the normal parity space is partitioned into irreps of pseudo SU(3) while a seniority coupling scheme is used to describe the companion unique parity configurations. If the residual proton-neutron interactions, $\mathbf{V}^{\pi}$ and $\mathbf{V}^{\nu}$, are predominantly of the quadrupole-quadrupole type, then yrast states can be approximated by strong coupled pseudo SU(3) wavefunctions coupled to seniority zero unique parity configurations.

$$\mathbf{|\psi^L\rangle} = \{[\mathbf{|n^\pi\rangle}|_{L^\pi}]\bar{\mathbf{\phi}}_{\pi}(\lambda, \mu, \nu, \lambda, \mu, \nu, \lambda, \mu, \nu) \times [\mathbf{|n^\nu\rangle}|_{L^\nu}]\bar{\mathbf{\phi}}_{\nu}(\lambda, \mu, \nu, \lambda, \mu, \nu, \lambda, \mu, \nu)\}^{\otimes L = L^L} \times \{[\mathbf{|n^\beta\rangle}|_{L^\beta}]\times [\mathbf{|n^\gamma\rangle}|_{L^\gamma}]\}^{L^L}.$$  

The spin degrees of freedom are suppressed as it is assumed that there are an even number of particles coupled to spin zero or pseudo-spin zero, as appropriate, in each of the subspaces. The symbols $\alpha$ and $\beta$ are multiplicity labels for the reductions $\bar{U}(N) \rightarrow \tilde{SU}(3)$ and $\text{Sp}(2J+1) \rightarrow \text{SO}(3)$, respectively.

3. TRIAXIAL QUANTUM ROTOR

The Hamiltonian of the triaxial quantum rotor is given by

$$\mathbf{H}_{\text{ROT}} = A_1 I_1^2 + A_2 I_2^2 + A_3 I_3^2,$$

where $I_\alpha$ ($\alpha = 1, 2,$ and $3$) is the projection of the total angular momentum of the rotor on the $\alpha$-th body-fixed symmetry axis and $A_\alpha$ is the corresponding inertia parameter. This familiar form for the rotor can be rewritten in a frame-independent representation by introducing three special scalars.\textsuperscript{19}
\[ L^2 = \sum_{\alpha} l_{\alpha} l_{\alpha} = \sum_{\alpha} l_{\alpha}^2. \]
\[ X_S = \sum_{\alpha, \beta} L_{\alpha} Q_{\alpha, \beta} l_{\beta} = \sum_{\alpha} \lambda_{\alpha} l_{\alpha}^2. \]
\[ X_S = \sum_{\alpha, \beta} L_{\alpha} Q_{\alpha, \beta} l_{\beta} = \sum_{\alpha} \lambda_{\alpha} l_{\alpha}^2. \]

The \( L_{\alpha} \) and \( Q_{\alpha, \beta} \) in (6) are cartesian forms of the total angular momentum and collective quadrupole operators, respectively. The "c" for collective is appended to the \( Q \) to explicitly denote that it is the \( r^2Y_2(\theta, \phi) \) operator which has nonvanishing matrix elements between major shells, \( n' = n, n \pm 2 \). The last expression given for each is the form that these operators take in the body-fixed, principal-axis system: \( \langle Q_{\alpha, \beta} \rangle^{BF/PA} = \lambda_{\alpha} \delta_{\alpha, \beta} \). This result presumes that for the rotor geometry the eigenvalues of the quadrupole operator are sharp. These equations can be inverted to yield an expression for the \( l_{\alpha}^2 \) in terms of \( L^2 \) and the \( X_S \)s:

\[ l_{\alpha}^2 = (l_1 \lambda_2 \lambda_3) L^2 + (l_2 \lambda_1 X_S + (l_3 \alpha X_S)) / D_{\alpha} \]
where \( D_{\alpha} = 2 \lambda_{\alpha}^2 + l_1 \lambda_2 \lambda_3 . \) 

(7)

Substituting this result for the \( l_{\alpha}^2 \) into (5) yields

\[ H_{\text{ROT}} = aL^2 + bX_S + cX_S, \]

(8)

where the parameters \( a, b \) and \( c \) depend on the inertia parameters and the eigenvalue of \( Q_{\alpha} \),

\[ a = \sum_{\alpha} a_{\alpha} A_{\alpha}, \quad a_{\alpha} = \lambda_1 \lambda_2 \lambda_3 / D_{\alpha}; \]
\[ b = \sum_{\alpha} b_{\alpha} A_{\alpha}, \quad b_{\alpha} = \lambda_{\alpha}^2 / D_{\alpha}; \]
\[ c = \sum_{\alpha} c_{\alpha} A_{\alpha}, \quad c_{\alpha} = \lambda_{\alpha} / D_{\alpha}; \]

(9)

The eigenvalues and eigenstates of hamiltonians (5) and (8) with parameters related through (9) are the same.

The reason for casting the rotor hamiltonian into a frame-independent form is so its shell-model image can be readily identified. The many-body expressions for the \( L_{\alpha} \) and \( Q_{\alpha, \beta} \) operators are simply their single-particle forms summed over all particles.
\[ L_\alpha = \sum_i l_{\alpha}(i) \quad \text{and} \quad Q^c_{\alpha,\beta} = \sum_i q^c_{\alpha,\beta}(i). \] (10)

It would seem from this that (8), with the \( L_\alpha \) and \( Q^c_{\alpha,\beta} \) operators interpreted as in (10), is the sought after shell-model image of the rotor hamiltonian. However, this interpretation ignores the underlying shell structure and the fermion character of the many-body system. In particular, it is important to note that whereas the \( L_\alpha \) have non-vanishing matrix elements only within a major oscillator shell, the \( Q^c_{\alpha,\beta} \) couple shells differing by two quanta, \( n' = n \pm 2 \). Indeed, the \( n' = n \pm 2 \) off-diagonal matrix elements of the \( Q^c_{\alpha,\beta} \) are about equal in size to the \( n' = n \) diagonal ones. It follows from this that operators like \( Q\bar{Q} \) and the \( X_{\alpha} \)'s, even if only used as residual interactions, can actually destroy the shell structure. One way around this dilemma is to simply set those matrix elements of \( Q^c \) that couple to different major oscillator shells to zero. This transforms the collective model quadrupole operators, \( Q^c_{\alpha,\beta} \), into their algebraic ('a') counterparts, \( Q^a_{\alpha,\beta} \). Elliott noted that these \( Q^a_{\alpha,\beta} \) operators along with the \( L_\alpha \)'s generate \( \text{SU}(3) \), the symmetry algebra of the isotropic harmonic oscillator hamiltonian. It has been shown by numerous examples that this substitution does indeed yield a shell-model hamiltonian that reproduces rotor results and therefore observed rotational phenomena in nuclei\(^{20}\).

\[
H_{\text{SU3}} = H_0 + aL^2 + bX^2 + cX^3. \] (11)

The correspondence extends beyond spectra to electromagnetic transition rates, see Figure 2. In fact, the theory shows that the shell model has all four rotor symmetry types, \( A \) and \( B_\alpha \) with \( \alpha = 1, 2, \& 3 \), of the so-called Vierergruppe (D\(_2\)), not just \( A \) as is traditionally accepted for a collective model description of even-even nuclei.

Since the parameters \( a, b \) and \( c \) in (11) are given by (9), it is necessary to have shell-model values for the \( \lambda_\alpha \)'s. This can be achieved by requiring a correspondence between invariants of the rotor and shell-model theories. Since \( \text{SU}(3) \) is a rank two group, it has two invariants: \( C_2 \) with eigenvalue \( \lambda^2 + \lambda \mu + \mu^2 + 3(\lambda + \mu) \) and \( C_3 \) with eigenvalue \( (\lambda - \mu)(\lambda - 2\mu + 3)(2\lambda + \mu + 3) \) where \( \lambda \) and \( \mu \) are the usual \( \text{SU}(3) \) irrep labels, that is, \( \lambda + \mu \) and \( \mu \) specify the number of boxes in the first and second rows, respectively, of a Young pattern labelling of the irreps. Note that \( C_2 \) is of degree two in the generators of \( \text{SU}(3) \) while \( C_3 \) is of degree three. The symmetry group of the rotor, which is \( T_5 \times \text{SO}(3) \), also has two invariants: traces of the square, \( \text{tr}[Q^2] \) which has eigenvalue
FIGURE 2

A comparison of rotor and SU(3) results for the gamma band to ground band transition ($^{2}\rightarrow 0_1$) in the leading irrep ($\lambda,\mu = (30,8)$) of $^{168}$Er for three values of the rotor asymmetry parameter: $\kappa = -1$ (prolate), $\kappa = 0$ (asymmetric), and $\kappa = +1$ (oblate).

$\lambda_1^2 + \lambda_2^2 + \lambda_3^2$, and the cube, $\text{tr}[Q^3]$ with eigenvalue $\lambda_1\lambda_2\lambda_3$, of the collective quadrupole matrix. The requirement of a linear correspondence between these two sets of invariants leads to the following relations\(^\text{21}\).

\[
\begin{align*}
\lambda_1 &= -(\lambda - \mu)/3, \\
\lambda_2 &= -(\lambda + 2\lambda + 3)/3, \\
\lambda_3 &= (2\lambda + \mu + 3)/3.
\end{align*}
\]

This correspondence sets up a direct relationship between the $(\beta,\gamma)$ shape variables of the collective model and the $(\lambda,\mu)$ irrep labels of SU(3).
\[ \beta^2 = \frac{4\pi}{5}(\lambda^2) - \frac{2(\lambda^2 + \lambda \mu + \mu^2 + 3\lambda + 3\mu + 3)}{2\lambda + \mu + 3}, \]

\[ \gamma = \tan^{-1}\left(\frac{\sqrt{3}(\mu + 1)}{2\lambda + \mu + 3}\right). \]

This in turn means, for example, that the potential energy surface concept can be given a shell-model interpretation\(^{22}\). A point to note is that the very simplest SU(3) theory reproduces the full dynamics inherent in a triaxial quantum rotor description of nuclear structure.

4. PSEUDO-SYMPELCTIC EXTENSION

As suggested in the introduction, the pseudo-symplectic scheme extends the pseudo SU(3) picture by allowing for inter-shell \(2\hbar\omega\) excitations of the monopole \((l=0)\) and quadrupole \((l=2)\) type just as the symplectic scheme does for the deformed ds-shell nuclei\(^{12}\). In this case, however, there are two open shells, one for protons \((\pi)\) and one for neutrons \((v)\) with each consisting of a set \((f)\) of normal parity orbitals and the abnormal or unique parity intruder level \(j = j_{\text{max}} + 1\) from the shell above. The \(0\hbar\omega\) space is first partitioned as indicated above for the pseudo SU(3) scheme into normal parity parts and within these into irreps of pseudo SU(3), and unique parity parts using seniority. Now since the proton-neutron interaction is known to be predominantly of the quadrupole-quadrupole type, the yrast states can be approximated by strong coupled pseudo SU(3) wavefunctions coupled to seniority zero unique parity configurations. Since multiple \(2\hbar\omega\) excitations are included, however, separating the full space into normal \((A_n\text{ nucleons})\) and unique \((A-A_n\text{ nucleons})\) parity parts might appear to be artificial. In particular, if normal parity \(2\hbar\omega\) excitations are allowed, should not unique parity \(2\hbar\omega\) excitations also be considered as well as mixing between these two modes? The answer to this important question lies in an understanding of the role the \(Q\bar{Q}\) interaction plays in dictating the dynamics of a deformed system. First note that the quadrupole-quadrupole interaction preferentially pulls in the most deformed of the excited configurations, and secondly that its matrix elements are strongly enhanced in the normal parity space due to the underlying oscillator symmetry but not in single \(j\)-shell configurations like those of the unique parity space. Because the matrix elements of the \(Q\bar{Q}\) interaction between normal parity state are typically much larger than those between normal and unique or unique and unique parity configurations it is valid to consider only multiple \(2\hbar\omega\) excitations in the normal parity and not in the unique parity space. This argument clearly breaks down for high-
lying states where backbending and hence pair alignment plays an important role, so the theory as presented is for the structure of low-lying states only.

The model Hamiltonian is taken to be the sum of the proton and neutron pseudo harmonic oscillator Hamiltonians plus a real quadrupole-quadrupole interaction. It is assumed that the protons and neutrons in the abnormal parity levels couple to seniority zero configurations and only contribute an additive constant to the binding energy which is suppressed in what follows. The general \( \Omega \omega \) picture is laid out above, see (3) & (4). To ensure that the underlying shell structure is preserved, a special operator \( \{ Q^c \Omega^c \Omega^c \}^{TE} \) that removes the trace of \( Q^c \Omega^c \) in all major shells of the oscillator is included\(^{23}\),

\[
H = \hbar \omega N - \frac{1}{2} \lambda \left[ \left( Q^c \Omega^c \right)^{TE} - \{ Q^c \Omega^c \}^{TE} \right] + H_{\text{ROT}}.
\]

where \( \tilde{N} = \tilde{N}_\pi + \tilde{N}_\nu \) and \( Q^c = Q^c_\pi + Q^c_\nu \).

To reiterate, the superscript "c" that is appended to \( \Omega \) denotes that it is the "collective" quadrupole operator that enters, \( Q^c = \sqrt{16 \pi \hbar^2} r^2 Y_2(\hat{r}) \), and not the "algebraic" one of Elliott, \( \Omega^a = \sqrt{16 \pi \hbar^2} \left[ \frac{p^2}{2} Y_2(\hat{p}) + r^2 Y_2(\hat{r}) \right] / 2 \). While the matrix elements of \( Q^c \) and \( \Omega^a \) are identical within an oscillator shell, \( Q^c \) has nonzero matrix elements between shells with quanta \( n' = n \pm 2 \) whereas \( \Omega^a \) does not.

The major shell separation energy, \( \hbar \omega \), is set by the simple empirical rule \( 41 \Lambda^{-1/3} \text{[Mev]} \). A small residual interaction, \( H_{\text{ROT}} \), like (8) is included in (14) to fix the inertia parameters and band splitting of the low-lying eigenstates.

The real mass quadrupole operators \( Q^c_\pi \) and \( Q^c_\nu \) can be expressed in terms of the generators of symplectic algebras for the protons and neutrons, that is, \( Q^c_\pi = Q^c_\alpha + \sqrt{6} / 2 (B_{2\alpha}^2 + B_{2\alpha}) \) with \( \alpha = \pi \) & \( \nu \). The operators \( B_{nm} (B_{nm}^{\dagger}) \) with \( l = 0 \) & 2 are the \( \hbar \omega \) raising (lowering) generators of the \( \text{SU}(3) \) algebras and the \( Q^c_\alpha \) are quadrupole generators of the \( \text{SU}_d(3) \) subalgebras. Through a second quantization formulation, these operators \( O = \Omega^a, B^+, B \) can be expanded in terms of pseudo SU(3) tensors,

\[
O = \kappa \tilde{O} + \ldots, \tag{15}
\]

where \( \tilde{O} \) has the same tensorial character as \( O \). The coefficient \( \kappa \) in (15) is always greater than unity, ranging from a high of about 1.4 for \( O = \Omega^a \) (\( \bar{n} = 0 \)) to a low of about 1.1 for \( O = B_{2m} \) (\( \bar{n} = 6 \)). The other terms in the series have a different tensorial character and an expansion coefficient that is typically less than ten percent of the leading term. For pseudo SU(3) applications these correction terms were found to yield, in aggregate, less than a one percent change in calculated results for eigenenergies and electromagnetic transition rates. In what follows they will therefore be ignored.
If higher order terms in the series expansion of $Q^C$ are neglected, the
hamiltonian, (12), can be rewritten as

$$H = \hbar \omega \tilde{N} - \frac{1}{2} \lambda \left[ \tilde{Q}^C \tilde{Q}^C - \langle \tilde{Q}^C \tilde{Q}^C \rangle \right] + a\tilde{L}^2 + b\tilde{X}_3^2 + c\tilde{X}_4^2. \quad (16)$$

In this expression the last three terms are an explicit form for $H_{\text{ROT}}$. The
operator $\tilde{L}$ is the angular momentum while the $\tilde{X}_3^a$ and $\tilde{X}_4^a$ terms are third
and fourth order rotational scalars: $\tilde{X}_3^a = (Lx \tilde{Q}^+ \tilde{x}^a)$ and $\tilde{X}_4^a = [(Lx \tilde{Q}^+ \tilde{x}^a \times \tilde{x}^a \times \tilde{L})]$. This
form for the residual interaction, see (3), allows the moment of inertia and band splitting of the low-lying states to be adjusted without otherwise affecting the dynamics. The matrix elements of this $H_{\text{ROT}}$ term vanish in $0^+$ states. Actually, a boson expansion of the pseudo-symplectic generators
leads to a more tractable version of the theory. In this case

$$\tilde{B}^a_{\text{Im}} \to \sqrt{4/3} \tilde{N}_{\text{s}} \tilde{B}^a_{\text{Im}},$$

$$\tilde{N} \to \tilde{N}_{\text{s}} + 2\tilde{N}_{\text{b}},$$

$$\tilde{L}_{\text{m}} \to \tilde{L}_{\text{m}}^{\text{s}} + \tilde{L}_{\text{m}}^{\text{b}},$$

$$\tilde{Q}^a_{\text{m}} \to \tilde{Q}^a_{\text{s}} + \tilde{Q}^a_{\text{b}}, \quad (17)$$

where $l=0,2$ and the $\tilde{B}^a_{\text{Im}}$'s satisfy boson commutation relations. The "s" and "b" indices are used to distinguish between the $Q_{\text{b}}$ shell-model and boson parts, respectively, of the $N$, $L$ and $Q^a$ operators. The procedure used to obtain the corresponding expansion of the hamiltonian is straightforward.

First each operator is expressed in normal order form, that is, the $2\hbar \omega$ raising (lowering) generators are arranged to be in the leftmost (rightmost) position and then the substitutions (17) are made. This yields

$$H = \hbar \omega \tilde{N}_{\text{b}} - \frac{1}{2} \lambda \left[ \tilde{Q}^a \tilde{Q}^a + 2\sqrt{2} \tilde{N}_{\text{s}} \left( b^+ \tilde{Q}^a + \tilde{Q}^a b \right) - 20\sqrt{2} \tilde{N}_{\text{s}} \left( b_{00}^+ + b_{00} \right) \right] + 2\tilde{N}_{\text{s}} \left( b^+ \tilde{b} + \tilde{b}^+ b \right) + 4\tilde{N}_{\text{s}} \tilde{N}_{\text{d}} - \frac{120}{7} \tilde{N}_{\text{b}} - \frac{2}{5} C_2 (\tilde{a}_{\text{s}}, \mu_{\text{s}})$$

$$- \frac{10}{3} \tilde{N}_{\text{s}} \tilde{N}_{\text{b}} - \frac{25}{7} \tilde{N}_{\text{b}}^2 + a\tilde{L}^2 + b\tilde{X}_3^2 + c\tilde{X}_4^2, \quad (18)$$

where $C_2$ is the second order Casimir operator of the pseudo SU(3) algebra, $\tilde{N}_{\text{d}}$ counts the number of $l = 2$ bosons, the $(Q^C \tilde{Q}^C \text{TE})$ term is included, and all constant terms have been dropped. This hamiltonian can be diagonalized in basis states classified by the chain of groups:

$$U_{\text{S}(3)} \times U_{\text{b}(6)} \to SU_{\text{S}(3)} \times SU_{\text{b}(3)} \to SU(3) \to SO(3) \to SO(2)$$

$$N_{\text{s}} \quad N_{\text{b}} \quad (\lambda_{\text{s}}, \mu_{\text{s}}) \quad (\lambda_{\text{b}}, \mu_{\text{b}}) \quad p \quad (\lambda_{\mu}) \quad \kappa \quad L \quad M \quad (20)$$
FIGURE 3
Calculated eigenenergies for $^{24}$Mg plotted as a function of the strength $\chi$ of the quadrupole-quadrupole interaction. Observed E2 strengths can be reproduce with $\chi = 0.42$ and no effective charge. Calculated results for the low-lying spectrum and the centroid and width of the resonant mode (23) are indicated on the far right.

The quantum numbers that characterize the irreps of each group are given in the second line of (19). The indices $\kappa$ and $\rho$ are multiplicity labels of the indicated reductions.
The results of calculations for $^{24}\text{Mg}$ and $^{168}\text{Er}$ using this formulation are given in the next set of four figures. Figure 3 is a plot of the excitation spectrum of $^{24}\text{Mg}$ as a function of $\chi$ from zero out to the value $\chi = 0.042$ that is required to reproduce the observed $B(E2, 2f \rightarrow 0f)$ strength. In the calculation, $\hbar\omega = 12.6(\text{Mev})$, and the $(a, b, & c)$ parameters of $H_{\text{ROT}}$ were assigned best fit values for this value of $\chi$, namely (0.14137, 0.042417, & 0.0055368), respectively. This set of parameters also produced the correct $2f \rightarrow 0f$ (gamma-to-ground) and $2f \rightarrow 0f$ (resonant-to-ground) E2 strengths, see Figure 4. Histograms showing vertical mixing in the two

![Figure 4](image)

**FIGURE 4**

Calculated E2 transition strengths for $^{24}\text{Mg}$ plotted as a function of the strength $\chi$ of the real quadrupole-quadrupole interaction. The observed strengths, intraband and interband, are reproduced with $\chi = 0.42$ and no effective charge.
Intensity analysis of the calculated $0_1^+$ ground state (G.S.) and $0_2^+$ resonant state (R.S.) eigenstates for $^{24}\text{Mg}$ are shown for five values of the strength $\chi$ of the real quadrupole-quadrupole interaction. For $\chi = 0.42$, which is the value required to reproduce observed E2 transition strengths without an effective charge, the $0h\omega$ component of the ground state is less than about 70% while the resonant state extends out to $16h\omega$.

The lowest $0^+$ states of $^{24}\text{Mg}$ for five values of $\chi$ are shown in Figure 5. From these it follows that whereas an $8h\omega$ space suffices for a good description of the ground state ($0^+_1$), the resonant mode ($0^+_2$) requires about double this value, $16h\omega$. It is this vertical mixing, which accounts for approximately 25% of the total ground state intensity, that gives rise to the enhanced E2 rates. An interesting number is the ratio of the expectation value of $Q^c Q^c$
in the resonant mode and ground state: $\beta_r/\beta_g = 1.4$. This means that the resonant mode configuration has a deformation that is representative of the 3:2 axis ratio found for some superdeformed structures. Similar calculations have been carried out using the full group theoretical hardware introduced above for selected rare earth and actinide nuclei\(^{26}\). A comparison of experimental and theoretical $B(E2)$ rates for \(^{24}\text{Mg}\) and \(^{168}\text{Er}\) are given in Figure 6. It is important to note that no effective charges were used in either of these calculations.

![Figure 6](image)

**FIGURE 6**

Comparison of calculated and experimental $E2$ rates for the \(^{24}\text{Mg}\) and \(^{168}\text{Er}\) nuclei. The theoretical results [SU(3)] were generated using a boson approximation to the normal and pseudo symplectic models, respectively. Effective charges were not used in the calculations.

Although space does not allow for a full discussion of potential energy surfaces, it seems important to note that the formulation laid down here allows a direct connection to be made between collective and shell-model theories of nuclear structure\(^{22}\). Specifically, a plot of eigenenergies of the hamiltonian versus the expectation of $Q^C Q^C$ in eigenstates of $H$, (16) or its boson counterpart (18), shows that the sharp rise beyond $\beta = \beta_0$ is forced because configurations with $\beta > \beta_0$ only exist in the $n\omega\pi$ space where $n \neq 0$. In contrast, the $\beta < \beta_0$ domain is defined by $\omega\pi\pi$ irreps of lesser deformation.
than the leading one. This picture also suggests a means for gaining a shell-model realization of superdeformation\textsuperscript{27}. Specifically, symplectic bands built on excited particle-hole configurations will be more strongly deformed than those built on 0\hbar\omega configurations. For example, in the \textsuperscript{24}Mg case the leading 10-particle, 2-hole configuration has \((\lambda,\mu) = (14,2)\) as compared to \((\lambda,\mu) = (8,4)\) for the ground state band. The bandhead for this configuration has a deformation that is 1.36 times larger than that of the ground state band: \(\gamma_{\text{C}_2(14,2)}/\gamma_{\text{C}_2(8,4)} = 1.36\). Adding symplectic excitations generated by the \(B^+\) operators to this will yield a superdeformed configuration.

5. CONCLUSION

It seems appropriate at this point to suggest that the rotor be considered an approximation to the shell model rather than the other way around. This follows because: 1) the ROT \(\leftrightarrow\) SU(3) mapping has been established, i.e. any rotor result can now be easily duplicated within a shell-model framework, and 2) the full many-particle shell-model space can be partitioned into representations of SU(3), normal ones for light nuclei and pseudo irreps for strongly deformed heavy systems. Furthermore, because the splitting of pseudo spin-orbit doublets in heavy nuclei is very much smaller than that encountered in the ds-shell, the pseudo SU(3) model is expected to work better than the Elliott SU(3) model. In addition, the symplectic extension of the Elliott SU(3) model, which allows for multiple \(2\hbar\omega\) excitations of the monopole \((l=0)\) and quadrupole \((l=2)\) type, has been shown to apply to the pseudo SU(3) scheme for heavy deformed nuclei. This means, for example, that E2 transition rates, interband as well as intraband, can be reproduced without introducing effective charges.

Topics not discussed in any detail whatsoever, but ones that are of great current interest to the field, include superdeformation and a description of the physics of low-lying configurations in terms of potential energy surfaces. The latter can be given a shell-model interpretation because in the process of establishing the ROT \(\leftrightarrow\) SU(3) mapping a direct connection between the \((\beta,\gamma)\) shape variables and the \((\lambda,\mu)\) irrep labels of SU(3) has been established, see (13). This, in addition to the reported work on quadrupole resonance phenomena, demonstrates the importance of vertical excitations in nuclei. Typically, low-lying eigenstates include as much as 25\% admixtures from higher-lying shells. Whereas horizontal mixing plays a very important role for pairing correlations, the inclusion of vertical excitations is absolutely essential for a complete understanding of rotational motion. For quadrupole resonant modes the amount of vertical mixing is even larger, and contrary to popular belief, not necessarily dominated by \(2\hbar\omega\) shell-model configurations.
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REFERENCES


