Quantum rotor and identical bands in deformed nuclei*

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Received 16 February 1994; accepted 14 May 1994

ABSTRACT. A quantum rotor with intrinsic spin, where the rotation is about the \( L \)-axis instead of the \( J \)-axis, is proposed for the interpretation of identical band and spin alignment phenomena in deformed nuclei. This scheme models a many-particle, shell-model theory for strongly deformed nuclei with the total pseudo-spin of the system decoupled from the rotational motion. The scheme is applied to an analysis to the superdeformed bands in \(^{192}\text{Hg}\) and \(^{194}\text{Hg}\).

RESUMEN. Se propone un rotor cuántico con espín intrínseco, donde la rotación se hace alrededor del eje \( L \), en lugar del eje \( J \), para la interpretación de bandas idénticas y los fenómenos de alineamiento de espines en núcleos deformados. Este esquema es un modelo de la teoría del modelo de capas de muchos cuerpos, apropiado para núcleos muy deformados con el pseudospín total del sistema desacoplado del movimiento de rotación. El esquema se aplica al análisis de las bandas superdeformadas en \(^{192}\text{Hg}\) y \(^{194}\text{Hg}\).

PACS: 21.10.Re; 21.10.Ky; 21.60.Cs

1. INTRODUCTION

Two extensions of classical rigid rotor dynamics are found in quantum systems with non-zero intrinsic spin \( S \) [1, 2]. One (called the \( J \)-rotor in this paper) simply replaces the rotational angular momentum \( L \) by the total angular momentum \( J \), where \( J \) includes \( S \). This simple model, which has been thoroughly studied since the earliest days of quantum mechanics [3] has been used to interpret diverse rotational phenomena in both physics and chemistry. The other scheme — called the \( L \)-rotor in this paper — is less well studied because its physical relevance has heretofore not been fully realized. In this case, the rotation is about the \( L \)-axis rather than the \( J \)-axis; spin degrees of freedom are decoupled from the rotational motion. The \( L \)-rotor picture emerges as a natural limit for strongly deformed rare-earth and actinide nuclei when a many-particle, shell-model coupling scheme with good total pseudo-spin symmetry, which decouples from the rotational motion, is employed [4–7]. Identical bands are interpreted within this framework as pseudo-orbital angular momentum (\( \tilde{L} \)) rotational sequences (\( \tilde{L} \)-rotor series) associated with different pseudo-spin (\( \tilde{S} \)) projections.

* Supported in part by the U.S. National Science Foundation.
2. L-ROTOR MODEL

Consider the simple model of a rotor with core angular momentum \( L \), intrinsic spin \( S \), and total angular momentum \( J = L + S \) (Fig. 1a). The hamiltonian for this system is

\[
H = \sum_\alpha a_\alpha L_\alpha + b L \cdot S + c S^2, \tag{1}
\]

when a self-interaction term generated by \( L \cdot S \) is assumed and where \( a_\alpha \) is the rotational inertial parameter around the intrinsic \( \alpha \)-th axis (equal to \( 1/2I_\alpha \), where \( I_\alpha \) is the corresponding moment of inertia). For a fixed-spin system the \( S^2 \) term is a constant and is maintained for completeness. The second term can be replaced by the \( J^2 \) operator, since \( L \cdot S = \frac{1}{2}(J^2 - L^2 - S^2) \). For the case of an axially symmetric rotor \( (a_1 = a_2 = a \neq a_3) \)
this Hamiltonian reduces to
\[
H = aL^2 + (a_3 - a)L_3^2 + \frac{1}{2} (J^2 - L^2 - S^2) + cS^2. \tag{2}
\]
The energies of this elementary system are given by the simple result
\[
E^{(L,S)} = \left( a - \frac{b}{2} \right) L(L+1) + (a_3 - a)K_L^2 + \frac{b}{2} J(J+1) + \left( c - \frac{b}{2} \right) S(S+1), \tag{3}
\]
where $K_L$ is the eigenvalue of $L_3$. The corresponding eigenstates have the form $|\gamma K_L L S J M\rangle$ where $\gamma$ is a running integer index used to distinguish multiple occurrences of a given $K_L$ combination, and $M$ is the projection of $J$ on the laboratory-fixed $z$-axis. This $LS$-coupled scheme differs from the eigenstates $|\gamma K_S S K J M\rangle$ of the $J$-rotor which have the projections of $K_S$ of $S$ and $K$ of $J$ on the intrinsic symmetric axis of the system as good quantum numbers (Fig. 1b).

When a band label $D = J - L$ is introduced, the $E^{(L,S)}$ of Eq. (3) can be re-written in the form
\[
E^{(L,S)} = aL(L+1) + bLD + (a_3 - a)K_L^2 + \frac{b}{2} D(D+1) + \left( c - \frac{b}{2} \right) S(S+1). \tag{4}
\]
The quantity $D$ introduced in this description ($D = S, S-1, \ldots, -S$) indicates the (spin) alignment of the band relative to the reference ($D = 0$) band. For $S = 1$, there are three bands with alignments $D = 1, 0$ (reference), and $-1$. When $S = \frac{1}{2}$, only two bands exist and they are shifted by half-integer amounts from the reference structure; in nuclear physics these would be for the odd-A neighbor of an even-even parent, assuming the addition of the odd nucleon induces no change in the internal structure of the system. In general there are $2S + 1$ bands with total angular momentum $J = J_{\text{min}}, J_{\text{min}} + 2, \ldots$ when $K_L = 0$ and $J = J_{\text{min}} + 1, J_{\text{min}} + 2, \ldots$ when $K_L \neq 0$. The $J_{\text{min}}$ values are rather complicated functions of $K_L, S, D,$ and $x = \text{mod}(L, 2)$: $J_{\text{min}} = D + x + 2((S - D + 3 - 2x)/4)$ for $K_L = 0$ and $J_{\text{min}} = D + K_L + \max(0, [(S - D - 2K_L + 1)/2])$ for $K_L \neq 0$ where $[w]$ is the greatest integer function. If the coefficient $b$ of $L \cdot S$ is positive (negative), the $D = S$ band lies highest (lowest) while the $D = -S$ band lies lowest (highest) for a given $L$ value. For the special case when $b = 2a$, Eq. (3) reduces to
\[
E^{(L,S)} = aJ(J+1) + (a_3 - a)K_L^2 + (c - a)S(S+1), \tag{5}
\]
which gives the energies of a $J$-rotor with the projection governed by $K_L$ rather than $K$. Since the projection quantum numbers are not measurable, when $b = 2a$ it is impossible to distinguish the $L$-rotor and the $J$-rotor pictures based solely on excitation energies.

Stretched intraband (interband) electric quadrupole ($E2$) transitions in deformed nuclei are strongly enhanced (inhibited). All $E2$ transitions with $\Delta D \neq 0$ must therefore be strongly suppressed if the $L$-rotor picture is to describe nuclear physics phenomena. The
de-excitation energies within a band (assuming $\Delta L = 2$ transitions and a prolate rotor geometry) are given from Eq. (4) as

$$
\Delta E^{(L,D)}(a,b) = E^{(L+2,S)}J+2 - E^{(L,S)}J = 2a(2L + 3) + 2bD,
$$

or for the special $b=2a$ case from Eq. (5),

$$
\Delta E^{J}(a) = E^{(L+2,S)}J+2 - E^{(L,S)}J = 2a(2J + 3).
$$

A plot of intraband $\Delta E^{(L,D)}(a,b)$ versus $L$ values therefore yields lines with identical slopes (4a) but with different intercepts ($6a + 2bD$). The $E2$ selection rules are obtained from the expression

$$
B(E2; J' \rightarrow J) = \frac{1}{2J'+1}|\langle \gamma K_LLSJ|Q^e||\gamma' K'_LL'S'J' \rangle|^2,
$$

where $Q^e$ is the electric quadrupole operator [10] and

$$
\langle \gamma K_LLSJ|Q^e||\gamma' K'_LL'S'J' \rangle = \delta_{SS'}\sqrt{(2J + 1)(2J' + 1)}
\times W(S' JL2; L'J)\langle \gamma K_LL||Q^e||\gamma' K'_LL' \rangle.
$$

The $W$-function in Eq. (7) denotes a Racah recoupling coefficient. For intraband $B(E2)$ transitions, this equation reduces to

$$
B(E2; J' \rightarrow J) = \frac{25}{4\pi}\left|\begin{array}{cc} L & 2 & L' \\ -K_L & 0 & K_L \end{array}\right| W(S' JL2; L'J) Z^2 \beta^{(S)}^2 W.u.
$$

where $Z$ is the atomic number and $\beta^{(S)}$ the usual collective model deformation parameter which may be spin dependent. Representative results are displayed in Fig. 2. Note that as the value of $J$ increases, the intraband $B(E2)$ transition strengths for the same $J$ values for the $D = -1$ and $D = +1$ bands become equal.

3. Pseudo-spin realization

The $L$-rotor picture emerges in the context of a many-particle, shell-model theory whenever space-like and spin-like degrees of freedom decouple. The many-particle, pseudo-space/spin coupling scheme for heavy deformed nuclei is an example [4-6]. In this case, replacing the normal single-particle orbital angular momentum ($l$) and spin ($s$) operators by their pseudo-orbital ($\tilde{l}$) and pseudo-spin ($\tilde{s}$) counterparts transforms the one-body orbit-orbit ($v_{ll}$) and spin-orbit ($v_{ls}$) interactions into their corresponding pseudo forms $\tilde{v}_{ll} = v_{ll}$ and $\tilde{v}_{ls} = 4v_{ll} - v_{ls}$ [7]. (While a whole class of such transformations can be identified, see
\[ B(E2) (Z^2 β^{(0)}_2)^2 \text{ W.u.} \]

\[ B(E2) (Z^2 β^{(1)}_2)^2 \text{ W.u.} \]

Figure 2. a)
Figure 2. $B(E2)$ strengths for a $K_L = 0$ scenario with $b = 2a$: a $S = 0$ $L$-rotor and the corresponding $S = 1$ series. Strength for $L < 16\hbar$ are shown in a) (previous page), while those for $L > 16\hbar$ in b) (this page). Transition strengths less than 0.001$Z^2\beta^2$W.u. are suppressed for clarity.
for example Refs. [8,9], only the latter special one preserves the oscillator structure and therefore admits a many-particle, shell-model theory with known group properties.) As a consequence of the fact that $\tilde{v}_{\lambda \beta} \approx 0$, the many-particle $\hat{S}$ is a good quantum number in such a theory [5]. Furthermore, since this special transformation carries the harmonic oscillator Hamiltonian into a pseudo-oscillator ($H_0 \rightarrow \hat{H}_0 + \hbar \omega$) and the deformation inducing quadrupole-quadrupole interaction ($Q \cdot Q$), which dominates the residual interaction, into its pseudo (quadrupole-quadrupole) counterpart ($Q \cdot Q$) with (at most) small correction terms, the $L$-rotor picture models a many-particle, shell-model theory with a strong deformation inducing residual interaction and good many-particle pseudo-spin symmetry under the replacement $L \rightarrow \hat{L}$ and $S \rightarrow \hat{S}$ (where $\hat{L}$ is the sum of the pseudo-orbital angular momenta of the valence particles and $\hat{S}$ is the corresponding many-particle pseudo-spin) which couple to the total angular momentum $J$. Note that the total many-particle angular momentum $J$ is left invariant under the pseudo-spin transformation. The $L$-rotor picture models a many-particle pseudo-$LS$ coupling scheme.

Valence protons and neutrons in heavy nuclei fill different major oscillator shells. An appropriate scheme for simulating a many-particle, shell-model theory in this case is therefore two interacting $L$-rotors. (The superposition of these two rotors does not violate the Pauli Principle because they refer to different particle types.) The model can assume various forms depending upon whether $\hat{L}_\pi$ of the protons and $\hat{L}_\nu$ of neutrons first couple to their own spins ($J = (\hat{L}_\pi + \hat{S}_\pi) + (\hat{L}_\nu + \hat{S}_\nu) = J_\pi + J_\nu$) or first couple with each other with the spin coupling done last ($J = (\hat{L}_\pi + \hat{L}_\nu) + (\hat{S}_\pi + \hat{S}_\nu) = \hat{L} + \hat{S}$). Since the proton-neutron quadrupole-quadrupole field favors a product configuration displaying the maximum deformation, the second scenario is preferred in nature. In the pseudo-space/spin
picture, this is accomplished through the strong coupling of pseudo-SU(3) representations, \((\hat{\lambda}_\tau \hat{\mu}_\tau) \times (\hat{\lambda}_\nu \hat{\mu}_\nu) \rightarrow (\hat{\lambda} \hat{\mu})\); the pseudo-spins \((\hat{S}_\tau \text{ and } \hat{S}_\nu)\) are coupled to total \(\hat{S}\) in the usual way (see Fig. 3).

In its simplest form the pseudo-space/spin coupling scheme is not a complete shell-model theory because it only takes direct account of nucleons occupying normal parity orbitals. Nucleon pairs distributed in the unique parity intruder orbitals (one for protons and one for neutrons) are treated as spectators which (at most) contribute to the many-particle dynamics in an adiabatic way. This assumption can obviously only be valid for levels well below and high above the backbending region which signals the importance of strong pair alignment phenomena. Similarly, the \(L\)-rotor picture can only be considered applicable in a regime where pair alignment effects do not dominating the dynamics. The normal and unique parity parts of the proton and neutron spaces are then considered to be weakly coupled with nucleons in the unique parity orbitals serving only to renormalize the collective dynamics generated by interactions among the particles in the unique parity orbitals. Though this picture is an over simplification, the scheme has been shown to work reasonably well so long as there is no pair breaking nor pair scattering of nucleon from the unique parity levels to the normal ones or vice versa [11].

4. Superdeformation application

Rotational bands in several deformed nuclei have been found to have identical (within \(\pm 2\%\) for normal deformed nuclei [12] and \(\pm 1\%\) for superdeformed nuclei [13–18]) transition energies. For example, certain superdeformed bands in \(^{194}\text{Hg}\) appear to be nearly identical to a superdeformed band in \(^{192}\text{Hg}\) [13–18]. The many-particle, pseudo-spin scheme can be applied in this case by assigning \(\hat{S} = 0\) to the superdeformed band in \(^{192}\text{Hg}\), \(\hat{S} = 0\) to superdeformed band (1) in \(^{194}\text{Hg}\), and \(\hat{S} = 1\) to superdeformed bands (2) and (3) in \(^{194}\text{Hg}\). The non-zero pseudo-spin is then associated with neutron alignment and different \(\hat{S}\) values have the possibility of distinct deformation parameters: \(\beta^{(0)}[^{192}\text{Hg}], \beta^{(0)}[^{194}\text{Hg}, \text{ band (1)}], \beta^{(1)}[^{194}\text{Hg}, \text{ band (2) and (3)}]\). These band assignments, which assumes a many-particle, pseudo-spin dynamics with the two additional neutrons in \(^{194}\text{Hg}\) as compared with \(^{192}\text{Hg}\) occupying normal parity orbitals, are different from earlier analyses made within the context of a single-particle picture, Refs. [19–21]. Regarding this difference it is indeed unfortunate that the available data, which are limited to the transition energies only, provide no guidance as to whether the extra neutrons are expected to lie in the normal or unique parity orbitals. Assigning the extra neutrons to the normal parity part of the space is consistent with the fact that the experimental values for the transition energies within bands (2) and (3) are almost identical, differing from one another only by a \(h\) shift in the total angular momentum. The deformation parameters \(\beta^{(3)}\) can be determined by fitting to the measured \(B(E2)\) transition strengths, or derived microscopically using the appropriate pseudo-SU(3) configuration. This letter follows the assignment for final total angular momentum \(J_f\) given in the Ref. [16].

The \(\hat{L}\)-rotor picture predicts three \(\hat{D}\)-bands for an \(\hat{S} = 1\) configuration; however, some of the bands may not be separately distinguishable. For example, if \(b = 2a\) the transition energies of the \(\hat{D} = +1\) band match those of the \(\hat{D} = -1\) band identically. Since the
Superdeformed bands for Hg

\( \tilde{E}_L = 10 \)\n\( \tilde{E}_L = 8 \)\n\( \tilde{E}_L = 10 \)\n\( \tilde{E}_L = 8 \)

\( ^{124}_{1} \text{Hg} (2) \)
\( \tilde{D} = \pm 1, -1 \)
\( S = 1 \)

\( ^{194}_{0} \text{Hg} (3) \)
\( \tilde{D} = 0 \)

\( ^{194}_{0} \text{Hg} (1) \)
\( \tilde{D} = 0 \)
\( S = 0 \)

\( ^{192}_{0} \text{Hg} \)
\( \tilde{D} = 0 \)
\( S = 0 \)

**Figure 4.** Transition energies for an \( \tilde{L} \)-rotor system with \( \tilde{D} \)-band assignments. The \( \tilde{L} \)-rotor parameters are \( a = 4.5 \text{ keV} \), \( b = 9.0 \text{ keV} \), and \( k = 92 \text{ keV} \) and \( 74 \text{ keV} \) for \( \tilde{S} = 0 \) and \( \tilde{S} = 1 \), respectively. The top band (heavy lines) for each pair gives the experimental results for superdeformed bands in the identified Hg isotopes while the bottom one (light lines) is the \( \tilde{L} \)-rotor description.

\( B(E2) \) strengths are also equal for large \( J \) values (see Fig. 2), the transition strength for states belonging to the \( \tilde{D} = \pm 1 \) pair appear to be twice as strong as for the \( \tilde{D} = 0 \) band, a result that appears to be in agreement with the experiment.

Equation (5) was fit to each of the four superdeformed Hg bands for \( \tilde{L} \) values in the range \( \tilde{L} = 10-38h \) corresponding to \( E_\gamma = 255-735 \text{ keV} \) where the complete experimental results are available. The bands have nearly the same inertial parameter \( a = 4.5 \text{ keV} \). The \( L \cdot S \) self-interaction strength for \( ^{194}_{0} \text{Hg} \) can be deduced from the energy shift of band (2) compared to the reference band (3), and is \( b = 9.0 \text{ keV} \). Since the kinematic \( (I^{(1)} = L/\omega) \) and dynamic \( (I^{(2)} = (d^2E/d^2L)^{-1}) \) moments of inertia are not precisely equal and the transition energies not equally spaced (as opposed to the case of rigid rotors), one cannot deduce unambiguously the intercept of the \( \Delta E \) versus \( L \) curves. However, if a constant \( k \) replaces the \( 6a \) term in Eqn. (5), this constant \( k \) turns out to be the same as for the same \( \tilde{S} \) value (see Fig. 4). Within the framework of this model, the integer alignment in the Hg isotopes occurs as a consequence of the \( b = 2a \) condition.

5. Conclusions

An \( L \)-rotor picture is important when strongly deformed configurations are favored and the coupling between spatial and spin degrees of freedom is weak. The model gives rise to \( L(L + 1) \) rotational sequences associated with each of the \( (2S + 1) \) spin orientations, and this allows an identification of a spin alignment band label \( D = J - L \). The \( L \cdot S \) coupling determines any deviation from the reference band \( (D = 0) \)—which can be significant for
low $L$ values but is negligible for high $L$ values—and its strength can be extracted from shifts in the excitation spectra. A value $b = 2a$ for the $L \cdot S$ strength leads to integer alignment and produces $J(J + 1)$ rotational sequences.

We have shown that the appearance of identical superdeformed bands in $^{194}$Hg is consistent with an $\hat{L}$-rotor picture which emerges naturally within the context of many-particle theory with good total pseudo-spin symmetry and its pseudo-$SU(3)$ and pseudo-symplectic extensions. The occurrence of only $\bar{S} + 1$ ($\bar{S} + \frac{1}{2}$) numbers of $\bar{D}$ bands for integer (half-integer) pseudo-spin nuclei, instead of the expected $2\bar{S} + 1$ distinct bands, happens when $b = 2a$. For members of the same pseudo-spin multiplet, this band degeneracy doubles the $B(E2)$ transition strengths of $\bar{D} \neq 0$ bands with respect to those of the reference ($\bar{D} = 0$) band. Measurements of $B(E2)$ rates for heavy deformed nuclei are crucial for proving or disproving the model. Deviations from this simple picture, such as differences in the kinematic and dynamic moments of inertia, are expected to teach us additional physics concerning, for example, the alignment of particles in the unique parity intruder levels.

Acknowledgments

Comments and suggestions from R.V. Janssens are gratefully acknowledged. The authors also acknowledge helpful discussions with H. Naqvi and J. Escher as well as comments on the manuscript from D. Troltenier and A. Blokhin.

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