Р. М. Ашерова, Дж. П. Драер, Ю. И. Харитонов, Ю. Ф. Смирнов

О каноническом решении проблемы кратности для алгебр $\mathfrak{u}(3)$ и $\mathfrak{u}_q(3)$

Обнинск — 1997
On the Canonical Solution of the Multiplicity Problem for the $U(3)$ and $U_3(3)$ Algebras

R.M. Aferova
Institute of Physics and Power Engineering, Obninsk

I.P. Drayer
Louisiana State University, Baton Rouge

Yu.A. Khuritonov
Peterburg Institute of Nuclear Physics, Gatchina

Yu.F. Sininov
Moscow State University - Institute of Nuclear Physics Instituto de Fisica, Mexico
The canonical solution of the multiplicity problem in the tensor product of the U(3) irreducible representations, developed by L.C. Biedenharn et al. [1-7] and realized through the well-known Draayer-Akiyama (DA) computer code [8], is extended to the quantum algebra $U_q(3)$. An analytical formula for the special stretched Wigner coefficients, 
\[ \langle (\lambda_1 \mu_1) H_1, \langle \lambda_2 \mu_2 \rangle e_2 A_2 M_2, (\lambda_3 \mu_3) H_3 \rangle_{\text{bar}} \]
is derived using the projection operator method [9-10]. In this expression $H_j$ denotes the highest weight vector of the $(\lambda_j \mu_j)$ irrep; the subscript "max" means the coefficients corresponding to the unit tensor operator with a maximal characteristic null space, and $q$ is the usual quantum label so the standard $U(3)$ Wigner coefficient, which is what is required in the DA code, can be obtained by going to the $q \to 1$ limit of the theory. As an illustration the some $q$-Wigner coefficients for the tensor product (22) $\times$ (22) are calculated. The procedure for evaluating the non-highest Wigner $U_q(3)$ coefficients is similar to that given in [8].

О каноническом решении проблемы кратности для алгебр $U(3)$ и $U_q(3)$

Р. М. Ашерова, Д. П. Драер, Ю. И. Харитов и Ю. Ф. Смирнов

Каноническое решение проблемы кратности в тензорном произведении неприводимых представлений (НП) алгебры $U(3)$, предложенное Л. С. Биденхарном и др. [1-7] и реализованное в компьютерных кодах Драера и Акиямы (ДА) [8], обобщается на случай квантовой алгебры $U_q(3)$. Методом проекционных операторов [9-10] получена аналитическая формула для специального вида "затравочных" $U_q(3)$ коэффициентов Вигнера:

\[ \langle (\lambda_1 \mu_1) H_1, (\lambda_2 \mu_2) e_2 A_2 M_2, (\lambda_3 \mu_3) H_3 \rangle_{\text{bar}} \]

Здесь через $H_i$ обозначен вектор старшего веса НП $(\lambda_i \mu_i)$; индекс "max" означает, что коэффициент соответствует единичному тензорному оператору с максимальным характеристическим нулево-пространством, и $q$ есть обычный квантовый индекс параметра деформации алгебры $U_q(3)$. Стандартный $U(3)$ коэффициент Вигнера может быть получен из этой теории в пределе $q \to 1$. В качестве примера вычисляются некоторые $U_q(3)$ коэффициенты Вигнера для тензорного произведения НП (22) $\times$ (22). Процедура вычисления остальных коэффициентов (не старшего веса) подобна приведенной в [8].

(С) Физико-энергетический институт (ФЭИ), 1977
1 Introduction

The problem of the decomposition of a tensor product of $U(3)$ irreps into irreducible components and a canonical definition for the outer multiplicity, which distinguishes multiple occurrences of an irrep in a product of $U(3)$ irreps, has been thoroughly investigated by Biedenharn, Louck, Hecht (BLH) and co-workers [1-7]. Their mathematically canonical definition puts the outer multiplicity on a sound group theoretical basis through the use of an upper Gelfand pattern for a Wigner operator with the $U(3)$ irreducible tensor character under consideration. Practical aspects of this choice are the vanishing of certain Wigner (Chebsch-Gordan) and Racah coefficients, very simple symmetry relations under conjugation, and attractive limiting forms for the $SU(3) \supset SU(2) \times U(1)$ Wigner coefficients.

Using the essential features of the BLH definition, Draayer and Akiyama (DA) [8] developed a practical algorithm for evaluating $SU(3) \supset SU(2) \times U(1)$ Wigner coefficients. The starting point of the DA prescription is a set of special Wigner coefficients,

$$\langle (\lambda_{1} \mu_{1}) H_{1}, (\lambda_{2} \mu_{2}) e_{2} A_{2} M_{2} | (\lambda_{3} \mu_{3}) H_{3} \rangle_{\text{plan}},$$

which are generated using an empirically deduced, Eq.(20) of Ref.[8], that up until the now has escaped analytic verification.

This contribution gives an alternative analytic result for the starting coefficients that are used in the DA algorithm. The derivation employs the projection operator (PO) method developed by the Moscow group [9-10]. In this regard, it should be noted that the Moscow group also considered, early on, the decomposition of the tensor product of $U(3)$ irreps into irreducible components [9]. Their approach employed the PO method [10] in conjunction with the introduction of an additional integral of motion for a resolution of the multiplicity, in accordance with a prescription recommended by Racah and Moshinsky [11]. Some final results can be found in [12].

Based on these successes, a unification of the BLH tensor approach and the PO method appears to be a recommended general strategy. This was first realized by Alisauskas for the usual $SU(3)$ algebra [13] (and later for the quantum $U_q(3)$ algebra [14]). Here the question of the applicability of this approach to a derivation of results for a Wigner-Racah algebra for q-deformed objects is considered. The present discussion is restricted to the calculation of starting Wigner coefficients for the $U_q(3)$ algebra, leaving to the future a derivation of more general results. As noted above, the corresponding Wigner...
coefficients for the standard $U(3)$ case can be obtained by going to the $q \to 1$ limit.

2 Wigner coefficients in the PO approach

Let $A_{kl}$ with $i, k = 1, 2, 3$, be a Cartan-Weyl basis for the $U_q(3)$ algebra [15] and $D^{(\lambda, \mu)}$ an irrep of this algebra with highest weight $f = (f_1, f_2, f_3)$. (Also, $D^{(\lambda, \mu)}$ will be used to denote an irrep of $SU_q(3)$ where $\lambda = f_1 - f_2$ and $\mu = f_2 - f_3$). On the space of an irrep $D^{(\lambda, \mu)}$, the canonical Gelfand-Zetlin basis can be realized:

$$| f_i \rangle_q \equiv (\lambda \mu) \epsilon \Lambda M_\lambda \in \{0, 1, \ldots, \lambda \mu \}$$

with

$$\epsilon = -(\lambda + 2\mu) + 3(l + k),$$

$$\Lambda = \frac{1}{2}(\lambda + l - k),$$

$$M_\lambda = f_1 - \frac{1}{2}(f_2 + f_3) = r - \Lambda,$$

where the integers $l, k, r$ satisfy $0 \leq k \leq \lambda, \ 0 \leq l \leq \mu, \ 0 \leq r \leq 2\lambda$.

A decomposition of the tensor product of irreps into reducible components can be written as:

$$D^{(\lambda, \mu)} \times D^{(\lambda, \mu)} = \sum_{(\lambda, \mu)} v(\lambda, \mu) D^{(\lambda, \mu)}.$$  

The outer multiplicity label $\rho = 1, 2, \ldots, v(\lambda, \mu)$ is used to distinguish multiply occurring irreps $D^{(\lambda, \mu)}$. The Wigner coefficients $\langle (\lambda_1, \mu_1) | (\lambda_2, \mu_2) \rangle$ are by definition the elements of a unitary transformation between coupled and uncoupled representations of $SU_q(3)$ in the $\alpha$-scheme,

$$\langle (\lambda_1, \mu_1, \lambda_2, \mu_2) : \rho(\lambda, \mu) \alpha \rangle_q = \sum_{n_{\lambda_1}, n_{\mu_1}} \langle (\lambda_1, \mu_1, \alpha_1, (\lambda_2, \mu_2) \alpha_2) | (\lambda \mu) \alpha \rangle_q \left| (\lambda_1, \mu_1) \alpha_1 \right\rangle_q \left| (\lambda_2, \mu_2) \alpha_2 \right\rangle_q.$$  

Building on the BLH scheme for a resolution of the outer multiplicity, DA developed a practical algorithm for evaluating all $SU(3) \supset SU(2) \times U(1)$ Wigner coefficients. The DA prescription includes a straightforward Schmidt
orthogonalization procedure for producing independent orthogonal vectors in the multiplicity subspace. The scheme requires a set of special starting (seed) Wigner coefficients with \( \rho = p_{\text{max}} = \nu(\lambda, \mu) \).

\[
(\lambda \mu_1, (\lambda_2 \mu_2 \epsilon_2 \Lambda_2 m_2 | (\lambda \mu) H)_{p_{\text{max}}}. \tag{9}
\]

These seeds are associate with a unit tensor operator characterized by a maximal null space and includes the highest weight states \( | H = (f_1, f_2 f_3) \) and \( | H_1 \) of the irreps \( (\lambda \mu) \) and \( (\lambda_1 \mu_1) \), respectively. As indicated above, DA determined a general expression for these special Wigner coefficients by employing empirical methods. Below an alternative analytic expression is introduced, one derived using the PO method [9,15]. The latter, therefore, provides an analytically proven result, missing in all previous analyses, for the seed coefficients upon which the DA prescription rests.

To follow the argument, consider the PO defined by

\[
P^{(\lambda \mu)}_{\alpha \alpha'} = \sum_\rho |(\lambda \mu)\rho, (\lambda \mu')\rho'\rangle \langle (\lambda \mu)\rho, (\lambda \mu')\rho'\rangle, \tag{10}
\]

where \( \alpha = \epsilon_\Lambda M, \). Then the matrix element (ME) of the PO, \( P^{(\lambda \mu)}_{\alpha \alpha'}, \) has the form

\[
(\lambda \mu_1)_{\alpha_1}, \epsilon_{\lambda_2 \mu_2} \alpha_2 | P^{(\lambda \mu)}_{\alpha \alpha'} | (\lambda \mu_1)_{\alpha_1}, \epsilon_{\lambda_2 \mu_2} \alpha_2 \langle \lambda \mu \alpha' \rangle_ho,
\]

\[
\sum_\rho |(\lambda \mu_1)_{\alpha_1}, \epsilon_{\lambda_2 \mu_2} \alpha_2 | (\lambda \mu) \alpha \rangle \rho \langle (\lambda \mu) \alpha | (\lambda \mu) \alpha' \rangle_{\rho'}, \tag{11}
\]

In particular, for the ME of the PO \( P^{(\lambda \mu)}_{\beta \beta'} \)

\[
(\lambda \mu_1)_{\beta_1}, \epsilon_{\lambda_2 \mu_2} \beta_2 | P^{(\lambda \mu)}_{\beta \beta'} | (\lambda \mu_1)_{\beta_1}, \epsilon_{\lambda_2 \mu_2} \beta_2 \langle \lambda \mu \beta' \rangle_{\rho_{\text{max}}},
\]

\[
\times |(\lambda \mu) \beta \rangle_{\rho_{\text{max}}} \langle (\lambda \mu) \beta | (\lambda \mu) \beta' \rangle_{\rho'_{\text{max}}}, \tag{12}
\]

where \( M = (f_1, f_2 f_3) \) is the dominant weight corresponding to the state with \( k = 0, l = \mu, \) and \( \tau = \lambda + \mu \) in (4)-(6), namely,

\[
| (\lambda \mu) \beta \rangle_{\rho_{\text{max}}} \equiv | (\lambda \mu) \epsilon_M \Lambda_M M, M \rangle, \tag{13}
\]

where \( \Lambda_M \) and \( \epsilon_M \) are given by

\[
\Lambda_M = \frac{1}{2}(\lambda + \mu), \quad \epsilon_M = -(\lambda - \mu). \tag{14}
\]
This configuration satisfies the relations
\[ A_{12} | (\lambda \mu) M_M \rangle = A_{13} | (\lambda \mu) M_L \rangle = A_{32} | (\lambda \mu) M_S \rangle = 0. \]  
(15)
Similarly, \( M_2 \) is the dominant weight in the irrep \((\lambda_2 \mu_2)\), i.e.
\[ \Lambda_{M_2} = \frac{1}{2} (\lambda_2 + \mu_2), \quad \epsilon_{M_2} = - (\lambda_2 - \mu_2). \]  
(16)
The values of quantum numbers \( A_1 \) and \( \epsilon_1 \) are chosen as follows
\[ A_1 = \frac{1}{2} (\lambda + \mu) - \frac{1}{2} (\lambda_2 + \mu_2) \geq 0, \quad \epsilon_1 = - (\lambda - \mu) + (\lambda_2 - \mu_2). \]  
(17)
The last Wigner coefficient on the r.h.s. of (12) corresponds to the maximum possible value of \( A_M \), and the maximum change,
\[ \Delta \Lambda = | \Lambda - A_1 |, \]  
(18)
in the angular momentum, \((\Delta \Lambda = \frac{1}{2} (\lambda_2 + \mu_2))\).

In accordance with the BLH procedure, the only nonvanishing Wigner coefficients with maximum \( \Delta \Lambda \) in the \((\lambda \mu)\) irrep have \( p = p_{\text{max}} \). In other words, they are different from zero only for tensors \( T^{(\lambda \mu)} \) with maximal null space. Therefore, in the particular case (12), the sum over \( p \) on the r.h.s. of (11) reduces to the single term. It follows that formula (12) can be used to obtain the starting coefficients (9) (up to a normalization factor) if the ME of the projector in the l.h.s. of this formula can be found. However, it should be noted that for (12) to apply, the condition
\[ (\lambda + \mu) \geq (\lambda_2 + \mu_2) \]  
(19)
must be satisfied. If (19) does not hold, the symmetry property of Wigner coefficients under \((\lambda \mu)\) and \((\lambda_1 \mu_1)\) interchange must be applied.

As mentioned above, the last coefficient in the r.h.s. of (12) can be considered to be the normalizing factor. It can be calculated using the PO \( P^{(\lambda \phi)}_{M,M} \) which projects out the dominant weight \((M = (f_1 f_2 f_3))\) vector belonging to the \( D^{(\lambda)} \) irrep with \( p = p_{\text{max}} \):
\[ \langle (\lambda_1 \mu_1) \varepsilon_1 A_2 A_1, (\lambda_2 \mu_2) M_2 | (\lambda \mu) M \rangle_{p_{\text{max}}}^2 = \langle (\lambda_1 \mu_1) \varepsilon_1 A_2 A_1, (\lambda_2 \mu_2) M_2 | P^{(\lambda \phi)}_{M,M} | (\lambda_1 \mu_1) \varepsilon_1 A_2 A_1, (\lambda_2 \mu_2) M_2 \rangle. \]  
(20)
The PO \( P^{(\lambda \phi)}_{M,M} \), in turn, can be obtained in factorized form from the extremal projector [10]
\[ P^{(\lambda \phi)}_{M,M} = P_{12} P_{13} P_{23}. \]  
(21)
\[ P_{ik} = \sum_{\tau_{i}^{k}} (-1)^{\mu_{i}} q_{(k-i-1)\tau_{i}}^{\mu_{i}} [f_i - f_k + k - i]!! \cdot A_{ik}^{\tau_{i}} A_{ik}^{i}, \]  
for the highest weight vector
\[ |(\lambda \mu)(f_1 f_2 f_3)\rangle_q \]
by the substitution 2 \(\rightarrow\) 3 [13, 14]:
\[ P^{(i \mu)_{23}}_{M,M} = P_{12} P_{13} P_{23} = \]
\[ \sum_{n_1, n_2, n_3} \frac{(-1)^{n_1 + n_2 + n_3} q^{\lambda + 1} [\lambda + 1][\mu + 1][\lambda + \mu + 2]!}{[n_1][n_2][n_3][\lambda + 1 + n_1][\mu + 1 + n_2][\lambda + \mu + 2 + n_3]!} \times A_{12}^{n_2} A_{13}^{n_3} A_{23}^{n_1}, \]
where \( n_1, n_2, n_3 \) are non-negative integers.

By using the commutation relations for powers of the generators of \( U(3) \) [15], a more convenient form for calculating ME of the projector in the r.h.s. of (20) can be found:
\[ P^{(i \mu)_{23}}_{M,M} = \sum_{n_1, n_2, n_3} \frac{(-1)^{n_1 + n_2 + n_3} q^{\lambda + 1} [\lambda + 1][\mu + 1][\lambda + \mu + 2]!}{[n_1][n_2][n_3][\lambda + 1 + n_1][\mu + 1 + n_2][\lambda + \mu + 2 + n_3]!} \times \frac{[\lambda + \mu + 2 + n_1 + n_2 + n_3]!}{[\lambda + \mu + 2 + n_1 + n_3][\lambda + \mu + 2 + n_2 + n_3]!} \times A_{12}^{n_2} A_{13}^{n_3} A_{23}^{n_1}. \]

Then by expanding the powers of generators in this expression in accordance with coproduct rules [15], the ME of the PO \( P^{(i \mu)_{23}}_{M,M} \) can be deduced:
\[ \langle (\lambda_1 \mu_1) e_1 A_1 A_2, (\lambda_2 \mu_2) M_2 | P^{(i \mu)_{23}}_{M,M} | (\lambda_1 \mu_1) e_1 A_1 A_2, (\lambda_2 \mu_2) M_2 \rangle \]
\[ = \sum_{n_1, n_2} \frac{(-1)^{n_1 + n_2} q^{\lambda_1 \lambda_2} [\lambda_1 \mu_1 + 1][\lambda_1 + 1][\mu_1 + 1][\lambda_2 + \mu_2 + 2 + n_1 + n_2]!}{[n_1][n_2][\lambda_1 + 1 + n_1][\mu_1 + 1 + n_2][\lambda_1 + \mu_2 + 2 + n_1 + n_2]!} \times \frac{[\lambda_1 + \mu_2 + 2 + n_1 + n_2]!}{[\lambda_1 + \mu_2 + 2 + n_1 + n_2][\lambda_1 + \mu_2 + 2 + n_1 + n_2]!} \times A_{12}^{\lambda_2} A_{13}^{\lambda_1} A_{23}^{\mu_2}. \]

Let the last ME be denoted \( G_{n_1 n_2} \), and recall that the ME of the \( U_q(3) \) generators in the canonical Gelfand-Zetlin basis were found in [16]. Using those formulae yields the following result for \( G_{n_1 n_2} \):
\[ G_{n_1 n_2} = \frac{[k_1]!![\lambda_1 + l_1 - k_1 + 1][\lambda_1 - k_1 + n_1]!}{[\lambda_1 - k_1]!![\lambda_1 + l_1 - k_1 + n_1 + n_2 + 1][k_1 - n_1]!} \times \]
\[
\frac{[\lambda_1 + l_1 + n_2 + 1][\lambda_1 + n_2][\lambda_1 + 1][\lambda_1 + n_1 + 1][\lambda_1 + 1 + n_1 + 1]}{[\lambda_1 + l_1][\lambda_1 + n_2][\lambda_1 + n_1][\lambda_1 + 1 + l_1][\lambda_1 + 1 + n_1 + 1]},
\]

where

\[
k_1 = \frac{1}{3}(-2\lambda - \mu + 2\lambda_1 + \mu_1 + 2\lambda_2 + \mu_2) = \mu - \mu_1 - \mu_2 + 2n,
\]

\[
l_1 = \frac{1}{3}(-2\lambda - \mu + \lambda_1 + \mu_1 - \lambda_2 - 2\mu_2) = \mu_1 - n,
\]

\[
n = \frac{1}{3}(-2\lambda - \mu + \lambda_1 + \mu_1 + \lambda_2 + 2\mu_2).
\]

The integers \(k_1, l_1, n_1, n_2\) satisfy the following restrictions: \(0 \leq k_1 \leq \lambda_1, 0 \leq l_1 \leq \mu_1, 0 \leq n_1 \leq k_1, 0 \leq n_2 \leq \mu_1 - l_1\).

To calculate the starting Wigner coefficient, consider relation (12). The projector \(P^{(\lambda_1 \mu_1)\nu}_{H, M}\) can be constructed as

\[
P^{(\lambda_1 \mu_1)\nu}_{H, M} = \frac{1}{[\nu]!} A^\nu_{\mu} P_{M, M}^{(\lambda_1 \mu_1)\nu}
\]

and reduced to [14]:

\[
P^{(\lambda_1 \mu_1)\nu}_{H, M} = [\lambda + 1][\mu + 1][\lambda + \mu + 2] \sum_{n_1, n_2, \nu, \tau} A^\nu_{\mu} [\mu + n_2 + 1][\lambda + 1 + n_1 + n_2]
\]

\[
\times [\lambda + \mu + 2 + n_1 + n_2 - u][\lambda + 1 + n_1 - u][\lambda + 1 + n_2 - u][\lambda + 1 + n_1 + n_2 - u][\lambda + \mu + 2 + n_2]
\]

\[
\times A^\nu_{\tau} A^\nu_{\mu} A^\nu_{\mu} A^\nu_{\mu} A^\nu_{\mu} A^\nu_{\mu} P_{12} (\Lambda = \Lambda_M).
\]

By substituting the PO (32) into (12) and expanding powers of the generators in accordance with coproduct rules [15], the ME in the l.h.s. of (12) can be shown to be given by the following expression:

\[
\{((\lambda_1 \mu_1)H_1, (\lambda_2 \mu_2)H_2)_{\nu}^{H_2^{(\lambda_1 \mu_1)\nu} ((\lambda_1 \mu_1)H_1, (\lambda_2 \mu_2)H_2)}\}
\]

\[
= [\lambda + 1][\mu + 1][\lambda + \mu + 2] \sum_{n_1, n_2, \nu, \tau} A^\nu_{\mu} [\mu + n_2 + 1][\lambda + 1 + n_1 + n_2]
\]

\[
\times [\lambda + \mu + 2 + n_1 + n_2 - u][\lambda + 1 + n_1 - u][\lambda + 1 + n_2 - u][\lambda + 1 + n_1 + n_2 - u][\lambda + \mu + 2 + n_2]
\]

\[
\times A^\nu_{\tau} A^\nu_{\mu} A^\nu_{\mu} A^\nu_{\mu} A^\nu_{\mu} A^\nu_{\mu} P_{12} (\Lambda = \Lambda_M).
\]

\[
\times \frac{[\mu - r + n_2][\mu - r + n_2 - s]}{[\nu]!} C_\nu D_\tau.
\]
where

\[
C_s = \langle (\lambda_1 \mu_1)_{H_1} | A_{22} A_{33} A_{32}^{(n)} | (\lambda_1 \mu_1)_{\Lambda_1 \Lambda_1} \rangle = N_{(\lambda_1 \mu_1)}^{(\lambda_2 \mu_2)} \left( \frac{s!}{s - n_2} \right) \left( \frac{\mu_1 + n_2 - s}{\mu_1} \right)!
\]

\[
D_r = \langle (\lambda_2 \mu_2)_{\Delta_2 \Delta_2} m_2 | A_{21} A_{31}^{(n-r)} A_{32}^{(n-r-2)} | (\lambda_2 \mu_2)_{M_2} \rangle
\]

\[
= N_{(\lambda_2 \mu_2)}^{(\lambda_1 \mu_1)} \left( \frac{[\lambda_2 - m_2]!}{[2\lambda_2][\lambda_2 + m_2]} \right)^{\frac{3}{2}} \left( -1 \right)^{\lambda_1 - m_2 - r} q^\phi
\]

\[
	imes \frac{[N_1 - r][\mu - N_2 - r][\lambda_2 + m_2 + r]}{[\mu_2 - \mu + N_2 + r][N_1 - \lambda_2 + m_2][\lambda_2 - m_2 - r]}!
\]

(35)

In (34) and (35)

\[
N_{(\lambda_1 \mu_1)}^{(\lambda_2 \mu_2)} = \left( \frac{[\lambda][\mu][\lambda + \mu + 1][\lambda + l - 1][\lambda + \mu + 1 - k][k][l]}{[l - k][\mu - l][\lambda + \mu + 1 - k][\lambda + l + 1]} \right)^{\frac{1}{4}}
\]

(36)

\[
k_1 = n_1, \quad l_1 = s - n_2,
\]

(37)

\[
k_2 = N_1 - \lambda_2 + m_2, \quad l_2 = \mu_2 - \mu + N_2 + \lambda_2 - m_2,
\]

(38)

\[
N_1 = n_1 = 2n + \mu - \mu_1 - \mu_2, \quad N_2 = s - n_2 = \mu_1 - n.
\]

(39)

\[
\phi = - (\lambda_2 - m_2 - r)(\lambda_2 + \mu - \mu_1 + N_2 - m_2 + s - n_1 + 1).
\]

(40)

Recall that this result is valid under condition (19).

Note that expressions (27)-(31) suffice to determine the normalizing Wigner coefficient in (12). Furthermore, the l.h.s. of (12) is known through (33)-(40) and can be rewritten in the final form:

\[
\langle (\lambda_1 \mu_1)_{H_1}, (\lambda_2 \mu_2)_{\Delta_2 \Delta_2} m_2 | (\lambda_1 \mu_1)_{H_1} \rangle^{\frac{1}{2}}
\]

\[
\times \langle (\lambda_1 \mu_1)_{\epsilon_i \Lambda_1}, (\lambda_2 \mu_2)_{M_2} | (\lambda_1 \mu_1)_{\epsilon_i \Lambda_1} \rangle^{\frac{1}{2}}
\]

\[
= \langle (\lambda_1 \mu_1)_{H_1}, (\lambda_2 \mu_2)_{\epsilon_i \Lambda_2} m_2 | H_{\Lambda_1 \Lambda_1}, (\lambda_2 \mu_2)_{M_2} \rangle
\]

\[
= A \sum_{w,w'} B_{w w'},
\]

(41)

where

\[
A = (-1)^{\lambda_1 - N_1 + \lambda_2 - m_2} q^\phi \left( \frac{[\lambda + 1][\mu + 1][\lambda + \mu + 2]}{[\lambda + \mu + 2 + N_1]} \right)^{\frac{1}{2}}
\]

\[
\times \left( \frac{[\lambda_1][\mu_1][\lambda_1 + \mu_1 + 1][N_1][\mu_1 + N_2][\lambda_1 - N_1 + N_2 + 1]}{[N_2][\lambda_1 - N_1][\lambda_1 + \mu_1 + N_1 + 1][\lambda_1 + N_2 + 1]} \right)^{\frac{1}{2}}
\]
\begin{align}
&\times \left( \frac{[\lambda_2][\mu_2][\lambda_2 + \mu_2 + 1][N_2 + m_2][N_2 + \mu - \Lambda_2 - m_2][\Lambda_2 + m_2][\Lambda_2 + \Lambda_2 + m_2][\mu - \Lambda_2 + m_2 - N_2][\lambda_2 + \Lambda_2 - m_2 - N_1]}{[\Lambda_2 + m_2][\Lambda_2 + \Lambda_2 + m_2][\mu - \Lambda_2 + m_2 - N_2][\Lambda_2 + m_2][\Lambda_2 + \Lambda_2 + m_2][\mu - \Lambda_2 + m_2 - N_2][\lambda_2 + \Lambda_2 - m_2 - N_1]} \right)^{\frac{1}{2}},
\end{align}

(42)

\begin{align}
B_{u,v} &= \frac{(-1)^{\ast + q'}[\lambda_2 + m_2 + r]}{[\Lambda_2 - m_2 - r][\Lambda_2 - m_2 - r][\mu - r + u][\mu - r + u][\lambda + N_1 - u + 1][\mu - \mu_2 + N_2 + r]} \times \frac{[\lambda + \mu + N_1 - N_2 + s - u + 2][\mu - N_2 + s - r]}{[\mu - N_2 + s + 1][\lambda + \mu - N_2 + s - u + 2][\mu - s][\lambda - N_2 - s]}.
\end{align}

(43)

\begin{align}
\alpha &= \frac{1}{2}(N_1(\lambda_2 - \mu_1) + N_2(\mu_1 + \mu_2) - \mu_1) - (\Lambda_2 - m_2)(\Lambda_2 - m_2 + \lambda_2 + \mu_2 - \mu - N_1 + N_2 + 1),
\end{align}

(44)

\begin{align}
\beta &= u + r(\lambda_2 + \mu_2 - \mu_1 + N_2 + 1) - rs + s(\mu - \mu_2 - N_2),
\end{align}

(45)

\begin{align}
- \Lambda_2 \leq m_2 \leq \Lambda_2, \quad m_2 = \frac{1}{2}(\lambda - \lambda_1).
\end{align}

(46)

For \(\lambda + \mu < \lambda_2 + \mu_2\) it is necessary to use symmetry properties of the Wigner coefficients with respect to a permutation of \((\lambda_1, \mu_1)\) and \((\lambda \mu)\) irreps or to determine an analytical continuation of the relations found above into the \(\lambda + \mu < \lambda_2 + \mu_2\) region (see Ref. [14]). From this it follows that the seed Wigner coefficients, (9), required for an application of the DA prescription, are known in analytical form. Thus these results, coupled with the DA prescription, suffice to determine all Wigner coefficients.

As an example, consider Wigner coefficient for the \((\lambda_1 \mu_1) = (\lambda_2 \mu_2) = (22)\) and \((\lambda \mu) = (41)\) case. The coefficients

\begin{align}
X_1 &= \langle (22)H_1, (22)\epsilon_2\Lambda_2 m_2 \mid (41)H \rangle^{\ast},
\end{align}

\begin{align}
\times \langle (22)\epsilon_1\Lambda_1 \Lambda_1, (22)M_2 \mid (41)M \rangle^{\ast},
\end{align}

(47)

with \(\epsilon_2 = 0, \quad m_2 = 1, \quad \Lambda_2 = 1 (i = 1), \quad \Lambda_2 = 2 (i = 2)\) are calculated through (41)-(46):

\begin{align}
X_1 &= \frac{[2][2][2][4][4]}{[3][4][8][4]};
\end{align}

(48)

\begin{align}
X_2 &= \frac{[2][3][4][4][8][8]}{[3][4][8][4]} (2)[6] + q[4]^2.
\end{align}

(49)
The second Wigner coefficient in l.h.s. of (47) corresponds to the normalizing coefficient
\[
N = (22 - 3^{11}_{22})(22)022 \mid (41) - 3^{5}_{22}1^g
\]
It follows from Eqs. (27)-(31)
\[
N^2 = \frac{1}{[3][3][4][6][8]}(3[3][4][6][8] + [2][3][4][5][9] - q^2[2][3][4][5][8] - q^{-2}(2[4][6][8] + [2][2][5][10]) + q^{-4}[2][5][6][8]).
\] (50)
These Wigner coefficients are all in agreement with the tables of Ref. [5] for the classical \( q = 1 \) limit.

3 Conclusions

An analytical formula for (normalized) starting Wigner coefficients, (1), which are required input in the DA prescription for generating Wigner coefficients, and their \( q \)-analog, (9), has been derived. The method has been incorporated into a new DA code. The solution was tested by comparing results with those from the standard DA code for a large number of \( (\lambda_1 \mu_1), (\lambda_2 \mu_2) \) and \( (\lambda \mu) \) irreps. In all cases the formulae given above reproduce results from the standard DA code. This exercise confirms the correctness of the empirical result expressed by Eq.(20) of Ref.[8] and at the same time demonstrates the correctness of an alternative starting point that is based on analytically proven results.

Advantages of the results obtained in this work are: 1) they give normalized results for Wigner coefficients, and 2), they are valid for both the usual \( U(3) \) and \( q \)-deformed \( U_q(3) \) algebras. The latter means, for example, that it should now be possible to develop a new generation algebraic computer code for calculating \( U(3) \) as well as \( U_q(3) \) Wigner coefficients.

This work was inspired by Larry Biedenharn, to whom one of the authors (J.D.) owes a great deal of gratitude for the questions he raised and answers he provided on issues of common interest. Professor Biedenharn was particularly helpful during the development of the DA code, which draws heavily from his work. Two of the authors (R.A. and Yu.S.) are grateful for support from the US National Science Foundation for their visit to Louisiana State University (LSU) and to the scientists and staff members of the Department of Physics and Astronomy at LSU for their kind hospitality. This work was
also supported by the Russian Foundation for Fundamental Research, Grant No:96-01-01421.

References


