On the validity of the pseudo-spin concept for axially symmetric deformed nuclei

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\textbf{ABSTRACT.} The average single-particle field shows a very small pseudo-spin-orbit splitting in the pseudo-spin representation. If this splitting is neglected, pseudo-spin becomes a good quantum number and the resulting scheme has a very simple interpretation. In spite of many applications of the pseudo-spin idea, until the present a detailed test of the single-particle, pseudo-Nilsson scheme has not been done. It is the purpose of this contribution to fill that gap. The pseudo-spin symmetry embodied in the realistic deformed average field is explored by comparing the single-particle energies and wave-functions of the deformed Woods-Saxon model with the corresponding results of the pseudo-Nilsson model. This nicely confirms the main assumption of the pseudo-spin model: the weak coupling of intruder shells to the Hilbert space of the pseudo-Nilsson states.

\textbf{RESUMEN.} El campo medio de partícula individual muestra un rompimiento pseudo-espin-órbita muy pequeño en la representación del pseudo-espin. Si este rompimiento se omite, el pseudo-espin se convierte en un buen número cuántico y el esquema resultante tiene una interpretación simple. A pesar de muchas aplicaciones de la idea del pseudo-espin, hasta el presente no ha sido hecha una prueba detallada del esquema del pseudo-Nilsson de partícula individual. El propósito de esta contribución es llenar este hueco. Se explora la simetría del pseudo-espin incorporada en el campo medio deformado realista, se comparan las energías de partícula individual y las funciones de onda del modelo Woods-Saxon deformado con los resultados correspondientes del modelo de pseudo-Nilsson. Estos resultados confirman agradablemente la principal suposición del modelo del pseudo-espin: el acoplamiento débil de las capas intrusas al espacio de Hilbert de los estados del pseudo-Nilsson.

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1. INTRODUCTION

The average single-particle field shows a very small pseudo-spin-orbit splitting in the pseudo-spin representation [1,2]. If this splitting is neglected, pseudo-spin becomes a good quantum number and the resulting scheme has a very simple interpretation. In spite of many applications of the pseudo-spin idea, until the present a detailed test of the single-particle, pseudo-Nilsson scheme (i.e., a scheme in which the spin and orbital angular momentum operators are replaced by their "pseudo-space" counterparts, cf. Sec. 3) has not been done and it is the purpose of this contribution to fill that gap. The pseudo-spin symmetry embodied in the realistic deformed average field is explored by comparing the single-particle energies and wave functions of the deformed Woods-Saxon model with the corresponding results of the pseudo-Nilsson model. Additionally we calculate the magnetic moments of deformed odd-\(A\) nuclei of the rare earth region. An extended and more complete version of this contribution can be found in [9].

After a short summary of the transformation to the pseudo-spin picture (Sec. 2) we introduce the pseudo-Nilsson model in Sec. 3; a brief review of the Woods-Saxon model and a comparison with the pseudo-Nilsson model is given in Sec. 4; calculated results of magnetic moments of deformed odd-\(A\) rare-earth nuclei are compared with experimental data in Sec. 5; finally summary and conclusions are given in Sec. 6.

2. PSEUDO-SPIN SYMMETRY

The single-particle part of the spherical shell-model Hamiltonian can be well approximated by

\[
h_{\text{s.p.}} = h_0 + Cl \cdot s + Dl^2, \tag{1}
\]

which consists of a spherical oscillator Hamiltonian \(h_0\), a spin-orbit interaction \(l \cdot s\), and an \(l^2\) term. It is well known that the combined effect of the \(l \cdot s\) and \(l^2\) terms becomes more and more dominant for heavier nuclei, actually destroying the simple spherical oscillator [i.e., \(\text{SU}(3)\)] structure of \(h_{\text{s.p.}}\) for all but very light systems. A relabelling scheme for the internal shell-model quantum numbers, however, allows a major simplification of the theoretical treatment for heavy nuclei \((A \geq 100)\) because the influence of the spin-orbit term can be strongly reduced.

Alternatively to the relabelling scheme one can use an explicit algebraic expression [6] to define a corresponding unitary transformation \(U\) to express normal-space quantities \(A\) (e.g., \(l, s, j, \ldots\)) in terms of their pseudo-space counterparts \(\tilde{A}\) (e.g., \(\tilde{l}, \tilde{s}, \tilde{j}, \ldots\)),

\[
\tilde{A} = UAU^+. \tag{2}
\]

Under \(U\) the spherical shell-model quantum numbers transform as

\[
\tilde{j} = j, \quad \tilde{N} = N - 1, \quad \tilde{s} = s, \quad \tilde{l} = l \pm 1. \tag{3}
\]
The Nilsson Hamiltonian, Eq. (1), expressed in terms of pseudo-operators takes the form:

$$h_{s.p.} = h_0 + CI\cdot s + DL^2 \to \tilde{h}_0 + (4D - C)\tilde{L}\cdot\tilde{s} + DL^2 + (\hbar\omega + 2D - C).$$  \hspace{1cm} (4)

The normal-space $\rightarrow$ pseudo-space transformation only affects normal parity levels. It is assumed that particles in the unique parity orbitals (i.e., the abnormal-parity intruder subshells) contribute in an adiabatic fashion to the dynamics of the low-lying states. Evidence suggests that this is a reasonable assumption for low-energy properties, at least in the energy domain below the first bandcrossing. The issue of whether or not this assumption is really justified, i.e., questions about the mixing between the intruder states with the normal-parity orbitals, will be discussed further in Sec. 4.

The physical significance of the pseudo transformation comes from the fact that

$$4D - C \simeq 0$$ \hspace{1cm} (5)

holds for heavy nuclei, and hence, the pseudo-spin-orbit term almost disappears. This observation [i.e., Eq. (5)] not only emerges as a numerical result which is required for describing experimental data in the spherical shell model [3, 4], but is also consistent with the results of relativistic mean-field calculations [7].

Consequently, it seems to be a very good approximation to replace the single-particle, shell-model Hamiltonian (1) by its much simpler counterpart

$$\tilde{h}_{s.p.} = \tilde{h}_0 + DL^2$$ \hspace{1cm} (6)

for a description of excited single-particle states in heavy nuclei. The concept of pseudo-spin is illustrated in Fig. 1 which shows the single-particle neutron levels \((50 < N < 82)\) of the deformed Woods-Saxon potential (see Sec. 4) as functions of the quadrupole deformation $\beta_2$.

Certain normal parity spherical subshells form doublets \((s_{\frac{3}{2}} - d_{\frac{3}{2}})\) and \((d_{\frac{3}{2}} - f_{\frac{3}{2}})\) for \(N = 4\) and \(p_{\frac{3}{2}} - f_{\frac{3}{2}}\) and \(f_{\frac{3}{2}} - h_{\frac{3}{2}}\) for \(N = 5\) with a very small energy splitting within each doublet. This is precisely the consequence of Eqs. (1) and (5). The normal-parity subshells \([l_j - (l + 2)_{j+1}]\) forming a doublet can be treated as one pseudo-level with the same pseudo-orbital angular momentum $\tilde{l} = l_j + 1 = l_{j+1} - 1$ and the (almost) totally decoupled pseudo-spin $\tilde{s}$. Even though the pseudo-spin transformation operator $U$ does not commute with the distorting field, it nearly does and hence pseudo-spin remains an approximately good quantum number even at large shape distortions [4, 5]. Indeed, a closer examination of the single-particle diagrams (see Fig. 1) shows that some normal-parity states differing by $\Delta \Omega = 1$ ($\Omega$ is the third component of the single-particle angular momentum $j$: $\Omega = m_j$) group into doublets while several $\Omega = 1/2$ states appear as singlets. This feature seems to be pronounced for both oblate $\beta_2 < 0$ and prolate $\beta_2 > 0$ deformations.

Following the unitary transformation, Eq. (2), the relabelling of the physical states is quite straightforward for the case of spherical nuclei [7]. The deformed part of the average Hamiltonian mixes the spherical basis states \(|NJjm\rangle\). As demonstrated by Nilsson [10],
in the limit of very large quadrupole distortions the physical states can be classified by means of asymptotic quantum numbers \([N \tilde{n}, \Lambda, \Omega]\), where \(N\) is the total number of oscillator quanta, \(n_\pi\) is the number of quanta along the body-fixed symmetry axis (\(z\)-axis), \(\Lambda\) is the projection of the single-particle orbital angular momentum on the \(z\)-axis, while \(\Omega = \Lambda + \Sigma\) where \(\Sigma\) is the third component of the spin (\(\Sigma = \pm 1/2\)). For an axial average field, \([\hat{h}_{\text{as, p}}, \hat{j}_z] = 0\) and \(\Omega\) is the constant of motion. However, neither \(n_\pi\) nor \(\Lambda\) nor \(\Sigma\) are good quantum numbers, especially at small and moderate deformations (see, e.g., examples given in Ref. [11]).

In the pseudo-spin space, the deformed normal-parity Nilsson states can be classified by means of the pseudo-asymptotic quantum numbers \([\tilde{N} \tilde{n}, \tilde{\Lambda}, \tilde{\Omega}]\) [3], where \(\tilde{N} = N - 1\), \(\tilde{n}_\pi = n_\pi\), \(\tilde{\Lambda} = \Lambda \pm 1\), and \(\tilde{\Omega} = \Omega\), see Eq. (3). Within the pseudo-spin coupling scheme, the normal-parity single-particle states in Fig. 1 can thus be associated with the \(\tilde{N} = 3\) pseudo-oscillator shell containing only two spherical orbits with \(\tilde{l} = 1\) and \(\tilde{l} = 3\). At non-zero deformations, where the spherical \(2(2\tilde{l} + 1)\) degeneracy is lifted, the pseudo-spin label can still be kept. For the case of large prolate deformations this was discussed in Refs. [3, 4]. In these references the single-particle states were analyzed mostly in terms of the asymptotic pseudo-oscillator states.

In this contribution, however, we want to go beyond this limiting case and change the deformation continuously from oblate to prolate. A first attempt to investigate the validity of the pseudo-Nilsson scheme for prolate nuclei is given in Ref. [12]. Remarkable agreement of the single-particle energy levels in the normal versus the pseudo scheme was found for prolate deformed nuclei. Motivated by this success we want to extend these investigations
in a more systematic manner. Therefore, in the following we introduce the theoretical apparatus of a full pseudo-Nilsson model.

3. The pseudo-Nilsson model

The usual Nilsson Hamiltonian (modified harmonic oscillator, m.h.o.) is defined as [10]

\[ h_{\text{m.h.o.}} = h_0 + C l \cdot s + D l^2 + h_\delta, \]  

(7)

with

\[ h_0 = \frac{1}{2} \hbar \omega_0 \left( -\Delta + r^2 \right) \]

(8)

being the spherical oscillator Hamiltonian with eigenvalues \( E = \left( N + \frac{3}{2} \right) \hbar \omega_0 \). Here \( r^2 = x^2 + y^2 + z^2 \), and \( x_i \) (\( i = 1, 2, 3 \)) are dimensionless coordinates

\[ x_i = \sqrt{\frac{M \omega_0}{\hbar}} x'_i \]

(9)

(\( x'_i \) being the original real space coordinate). The oscillator frequency \( \omega_0 \) depends on the deformation \( \delta \) as [10]

\[ \omega_0(\delta) = \omega_0(0) \left( 1 - \frac{4}{3} \delta^2 - \frac{16}{27} \delta^3 \right)^{-1/6}. \]

(10)

The deformed quadrupole field \( h_\delta \) entering Eq. (7) is given by

\[ h_\delta = -\frac{1}{3} \delta \hbar \omega_0 (3z^2 - r^2) = -\frac{2}{3} \delta \hbar \omega_0 \sqrt{\frac{4\pi}{5}} r^2 Y_{20}(\theta, \phi), \]

(11)

with \( \delta \) being the quadrupole deformation parameter.

The modified harmonic oscillator Hamiltonian, Eq. (7), is approximated by its pseudo-spin counterpart:

\[ \tilde{h}_{\text{m.h.o.}} = \tilde{h}_0 + D \tilde{l}^2 + \tilde{h}_\delta, \]

(12)

where we have assumed the validity of the pseudo-spin symmetry, Eq. (5), and dropped the constant terms in Eq. (4). The eigenvalues of the spherical pseudo-oscillator \( \tilde{h}_0 \) are \( \tilde{E} = \left( \tilde{N} + \frac{3}{2} \right) \hbar \omega_0 \) and the pseudo-quadrupole field is given as

\[ \tilde{h}_\delta = -\delta \frac{2}{3} \hbar \omega_0 \sqrt{\frac{4\pi}{5}} r^2 Y_{20}(\tilde{\theta}, \tilde{\phi}), \]

(13)

i.e., it is assumed that the quadrupole interaction (13) is a quadrupole tensor in the pseudo-spin representation.
In the following, the pseudo-Nilsson Hamiltonian (12) is diagonalised in the basis

$$|\tilde{N} \tilde{I} \tilde{\Lambda}\rangle \equiv |\tilde{N} \tilde{I}\rangle |\tilde{I} \tilde{\Lambda}\rangle.$$  \hspace{1cm} (14)

Since Hamiltonian (12) does not depend on the pseudo-spin degrees of freedom, both the length of the pseudo-spin, $\tilde{s} = 1/2$ and its third component, $\tilde{\Sigma} = \pm 1/2$, are good quantum numbers. As a consequence of the axial symmetry of $h_{\text{m.h.o.}}$, $\tilde{\Lambda}$ also remains a good quantum number. That is, every level with $\tilde{\Lambda} > 0$ forms a pseudo-spin doublet with $\Omega = \tilde{\Lambda} \pm 1/2$. The $\tilde{\Lambda} = 0$ orbitals are pseudo-spin $\Omega = 1/2$ singlets.

The matrix elements of the Hamiltonian (12) can be written as

$$\left\langle \tilde{N} \tilde{\varphi} \tilde{\Lambda} \mid h_{\text{m.h.o.}} \mid \tilde{N} \tilde{I} \tilde{\Lambda}\right\rangle_{\hbar \omega_0} = \left[ \left( \tilde{N} + \frac{3}{2} \right) + v_U \tilde{I}(\tilde{I} + 1) \right] \delta_{\tilde{N} \tilde{\varphi}} + \frac{2}{3} \delta(-1) \tilde{\Lambda} \left\langle \tilde{N} \tilde{\varphi} \mid \rho^2 \mid \tilde{N} \tilde{I}\right\rangle \times \sqrt{2(\tilde{I} + 1)(2\tilde{\varphi} + 1)} \begin{pmatrix} \tilde{I} & 2 & \tilde{\varphi} \\ \tilde{\Lambda} & 0 & -\tilde{\Lambda} \end{pmatrix} \begin{pmatrix} \tilde{I} & 2 & \tilde{\varphi} \\ 0 & 0 & 0 \end{pmatrix}. \hspace{1cm} (15)$$

In the actual calculations the constant $v_U = D/\hbar \omega_0$ was kept fixed at $v_U = -0.03$, a typical value for heavy nuclei [4].

The description in terms of the pseudo-Nilsson scheme outlined above is expected to work well if several assumptions are met. Namely,

(i) the pseudo-spin-orbit splitting can be neglected,
(ii) $\Delta N = 2, 4, \ldots$ couplings can be ignored,
(iii) couplings to the abnormal parity subshell (intruder) with $j = N + \frac{1}{2}$ can be ignored,
(iv) differences in the radial matrix elements for the $j = \frac{1}{2}$ pseudo-spin-orbit partners can all be ignored, and in addition,
(v) the pseudo-quadrupole interaction can be approximated by a quadrupole tensor in the pseudo-spin representation.

Because the quadrupole fields $h_{\delta}$ and $\tilde{h}_{\delta}$ are different, the deformations $\delta$ entering Eqs. (11) and (13) are not identical. For the sake of comparison between the normal and pseudo-Nilsson schemes a relationship between the two deformations $\delta_{\text{m.h.o.}}$ and $\tilde{\delta}_{\text{m.h.o.}}$ needs to be introduced. By comparing the diagonal radial matrix elements of $h_{\delta}$ in the same shell, one obtains

$$\delta_{\text{m.h.o.}} \approx \frac{N + \frac{3}{2}}{N + \frac{1}{2}} \delta_{\text{m.h.o.}}, \hspace{1cm} (16)$$

see the discussion in Ref. [12].

The pseudo-Nilsson Hamiltonian (15) can be easily diagonalized, i.e.,

$$\hat{h}_{\text{m.h.o.}} \Psi_{\tilde{N} \tilde{\Lambda} k} = \varepsilon_{\delta}^{(\tilde{N} \tilde{\Lambda} k)} \Psi_{\tilde{N} \tilde{\Lambda} k}, \hspace{1cm} (17)$$
Figure 2. The spectrum of the pseudo-Nilsen model for $\tilde{N} = 3$ versus deformation $\delta (= \delta_{m.h.n})$. The Nilsson orbitals are labelled by means of the $\tilde{\Lambda}_k$ quantum numbers ($k$ orders the states energetically counting from the bottom. That is, $k = 1$ is the lowest level, $k = 2$ is the second level, and so on).

where the additional quantum number $k$ orders the eigenstates energetically (i.e., $k = 1$ for the lowest state, $k = 2$ for the second state, etc.). The expansion coefficients in the expression

$$\Psi_{\tilde{N}\tilde{\Lambda}k} = \sum_i a_{i\tilde{\Lambda}k} |\tilde{N}\tilde{\Lambda}k\rangle$$

(18)

define the pseudo-orbital content of the state $\Psi_{\tilde{N}\tilde{\Lambda}k}$. Particularly simple is the situation for $\tilde{N} = 3$, since in this case the diagonalization of the pseudo-Nilsen potential is reduced to a two-level problem. Indeed, the $\tilde{N} = 3$ shell consists of two subshells only, i.e., $\tilde{p}$ ($\tilde{\Lambda} = 0, 1$) and $\tilde{f}$ ($\tilde{\Lambda} = 0, 1, 2, 3$). There is only one pseudo-Nilsen orbital with $\tilde{\Lambda} = 2$ and one with $\tilde{\Lambda} = 3$. The deformation dependence of these levels is given by a simple expression

$$e_{\delta} - e_{\delta=0} = \frac{4}{3} \delta \hbar \omega_0 \frac{3\tilde{\Lambda}^2 - \tilde{\Lambda}(\tilde{\Lambda} + 1)}{(2\tilde{\Lambda} + 3)(2\tilde{\Lambda} - 1)} \left(\tilde{N} + \frac{3}{2}\right).$$

(19)

In particular the $\tilde{\Lambda} = 2$ orbital is not affected by deformation while the $\tilde{\Lambda} = 3$ orbital is strongly oblate driving. For the two $\tilde{\Lambda} = 0$ and $\tilde{\Lambda} = 1$ levels a simple $2 \times 2$ matrix must be diagonalized. The resulting level diagram, shown in Fig. 2, is to be compared with the Woods-Saxon diagram of Fig. 1. Remarkable agreement between the single-particle energies is found, for both oblate and prolate deformations. A slight curvature of the
Woods-Saxon levels with $\lambda = 2$ and 3 (i.e., $\frac{3}{2}-\frac{7}{2}$ and $\frac{3}{2}-\frac{5}{2}$ pseudo-spin-orbit doublets) results from a weak deformation dependence of $\omega_2$, see Eq. (10). The two levels with $\lambda = 1$ interact on the oblate side while those with $\lambda = 0$ cross at prolate deformations. This particular feature is also seen in the Woods-Saxon diagram.

4. Pseudo-spin symmetry in the Woods-Saxon model

The realistic average potential is often approximated through a Woods-Saxon deformed field. In this work we employ the axially-deformed, single-particle Woods-Saxon Hamiltonian of Refs. [13, 14] with the parameters of Ref. [15]. The nuclear surface is defined by means of the standard quadrupole deformation $\beta_2$:

$$R(\Omega) = c(\beta_2)R_o \left[ 1 + \beta_2 Y_{20}(\Omega) \right],$$

with $c(\beta_2)$ being determined from the volume-conservation condition and $R_o = r_o \times A^{1/3}$. The Woods-Saxon field is given by

$$V(\vec{r}, \beta_2) = \frac{V_0}{1 + \exp \left[ \text{dist}_\Sigma(\vec{r}, \beta_2)/a \right]}$$

with $\text{dist}_\Sigma(\vec{r}, \beta_2)$ being the distance between the nuclear surface $\Sigma$ and the space point $\vec{r}$ at a given deformation $\beta$. The single-particle Hamiltonian also contains a deformed spin-orbit term

$$V_{SO}(\vec{r}, \beta_2) = \lambda \left( \frac{\hbar^2}{mc} \right)^2 \left( \frac{V_0}{1 + \exp \left[ \text{dist}_{SO}(\vec{r}, \beta_2)/a \right]} \right) (\vec{\sigma} \times \vec{\sigma})$$

and a Coulomb potential generated by a uniform charge distribution of $(Z-1)$ protons inside the nuclear surface $\Sigma$. The single-particle Schrödinger equation is solved in a deformed oscillator basis $|n_p, n_z, \lambda, \Sigma\rangle$ to obtain the nuclear single-particle energy levels (presented earlier in Fig. 1) and wave functions.

It is worth emphasizing, that the deformed Woods-Saxon field discussed above differs in many respect from the modified harmonic oscillator model of Eq. (7). In particular, the spin-orbit interaction, Eq. (22), contains an explicit deformation dependence and there is no velocity-dependent $l^2$ term. (For a systematic comparison between these two models we refer the reader to Ref. [14].) In contrast to the pseudo-Nilsson model calculations

(i) the pseudo-spin-orbit splitting is not neglected,

(ii) $\Delta N = 2, 4, \ldots$ couplings are taken into account correctly,

(iii) coupling to the abnormal parity subshell (intruder) with $j = N + \frac{1}{2}$ are included,

(iv) radial matrix elements for the pseudo-spin-orbit partners $j = \tilde{l} \pm \frac{1}{2}$ are different.
Figure 3. The pseudo-Nilsson model amplitudes $|a_{\tilde{N}k}|^2$ of Eq. (18) for the $\tilde{N} = 3$ shell as functions of deformation $\delta_{m,h.o.}$. Only the amplitudes for the states $\tilde{A}_k = \tilde{0}_1$ and $\tilde{1}_1$ are shown, see the discussion in the text.

The striking similarity between the single-particle diagram of the Woods-Saxon potential and that of the pseudo-Nilsson model offers a very strong argument for the presence of a self-consistent pseudo-spin symmetry of realistic average potentials. However, in order to investigate this issue in detail, it is important to systematically compare the wave functions in the two different models. By doing so, we will be able to check explicitly the validity of the pseudo-spin concept.

In order to analyze the pseudo-spin content of deformed Woods-Saxon wave functions, the eigenstates of the Woods-Saxon Hamiltonian, $\Psi_{\Omega \nu k}$, were expanded in the spherical basis:

$$\Psi_{\Omega \nu k} = \sum_{N,l,j} a_{Nlj}^{\Omega \nu k} |Nlj\Omega\rangle.$$  \hspace{1cm} (23)

Summing over all contributions from different major shells one obtains the contribution to a given Woods-Saxon model eigenstate from a spherical subshell $l_j$. In the next step, carried out for the normal-parity orbitals only, we perform the transformation to the pseudo-spin representation. Formally, this is done by a simple relabelling of the spherical states in eq. (23). For instance, for the negative-parity states originating from the $p_{1/2}$, $p_{3/2}$, $f_{5/2}$, $f_{7/2}$, and $h_{11/2}$ spherical subshells* the following transformation can be made: $p_{1/2} \rightarrow \tilde{s}$, $(p_{3/2}, f_{5/2}) \rightarrow \tilde{d}$, and $(f_{7/2}, h_{11/2}) \rightarrow \tilde{g}$. By means of this simple relabelling, one is able to express

* In the modified harmonic oscillator model these are $N = 5$ states. However, in the Woods-Saxon model $N$ is not a good quantum number.
the structure of the Woods-Saxon eigenstates in the pseudo-spin representation, i.e., to express them in the basis of the pseudo-Nilsson model:

\[
\Psi_{\Omega \pi k} = \sum_l a_l^{\Omega \pi k} |I\Omega\rangle + \sum_{l,j \in \text{int.}} a_{lj}^{\Omega \pi k} |j\Omega\rangle
\]  \hspace{1cm} (24)

where the second term on the r.h.s. in Eq. (24) represents the contribution from intruder states.

It is now interesting to compare the deformation dependence of the expansion coefficients of the pseudo Nilsson model with those of the Woods-Saxon model, i.e., the squared amplitudes \( |a_j^{\Omega \pi k}|^2 \) in Eq. (18) and \( |a_l^{\Omega \pi k}|^2 \) in Eq. (24). In this contribution we only give a few examples and refer to [9] for a more systematic comparison. The pseudo-Nilsson model amplitudes for the \( \tilde{N} = 3 \) shell are shown in Fig. 3 as a function of deformation \( \delta \) (=\( \delta_{\text{m.h.o.}} \)). Only the amplitudes for the states \( \tilde{\Lambda}_k = 0_1 \) and \( \tilde{I}_1 \) are shown. Since the case of the \( \tilde{N} = 3 \) shell represents a two-level problem, the amplitudes for the \( \tilde{\Lambda}_k = 0_2 \) and \( \tilde{\Lambda}_k = 1_2 \) states are obtained by exchanging the \( \tilde{p} \) and \( \tilde{f} \) labels. The corresponding Woods-Saxon amplitudes are displayed in Figs. 4 and 5. In order to make a meaningful comparison, the Woods-Saxon quadrupole deformation was transformed into \( \delta_{\text{m.h.o.}} = 0.94 \beta_2 \) and then \( \delta_{\text{m.h.o.}} \) was found by means of Eq. (16). There is a striking similarity between the wave functions of the pseudo-Nilsson model and those of the Woods-Saxon model for all the \( \tilde{N} = 3 \) states. In particular, there is an almost exact degeneracy between the \( \tilde{p} \) and \( \tilde{f} \) amplitudes of the \( \frac{1}{2} - \frac{3}{2} \) (\( \tilde{\Lambda} = 1 \)) pseudo-spin doublets, see Fig. 5. It is remarkable that even the \( \Delta \tilde{I} = 2 \) crossings seen in the structure of the 0 and \( \tilde{I} \) orbitals, happen at very similar deformations in the two models.
The magnitude of the coupling to the $j=N+\frac{1}{2}$ intruder orbitals is displayed in Figs. 6 and 7 for the lowest $\tilde{A}=1$ and 2 pseudo-spin doublets of the $\tilde{N}=3$ and 4 shells, respectively. In our calculations these states are the most mixed. Nonetheless in the range of deformation considered the admixture of the $g_{\frac{7}{2}}$ (Fig. 6) and the $h_{\frac{11}{2}}$ (Fig. 7) components does not exceed 10%. The influence of the high-$j$ states originating from the higher shells (such as $i_{\frac{13}{2}}$ for $\tilde{N}=3$ and $j_{\frac{15}{2}}$ for $\tilde{N}=4$) is much weaker. We consider this result as strong microscopic support for the pseudo-spin coupling scheme, in which the high-$j$ unique parity orbitals are decoupled from the Hilbert space of the natural-parity states.

As another test of the pseudo-Nilsson model wave functions, in the next section the magnetic moments of odd-$A$ deformed nuclei in the rare-earth region are systematically calculated and compared with experimental data.

5. MAGNETIC MOMENTS OF ODD $A$ NUCLEI IN THE PSEUDO-NILSSON MODEL

The magnetic dipole operator

$$\mu = g_R I + (g_I - g_R) l + (g_S - g_R) s$$

(25)

transformed to the pseudo-spin coupling scheme takes the form [3]

$$\mu = g_R I + (g_S - g_R) j - \frac{1}{3}(g_S - g_I) s + (g_S - g_I) \sum_{\tilde{l}, \tilde{r}} \mu_{\tilde{l}\tilde{r}} t(\tilde{r}, \tilde{l}) \sum_{\tilde{l}\tilde{r}}$$

(26)
Woods–Saxon.

\[ \tilde{N} = 4 \quad h_{\frac{1}{2}} \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6}
\caption{The squared amplitudes $|a_{1l}|^2$ of the $g_{\frac{3}{2}}$ intruder states, Eq. (24), as functions of deformation $\delta$ for the $\tilde{N} = 3, \tilde{1}_1$ and $\tilde{2}_1$ Woods-Saxon pseudo-spin doublets of Fig. 1.}
\end{figure}

[The definitions of $\mu_{l/2}$ and $t^{(\tilde{l}, \tilde{s})}_{L=2, \tilde{s}=1, J=1}$ can be found in Ref. [3], Eqs. (6) and (A.9).]

Assuming the strong coupling limit of the particle-plus-rotor model, the magnetic moment for the ground state of a rotational band with $I = K = \Omega$ is given by [3]

\[ \langle \tilde{\Lambda} \Omega k | \mu_3 | \tilde{\Lambda} \Omega k \rangle = g_R \frac{\Omega}{\Omega + 1} + g_t \frac{\Omega^2}{\Omega + 1} + \frac{(-1)^{\tilde{N}}}{3} (g_t - g_R) \delta_{\tilde{\Lambda}0} \]

\[ - (g_s - g_t) \frac{\Omega}{2(\Omega + 1)} \frac{\tilde{N}}{\tilde{N}_3} \left( 2\alpha_{\tilde{N} \tilde{A}k} + \frac{1}{3} \right) \]

\[ + \frac{(-1)^{\tilde{N}}}{3} (g_s - g_t) \delta_{\tilde{\Omega}3} \left[ (\alpha_{\tilde{N} \tilde{A}k} - \frac{1}{3}) \delta_{\tilde{\Lambda}3} + \beta_{\tilde{N} \tilde{A}k} \delta_{\tilde{A}3} \right], \quad (27) \]

where

\[ \alpha_{\tilde{N} \tilde{A}k} \equiv \sum_{i, \tilde{l}} \mu_{\tilde{l}i} a_{i \tilde{A}k} a_{\tilde{l} \tilde{A}k} \frac{(-1)^{\tilde{N}}}{\sqrt{(2\tilde{l} + 1)}} \langle \tilde{l} \tilde{A}20 | \tilde{I} \tilde{A} \rangle \quad (28) \]

and

\[ \beta_{\tilde{N} \tilde{A}k} \equiv -\sqrt{6} \sum_{i, \tilde{l}} \mu_{\tilde{l}i} a_{i \tilde{A}k} a_{\tilde{l} \tilde{A}k} \frac{(-1)^{\tilde{N}}}{\sqrt{(2\tilde{l} + 1)}} \langle \tilde{l} 12 - 2 | \tilde{I} - 1 \rangle, \quad (29) \]

where the expansion coefficients $a_{i \tilde{A}k}$ are given by Eq. (18).
FIGURE 7. The squared amplitudes $|a_{\lambda I}|^2$ of the $h_{11}$ intruder states, Eq. (24), as functions of deformation $\delta$ for the $N = 4$, $\tilde{1}_1$ and $\tilde{2}_1$ Woods-Saxon pseudo-spin doublets of Fig. 1.

Using Eq. (27), the magnetic moments of odd-$A$ deformed rare-earth nuclei were determined. The wave functions of the pseudo-Nilsson model were calculated at experimental deformations of the even-even cores taken from the compilation [16] (in most cases, the dependence of the magnetic moments on $\delta$ was found to be rather weak). For the gyromagnetic ratios, the standard values $g_R = Z/A$, $g_I = g_I^{\text{free}}$, and $g_s = 0.8 g_s^{\text{free}}$ were employed. The results are displayed in Tables 1 and 2 for protons ($N = 3$) and neutrons ($N = 4$), respectively. One should mention that the above formulae are much simpler as compared to those employing the full Nilsson model. Nevertheless, a rather close agreement with experimental values (taken from the compilation [17]) was found.

Finally it is interesting to note, that the pseudo-spin model predicts an exact degeneracy of the magnetic moments for the $\Lambda = 1$, $\Sigma = \frac{1}{2}$, $n = 1, 2$ levels and for the the $\Lambda = 0$, $\Sigma = \frac{1}{2}$, $n = 1, 2, 3$ levels —independent of $\delta$. However, there is insufficient experimental data available to finally check this prediction.

6. SUMMARY

The pseudo-spin concept has been extended to prolate and oblate nuclei with finite deformation by means of the pseudo-Nilsson model. Calculations were performed for axially symmetric, quadrupole-deformed systems with $N = 3, 4$.

Detailed comparison between single-particle energies and wave functions of the pseudo-Nilsson model and the deformed Woods-Saxon model confirms the presence of the pseudo-spin symmetry in the realistic average potential at both oblate and prolate deformations.
Table 1. Comparison of experimental ($\mu_{\text{Ex}}$) and theoretical ($\mu_{\text{Th}}$) magnetic moments for odd-$N$ nuclei in the $\bar{N} = 4$ shell. $\hat{A}$ is the projection of the neutron orbital angular momentum in the pseudo space, $\Omega$ is the projection of the total angular momentum, and $\beta_2$ is the quadrupole deformation of the even-even core. A "±"-sign in the $\mu_{\text{Ex}}$ column indicates insufficient experimental information about the magnetic moment sign.

<table>
<thead>
<tr>
<th>$\hat{A}$</th>
<th>$\Omega$</th>
<th>$\beta_2$</th>
<th>$\mu_{\text{Ex}} [\mu_B]$</th>
<th>$\mu_{\text{Th}} [\mu_B]$</th>
<th>Nucleus</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>7/2</td>
<td>0.33</td>
<td>±0.40</td>
<td>+0.85</td>
<td>$^{175}$Yb</td>
</tr>
<tr>
<td>3</td>
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<td>+0.79</td>
<td>+0.86</td>
<td>$^{177}$Hf</td>
</tr>
<tr>
<td>3</td>
<td>5/2</td>
<td>0.34</td>
<td>±0.66</td>
<td>-0.21</td>
<td>$^{171}$Er</td>
</tr>
<tr>
<td>3</td>
<td>5/2</td>
<td>0.33</td>
<td>-0.68</td>
<td>-0.21</td>
<td>$^{173}$Yb</td>
</tr>
<tr>
<td>2\textsuperscript{1}</td>
<td>5/2</td>
<td>0.15</td>
<td>±0.58</td>
<td>+0.40</td>
<td>$^{147}$Nd</td>
</tr>
<tr>
<td>2\textsuperscript{1}</td>
<td>5/2</td>
<td>0.20</td>
<td>±0.35</td>
<td>+0.37</td>
<td>$^{149}$Nd</td>
</tr>
<tr>
<td>2\textsuperscript{1}</td>
<td>5/2</td>
<td>0.34</td>
<td>+0.67</td>
<td>+0.35</td>
<td>$^{163}$Dy</td>
</tr>
<tr>
<td>2\textsuperscript{1}</td>
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<td>0.32</td>
<td>+0.56</td>
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<tr>
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<td>±0.38</td>
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<td>$^{153}$Gd</td>
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<td>3/2</td>
<td>0.31</td>
<td>-0.26</td>
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<td>0.25</td>
<td>-0.30</td>
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<tr>
<td>2\textsuperscript{1}</td>
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<td>0.30</td>
<td>-0.37</td>
<td>+0.19</td>
<td>$^{161}$Er</td>
</tr>
<tr>
<td>2\textsuperscript{1}</td>
<td>3/2</td>
<td>0.22</td>
<td>-0.33</td>
<td>+0.19</td>
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</tr>
<tr>
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<td>0.26</td>
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<td>+0.18</td>
<td>$^{163}$Yb</td>
</tr>
<tr>
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<td>0.22</td>
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<td>+0.66</td>
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<tr>
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<td>0.19</td>
<td>±0.66</td>
<td>+0.66</td>
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</tr>
<tr>
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<td>+0.51</td>
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</tr>
<tr>
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<td>+0.49</td>
<td>+0.51</td>
<td>$^{171}$Tm</td>
</tr>
</tbody>
</table>
Table 2. Similar to Table 1 but for odd-Z nuclei in the $\tilde{N} = 3$ shell.

<table>
<thead>
<tr>
<th>$\tilde{A}$</th>
<th>$\Omega$</th>
<th>$\beta_2$</th>
<th>$\mu_{\text{Ex}}$ [$\mu_B$]</th>
<th>$\mu_{\text{Tb}}$ [$\mu_B$]</th>
<th>Nucleus</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>7/2</td>
<td>0.15</td>
<td>+2.58</td>
<td>+1.98</td>
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</tr>
<tr>
<td>3</td>
<td>7/2</td>
<td>0.20</td>
<td>±3.3</td>
<td>+1.98</td>
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</tr>
<tr>
<td>3</td>
<td>7/2</td>
<td>0.30</td>
<td>+2.40</td>
<td>+1.98</td>
<td>$^{161}$Tm</td>
</tr>
<tr>
<td>3</td>
<td>7/2</td>
<td>0.32</td>
<td>±2.03</td>
<td>+1.98</td>
<td>$^{171}$Lu</td>
</tr>
<tr>
<td>3</td>
<td>7/2</td>
<td>0.33</td>
<td>+2.34</td>
<td>+1.98</td>
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<td>3</td>
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<td>3</td>
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<td>+2.24</td>
<td>+1.98</td>
<td>$^{177}$Lu</td>
</tr>
<tr>
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</tr>
<tr>
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<td>+1.98</td>
<td>$^{177}$Ta</td>
</tr>
<tr>
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<td>+1.98</td>
<td>$^{181}$Ta</td>
</tr>
<tr>
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<td>0.21</td>
<td>±3.5</td>
<td>+3.03</td>
<td>$^{153}$Tb</td>
</tr>
<tr>
<td>3</td>
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<td>0.25</td>
<td>±3.42</td>
<td>+3.03</td>
<td>$^{159}$Tm</td>
</tr>
<tr>
<td>3</td>
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<td>0.25</td>
<td>±3.19</td>
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<tr>
<td>3</td>
<td>5/2</td>
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<td>+3.03</td>
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<tr>
<td>3</td>
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<td>+3.03</td>
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<tr>
<td>2</td>
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<td>0.28</td>
<td>±1.8</td>
<td>+1.66</td>
<td>$^{151}$Pm</td>
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<td>2</td>
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<td>±2.0</td>
<td>+1.49</td>
<td>$^{155}$Tb</td>
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<tr>
<td>2</td>
<td>3/2</td>
<td>0.34</td>
<td>±2.0</td>
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<td>$^{157}$Tb</td>
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<tr>
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<td>+2.0</td>
<td>+1.49</td>
<td>$^{159}$Tb</td>
</tr>
<tr>
<td>2</td>
<td>3/2</td>
<td>0.35</td>
<td>±2.2</td>
<td>+1.49</td>
<td>$^{161}$Tb</td>
</tr>
<tr>
<td>1_2</td>
<td>3/2</td>
<td>0.19</td>
<td>±0.13</td>
<td>+0.32</td>
<td>$^{189}$Ir</td>
</tr>
<tr>
<td>1_2</td>
<td>3/2</td>
<td>0.18</td>
<td>+0.15</td>
<td>+0.32</td>
<td>$^{191}$Ir</td>
</tr>
<tr>
<td>1_2</td>
<td>3/2</td>
<td>0.16</td>
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<tr>
<td>1_2</td>
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<td>0.20</td>
<td>+0.54</td>
<td>+1.80</td>
<td>$^{187}$Au</td>
</tr>
<tr>
<td>1_2</td>
<td>1/2</td>
<td>0.18</td>
<td>+0.49</td>
<td>+1.80</td>
<td>$^{189}$Au</td>
</tr>
<tr>
<td>1_1</td>
<td>1/2</td>
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<td>+0.75</td>
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<tr>
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<td>+0.75</td>
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</tr>
<tr>
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<td>+1.46</td>
<td>+0.76</td>
<td>$^{127}$Cs</td>
</tr>
</tbody>
</table>
able agreement was found between the wave functions, even at oblate deformations the $\Delta \lambda = 2$ coupling is strong.

One result of the calculations is the observation that the impurities in the normal-wave functions due to admixtures of high-$\lambda$ intruder subshells are small. This nicely as the main assumption of the pseudo-spin model, i.e., the weak coupling of intruder to the Hilbert space of the pseudo-Nilsson states.

To test the wave functions of the pseudo-Nilsson model, magnetic moments of deformed nuclei were computed within the strong coupling limit of the particle-plus-rotor model. In the simplicity of the pseudo-Nilsson model, general agreement was found between theoretical and experimental magnetic moments. This positive result strongly suggests that the wave functions of the pseudo-Nilsson model can be very useful when analysing single-particle data for the normal-parity states in deformed nuclei*.


enowledgments

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tences


* Many examples, e.g., magnetic moments, spectroscopic factors, etc., were discussed in Ref. [3] where, however, the main emphasis was on the asymptotic limit of the pseudo-Nilsson model.