INDUCTION AND SUBLUTION COEFFICIENTS OF HECKE AND COUPLING AND RE-COUPLING COEFFICIENTS OF U_q(N)

Feng Pan, a, b and J. P. Draayer

a Department of Physics, Liaoning Normal University, Dalian 116029, P. R. China
b Department of Physics & Astronomy, Louisiana State University, Baton Rouge LA 70803

1. Introduction

It is well-known that the symmetric (or permutation) groups and classical Lie algebras are very useful in dealing with quantum many-body problems. The interrelations between representations of the symmetric groups and those of the general linear groups is manifested by the Schur-Weyl duality, which enables us to calculate various coupling and re-coupling coefficients of unitary groups (or A-type Lie algebras) from those of the symmetric groups. Based on the Schur-Weyl duality, relations between CG coefficients of $S_f$ and various coupling coefficients of $SU(mn) \rightrightarrows SU(m) \times SU(n)$, those between Induction Coefficients (IDCs) of $S_{f_1} \times S_{f_2} \uparrow S_f$ and CG coefficients of $U(n)$, and those between Subduction Coefficients (SDCs) of $S_f \downarrow S_{f_1} \times S_{f_2}$ and Racah Coefficients (RCs) of $U(n)$ were extensively studied by Chen and his collaborators. Hence, one needs first to derive various coefficients of the symmetric groups, namely CG coefficients, IDCs, and SDCs. Once these coefficients are known, one can use relations between these coefficients of the symmetric groups and those of unitary groups to obtain various coupling and recoupling coefficients of the unitary groups.

The Eigenfunction Method (EFM) was proposed by Chen and his collaborators in evaluating various coefficients of the symmetric groups. Actually, the EFM has proven to be effective in evaluating coupling and recoupling coefficients not only for the symmetric groups, but also for point groups and space groups. Using these coefficients and relations resulting from the Schur-Weyl duality, Chen and his collaborators derived various
coupling and recoupling coefficients\footnote{7} for $SU(mn) \supset SU(m) \times SU(n)$ and $U(m+n) \supset U(m) \times U(n)$. All of these achievements are summarized in Professor Chen’s book\footnote{6} which has been and continues to be used in classrooms and by researchers worldwide since its publication.

Since a direct extension of the EFM for Hecke algebras proved not to be feasible, a Linear Equation Method (LEM) was proposed by Pan and Chen\footnote{9}. This method is not only useful for evaluating SDCs and IDCs of Hecke algebras, it is also useful for evaluating these coefficients for Brauer and Birman-Wenzl algebras. The LEM can also be used for constructing irreducible representations (irreps) of Brauer or Birman-Wenzl algebras based on representation theory of the symmetric group or that of Hecke algebras. Using SDCs and IDCs derived from LEM with the Schur-Weyl-Brauer duality relations, one can obtain CG and Racah coefficients of quantum group $U_q(n)$ and the same coefficients of $O_q(n)$ and $Sp_q(2n)$\footnote{23-30}. In section 2, the LEM for evaluating SDCs and IDCs of Hecke algebras will be reviewed. The symmetrization method for the derivation of CG coefficients of $U_q(n)$ from IDCs of Hecke algebras will be introduced in section 3, which is different from the assimilation method proposed in Chen’s early work\footnote{5-6}.

2. SDCs and IDCs of Hecke algebras

2.1. Irreps of Hecke algebras $H_n(q)$

The Hecke algebra $H_n(q)$ is a special realization of the braid group, of which the standard basis has been studied by Jones, Wenzl, Jimbo, and many others\footnote{10} and \footnote{13}. Its standard generators satisfy the same relations as a set of simple reflections or adjacent permutations of the symmetric group $S_n$, except that the simple property $g_i^2 = 1$ is replaced by $g_i^2 = (q - q^{-1})g_i + 1$. It is known that $H_n(q)$ is isomorphic to the group algebra of $S_n$ if $q$ is not a root of unity. More precisely, let $H_n(q)$ be the Hecke algebra of type $A_{n-1}$ over $\mathbb{C}(q)$, the field of rational functions over $\mathbb{C}$. $H_n(q)$ is generated by \{\(g_1, g_2, \ldots, g_{n-1}\)\} satisfying the following braid relations

\[ g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \]

\[ g_i g_j = g_j g_i \quad \text{for} \quad |i - j| \geq 2, \quad (2.1) \]

and the quadratic constraint
\[ g_i^2 = g_i(q - q^{-1}) + 1. \] (2.2)

Let \( \lambda_m^{[\lambda]} \) be a standard Young tableau and \( |\lambda_m^{[\lambda]}\rangle_q \) an associated orthogonal basis vector so that the \( |\lambda_m^{[\lambda]}\rangle_q \) satisfy \( \langle \lambda_m^{[\lambda]} | \lambda_m^{[\lambda']} \rangle_q = \delta_{mn'} \), where \( |\lambda\rangle = [\lambda_1 \lambda_2 \cdots \lambda_n] \) with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \) and \( \sum \lambda_i = n \), the total number of boxes in the Young diagram, denotes an irrep of \( H_n(q) \) and \( |\lambda\rangle_m \) denotes a Young operator which acts on the indices \( (1, \cdots, n) \). Let \( g_i (Y_m^{[\lambda]}) \) be the standard Young tableau obtained by interchanging the numbers \( i \) and \( i + 1 \) in the standard tableau \( Y_m^{[\lambda]} \); if \( g_i (Y_m^{[\lambda]}) \) is not a standard tableau, one sets the corresponding vector equal to zero. Then an irrep of \( H_n(q) \) in the standard basis, i.e. the basis adapted to the chain \( H_n(q) \supset H_{n-1}(q) \supset \cdots \supset H_2(q) \), is given by

\[ g_i |\lambda_m^{[\lambda]}\rangle_q = \frac{q^d}{[d]} |\lambda_m^{[\lambda]}\rangle_q + \left( \frac{[d+1][d-1]}{[d]^2} \right)^{1/2} g_i (Y_m^{[\lambda]})_q, \] (2.3)

where for a given \( x \)

\[ [x] = \frac{q^x - q^{-x}}{q - q^{-1}} \] (2.4)

and \( d \) is the axial distance from \( i \) to \( i + 1 \) in the Young tableau \( Y_m^{[\lambda]} \). This representation in unitary when \( q^* = q^{-1} \) or \( q = 1 \). In the following we always assume that \( q \) is generic, i.e. it is not a root of unity.

2.2. SDCs

An irrep of \( H_n(q) \) is reducible with respect to its subalgebra \( H_{n_1}(q) \times H_{n_2}(q) \) with \( n_1 + n_2 = n \). The process of the reduction is the same as \( S_n \) and is denoted by

\[ |\lambda\rangle \downarrow (H_{n_1}(q) \times H_{n_2}(q)) = \sum_{\lambda} \{\lambda_1 \lambda_2 \lambda\} \langle [\lambda_1], [\lambda_2] | [\lambda] \rangle. \] (2.5)

The orthogonal subduced basis of \( H_n(q) \supset H_{n_1}(q) \times H_{n_2}(q) \) is known as the non-standard basis of \( H_n(q) \), which is denoted by

\[ |[\lambda], [\lambda_1], [\lambda_2] \rangle_{m_1 m_2} \rangle_q, \] (2.6)
where \( \tau = 1, 2, \ldots \), \( \{ \lambda_1, \lambda_2 \lambda \} \) is the multiplicity label, and the set of quantum numbers \( (\tau | \lambda_1 m_1 | \lambda_2 | m_2 ) \) serves as its components. One can expand the non-standard basis of \( H_n(q) \) in terms of the standard basis of \( H_n(q) \) with

\[
| \lambda \rangle, \tau [\lambda_1 | \lambda_2 ]_m \rangle = \sum_m | \lambda \rangle \langle \lambda | \tau [\lambda_1 | \lambda_2 ]_m \rangle .
\]

(2.7)

The expansion coefficient is called \( | \lambda \rangle \downarrow | \lambda_1 \times | \lambda_2 \rangle \) SDC, or a transformation coefficient between the standard and non-standard bases of \( H_n(q) \). Because the standard and non-standard basis vectors are orthogonal, the SDCs satisfy the unitarity conditions

\[
\sum_{\tau, \lambda_2, m_2} \langle \lambda \rangle \langle m | \tau [\lambda_1 | \lambda_2 ]_m \rangle \langle \lambda \rangle \langle \lambda | \tau [\lambda_1 | \lambda_2 ]_m \rangle = \delta_{m m'},
\]

(2.8a)

\[
\sum_m \langle \lambda \rangle \langle m | \tau [\lambda_1 | \lambda_2 ]_m \rangle \langle \lambda \rangle \langle \lambda | \tau [\lambda_1 | \lambda_2 ]_m \rangle = \delta_{\tau \tau'} \delta_{\lambda_2 \lambda_2'} \delta_{m m}'.
\]

(2.8b)

As is noted before, the EFM has proven to be a powerful tool in the evaluation of the CGCs, SDCs and IDCs for symmetric groups. These coefficients result from a diagonalization of the CSCC-II of the symmetric group in appropriate spaces. The advantage of this method is that it can easily be programmed so the actual calculations can be accomplished with a computer. However, the EFM can only be used to calculate these coefficients numerically. In the Hecke algebra case, the representations are \( q \)-dependent. One wishes to obtain these coefficients with a generic \( q \). Thus the usual diagonalization process cannot be used. The LEM is proposed as an alternative for evaluating SDCs of Hecke algebras.

Assume that \( \{ g_1, g_2, \ldots, g_{n-1} \} \) is a set of generators of \( H_n(q) \), and \( \{ g_1, g_2, \ldots, g_{n-1} \} \) and \( \{ g_{n+1}, \ldots, g_{n-1} \} \) are the generators of \( H_{n+1}(q) \) and \( H_{n-1}(q) \), respectively. By applying \( g_i \) with \( i = 1, 2, \ldots, n - 1 \) and \( g_j \) with \( j = n + 1, n + 2, \ldots, n - 1 \) to (2.7), and then multiplying the result from the left with \( q \langle \lambda | \lambda \rangle_m \), we get two sets of linear equations

\[
\left( \frac{q^{d_{ii}}}{d_{ii}} - \frac{q^{d_{ij}}}{d_{ij}} \right) \langle \lambda \rangle \langle m | \tau [\lambda_1 | \lambda_2 ]_m \rangle =
\]

(2.8b)
\[ -\left( \frac{d_{i+1}}{d_{i+1}^2} \right)^\frac{1}{2} \left\langle \frac{[\lambda]}{m} \mid [\lambda], \frac{\tau_{\lambda_1}[\lambda_2]}{m_1 m_2}_q \right\rangle, \quad (2.9a) \]

\[ \left( \frac{q^{d_{i+1}} - q^{d_{i+1}}}{[d_{i+1}]^2} \right) \left\langle \frac{[\lambda]}{m} \mid [\lambda], \frac{\tau_{\lambda_1}[\lambda_2]}{m_1 m_2}_q \right\rangle = \]

\[ \left( \frac{d_{i+1}[d_{i+1} - 1]}{[d_{i+1}]^2} \right)^\frac{1}{2} \left\langle \frac{[\lambda]}{m} \mid [\lambda], \frac{\tau_{\lambda_1}[\lambda_2]}{m_1 m_2}_q \right\rangle, \quad (2.9b) \]

where \( d_{i+1}(d_{i+1}) \) is the axial distance from \( i \) to \( i + 1 \) (\( j \) to \( j + 1 \)) in the Young tableau \( Y^{[\lambda]}_{m_{i+1}} \), and \( d_{i} \) (\( d_{i} \)) is the axial distance from \( i \) to \( i + 1 \) (\( j \) to \( j + 1 \)) in the Young tableau \( Y^{[\lambda]}_{m_{j}} \). The Young tableaux \( Y^{[\lambda]}_{m_{j}}, Y^{[\lambda]}_{m_{j}}, Y^{[\lambda_1]}_{m_{i+1}}, Y^{[\lambda_2]}_{m_{i+1}} \) are defined by \( Y^{[\lambda]}_{m_{j}} = g_{i}Y^{[\lambda]}_{m_{j}}, Y^{[\lambda]}_{m_{j}} = g_{i}Y^{[\lambda]}_{m_{j}}, Y^{[\lambda_1]}_{m_{i+1}} = g_{i}Y^{[\lambda_1]}_{m_{i+1}}, Y^{[\lambda_2]}_{m_{i+1}} = g_{i}Y^{[\lambda_2]}_{m_{i+1}} \).

Assume that the dimensions of \([\lambda], [\lambda_1], \) and \([\lambda_2] \) are \( N_1, N_1 \) and \( N_2, \) respectively, which can be calculated by using the Robinson formula. In the non-multiplicity-free case there are \( N_1 N_1 N_2 \) SDCs. Eq. (2.9) provides \( N_1 N_2 (N - 1) \) linearly independent relations among the SDCs. Then, using the unitarity condition (2.8b) with

\[ \sum_{m} \left\langle \frac{[\lambda]}{m} \mid [\lambda], \frac{\tau_{\lambda_1}[\lambda_2]}{m_1 m_2}_q \right\rangle \left\langle \frac{[\lambda]}{m} \mid [\lambda], \frac{\tau_{\lambda_1}[\lambda_2]}{m_1 m_2}_q \right\rangle = 1, \quad (2.10) \]

one can obtain all the SDCs for the given irreps \([\lambda_1], [\lambda_2], \) and \([\lambda] \) because there are exactly \( N_1 N_2 \) unitarity conditions for the SDCs. Thus, relations (2.9) and (2.10) are sufficient for solving the SDCs when the subduction is multiplicity-free. In the multiplicity case, similarly to the multiplicity-free case, (2.9) and (2.10) provide \( N_1 N_2 \) linearly independent relations for the fixed multiplicity label \( \tau \). These relations are sufficient to solve the SDCs with fixed multiplicity label. However, the same relations hold for any other multiplicity labels. In order to resolve this multiplicity ambiguity, we can use these relations to derive the SDCs for a fixed multiplicity label. Then, the SDCs with different multiplicity labels are chosen to be orthogonal to each other. In this case the solution to the SDCs is not unique, and also
depends on the phase convention and the symmetry properties of SDCs. The phase convention for SDCs of Hecke algebras is chosen to be the same as that for the symmetric groups given in 6.

2.3. IDCs

The representation of $H_f(q)$ induced from irreps $([\nu_1],[\nu_2])$ of its subalgebra $H_{f_1}(q) \times H_{f_2}(q)$ is called the induced representation, or outer-product representation of $H_f(q)$ denoted by $([\nu_1],[\nu_2]) \uparrow H_f(q)$, or $[\nu_1] \otimes [\nu_2]$, which can be reduced into the direct sum of irreps of $H_f(q)$ as

$$[\nu_1] \otimes [\nu_2] = \sum_{\nu} \{\nu_1 \nu_2 \nu\} [\nu], \quad (2.24)$$

where the integer $\{\nu_1 \nu_2 \nu\}$ is the number of occurrence of the irrep $[\nu]$ in the induced representation. More specifically, the standard basis vectors of $H_{f_1}(q)$ and $H_{f_2}(q)$ should be denoted by $[Y_{m_1}^{[\nu_1]}(\omega_0^0)]_q$ and $[Y_{m_2}^{[\nu_2]}(\omega_0^0)]_q$, respectively, where $\omega_0^0 = (1, 2, \cdots , f_1)$ and $\omega_0^0 = (f_1+1, f_2+1, \cdots , f_1+f_2)$ are indices filled in the standard tableaux $Y_{m_1}^{[\nu_1]}$ and $Y_{m_2}^{[\nu_2]}$. The product of two basis vectors is denoted by

$$[[\nu_1]_{m_1} [\nu_2]_{m_2}, (\omega_0^0)]_q \equiv [Y_{m_1}^{[\nu_1]}(\omega_0)]_q [Y_{m_2}^{[\nu_2]}(\omega_0)]_q, \quad (2.25)$$

where

$$(\omega_0^0) = (\omega_0^0, \omega_0^0) = (1, 2, \cdots , f_1 + f_2). \quad (2.26)$$

Define $h = \left( \begin{array}{c} f_1 \\ \end{array} \right)$ normal-order sequences,

$$(w) = (\omega_1, \omega_2), \quad (w) = (a_1, a_2, \cdots , a_{f_1}), \quad (w) = (a_{f_1+1}, a_{f_2+1}, \cdots , a_f) \quad (2.27)$$

with $a_1 < a_2 < \cdots < a_{f_1}, a_{f_1+1} < a_{f_2+1} < \cdots < a_f$, where $\omega_i$ represents any one of the numbers $1, 2, \cdots , f$. For example, $f_1 = 2, f_2 = 1, h = 3$, the three normal-order sequences are $(12,3), (13,2)$, and $(23,1)$.

The left coset decomposition of $H_f(q)$ with respect to the subalgebra $H_{f_1}(q) \times H_{f_2}(q)$ is denoted by

$$H_f(q) = \sum_{\omega=1}^{k} \oplus Q_\omega (H_{f_1}(q) \times H_{f_2}(q)), \quad (2.28)$$
where the left coset representatives \( q_\omega \) are the \( q \)-deformed order-preserving permutations with

\[
Q_\omega(\omega^0) = (\omega). \tag{2.29}
\]

Applying the \( h_{Q_\omega} \)'s to (2.25) gives \( h_{\nu_1} h_{\nu_2} h \) basis vectors, where \( h_{\nu_1} \) and \( h_{\nu_2} \) are the dimensions of the irreducible representations of \([\nu_1]\) and \([\nu_2]\), respectively.

\[
Q_\omega([\nu_1]m_1[\nu_2]m_2, (\omega^0))_q = [Y_{m_1}^{[\nu_1]}Y_{m_2}^{[\nu_2]}, (\omega)]_q = [Y_{m_1}^{[\nu_1]}(\omega_1), Y_{m_2}^{[\nu_2]}(\omega_2)]_q, \tag{2.30}
\]

where \( Y_{m_i}^{[\nu_i]}(\omega_i) \) denotes a generalized Young tableau formed by filling the Young diagram \([\nu_i]\) with the numbers \( \omega_i \) according to the order specified by the Yamanouchi symbol \( m_i \). We call the basis vectors (2.10) the normal-ordered uncoupled basis vectors of \( H_{f_1}(q) \times H_{f_2}(q) \).

It is known from the symmetric group case \(^6\) that the normal-ordered basis vectors are orthogonal in the indices \( m_1 \) and \( m_2 \), but not necessarily in \( \omega \). However, if we assume that the indices in each box of the Young diagram \([\nu_i]\) are state labels, then the basis vectors given by (2.30) are also orthogonal with respect to \( \omega \). In the following, we always assume that this is the case. So the normal-ordered basis vectors (2.30) satisfy

\[
\delta([\nu_1]m_1[\nu_2]m_2, (\omega)) = \delta_{m_1m_1'}\delta_{m_2m_2'}\delta_{\omega\omega'}. \tag{2.31}
\]

It should be noted that because of the property of the Hecke algebra given in (2.2), we should always write basis vectors \([Y_{m_1}^{[\nu_1]}(\omega_1), Y_{m_2}^{[\nu_2]}(\omega_2)]_q \) explicitly in the operator form \( Q_\omega([\nu_1]m_1[\nu_2]m_2, (\omega^0))_q \). For example, for \( f_1 = 2 \) and \( f_2 = 1 \), we have \( Q_\omega = \{1, g_2, g_1g_2\} \) with three normal-ordered basis vectors

\[
[2],[1],(123)]_q = [2][12],[1][3)]_q, \quad [2][13],[1][2)]_q = g_2[2][12],[1][3)]_q,
\]

\[
[2][23],[1][1)]_q = g_1g_2[2][12],[1][3)]_q. \tag{2.32}
\]

In the case \( q = 1 \) case, i.e. for the symmetric group \( S_f \), we know that

\[
g_2[2][13],[1][2)] = [2][12],[1][3)]_q, \quad g_1[2][23],[1][1)] = [2][13],[1][2)]. \tag{2.33}
\]

However, (2.33) is incorrect for \( q \neq 1 \) since
\[ g_2 [\{2\}(13), \{1\}(2)]_q = g_2^2 [\{2\}(12), \{1\}(3)]_q = \]

\[ (q - q^{-1}) [\{2\}(13), \{1\}(2)]_q + [\{2\}(12), \{1\}(3)]_q. \]  \hspace{1cm} (2.34)

Thus, basis vectors \([\nu]\tau, m_q)\) of the \(\tau\)-th irrep of \([\nu]\) of \(H_f(q)\) can be obtained from a linear combination of the normal-ordered uncoupled basis vectors of \(H_{f_1}(q) \times H_{f_2}(q)\),

\[ [\nu]\tau, m_q) = \sum_{m_1 m_2 \omega} C_{\nu \nu' m_1 \omega m_2 \omega}(q) [\nu_1] m_1 [\nu_2] m_2, (\omega))]_q, \]  \hspace{1cm} (2.35)

where \(\tau\) is the outer multiplicity label needed in the outer-product \([\nu_1] \times [\nu_2] \uparrow [\nu]\), and the coefficients

\[ C_{\nu \nu' m l m_1 \omega m_2 \omega}(q) \equiv C_{\nu \nu' m l m_1 \omega m_2 \omega}(q) \]  \hspace{1cm} (2.36)

is the \([\nu_1] \times [\nu_2] \uparrow [\nu]\) IDCs or the outer-product reduction coefficients. They satisfy the following unitarity conditions,

\[ \sum_{m_1 m_2 \omega} C_{\nu \nu' m l m_1 \omega m_2 \omega}(q) C_{\nu \nu' m l m_1 \omega m_2 \omega}(q) = \delta_{\nu \nu'} \delta_{\tau \tau'} \delta_{m m'}, \]  \hspace{1cm} (2.37a)

\[ \sum_{m_1 m_2 \omega} C_{\nu \nu' m l m_1 \omega m_2 \omega}(q) C_{\nu \nu' m l m_1 \omega m_2 \omega}(q) = \delta_{\omega \omega'} \delta_{m_1 m_1'} \delta_{m_2 m_2'}. \]  \hspace{1cm} (2.37b)

In the following, the LEM will be applied to the evaluation of IDCs of \(H_f(q)\). Let \(\{g_1, g_2, \cdots, g_{f-1}\}\) be a set of generators of \(H_f(q)\). Applying the operators \(g_i\) with \(i = 1, 2, \cdots, f - 1\) to (2.35), the left-hand side of (2.35) becomes

\[ g_i [\nu]\tau, m_q) = \sum_{m_1 m_2 \omega} \left( \frac{q^{d_i}}{d_i} C_{\nu \nu' m l m_1 \omega m_2 \omega}(q) + \left( \frac{[d_i - 1] [d_i + 1]}{[d_i]^2} \right)^{\frac{1}{2}} \right) \]

\[ C_{\nu \nu' m l m_1 \omega m_2 \omega}(q) \times Q_{\omega} [\nu_1] m_1 [\nu_2] m_2, (\omega^0))]_q \]  \hspace{1cm} (2.38a)

where \(d_i\) is the axial distance from the box \(i\) to \(i + 1\) in the Young tableau \(Y_{m_1}^{\nu}\), while \(m'\) is specified by \(Y_{m'}^{\nu} = g_i Y_{m}^{\nu}\). The last term in (2.38a) equals zero if \(Y_{m'}^{\nu}\) is not a standard Young tableau. While the right-hand side of (2.35) becomes
\[
\sum_{m_1, m_2, w_2} C^{[\nu_1 \tau, \nu_1 \tau, m_1 \omega_1, \nu_2 m_2 \omega_2]}_{\nu_2, m_1 \omega_1, \nu_2 m_2 \omega_2} (q) \left( g_1 \right) \left| \nu_1 \right| \left| \nu_2 \right| m_1 m_2, (\omega_0) \right) q. 
\] (2.38b)

Combining (2.38a) and (2.38b) we get
\[
\frac{q^d}{d_4} C^{[\nu_1 \tau, \nu_1 \tau, m_1 \omega_1, \nu_2 m_2 \omega_2]}_{\nu_2, m_1 \omega_1, \nu_2 m_2 \omega_2} (q) + \left( d_4 - 1 \right) \left( \frac{d_4 + 1}{d_4^2} \right) C^{[\nu_1 \tau, \nu_1 \tau, m_1 \omega_1, \nu_2 m_2 \omega_2]}_{\nu_2, m_1 \omega_1, \nu_2 m_2 \omega_2} (q) = f_i(q) C^{[\nu_1 \tau, \nu_1 \tau, m_1 \omega_1, \nu_2 m_2 \omega_2]}_{\nu_2, m_1 \omega_1, \nu_2 m_2 \omega_2} (q),
\] (2.39)

where \(f_i(q)\) is a polynomial in \(q\), and \(f_i(q) C^{[\nu_1 \tau, \nu_1 \tau, m_1 \omega_1, \nu_2 m_2 \omega_2]}_{\nu_2, m_1 \omega_1, \nu_2 m_2 \omega_2} (q)\) is the coefficient in the front of \(Q_\omega \left| \nu_1 \right| \left| \nu_2 \right| m_1 m_2, (\omega_0) \right) q\) after applying the operator \(g_i\) to the right-hand side of (2.35). In the induction \([\nu_1] \times [\nu_2] \leftarrow [\nu] \) there are \(h_\nu h_{\nu_1} h_{\nu_2} \left( f_{f_1} \right)\) normal-ordered uncoupled basis vectors of \(H_{f_1} (q) \times H_{f_2} (q)\).

In the multiplicity-free cases, for a given irrep \([\nu] \) of \(H_f(q)\), there are \(h_\nu h_{\nu_1} h_{\nu_2} \left( f_{f_1} \right)\) IDCs. However, Eq. (2.39) will yield \((f - 1) h_\nu h_{\nu_1} h_{\nu_2} \left( f_{f_1} \right)\) linear relations among the IDCs. Similar to the case of SDCs, there are \((f - 2) h_\nu h_{\nu_1} h_{\nu_2} \left( f_{f_1} \right) + 1\) trivial and redundant relations among the IDCs.

Hence, we only need to choose \(h_\nu h_{\nu_1} h_{\nu_2} \left( f_{f_1} \right)\) linear independent relations from (2.39). Using unitarity conditions (2.37), we can obtain all the IDCs for given irreps \([\nu_1], [\nu_2], \) and \([\nu] \). The relative phase for the IDCs is determined by the Yamanouchi phase convention, while the overall phase is fixed by requiring that the IDC with \(m_1 = m_2 = m = 1\) and with the smallest possible index \(\omega\) is chosen positive.

\[
C^{[\nu_1 \tau, 1]}_{[\nu_1] \left| 1 \right| \left| 1 \right| \omega} (q) \big|_{\omega = \text{min}} > 0,
\] (2.40)

which is consistent with the phase convention for the IDCs of symmetric groups. IDCs of \(H_f(q)\) with \(f \leq 4\) were shown in 14.

3. RCs and CGCs of quantum groups \(U_q(n)\)

3.1. Racah coefficients

The \(U_q(n)\) RCs are simply a generalization of the \(SU_q(2)\) RCs, which are the elements of a unitary matrix between bases with two different coupling.
orders of three irreps \( \nu_1, \nu_2 \) and \( \nu_3 \) of \( U_q(n) \),

\[
|\langle \nu_1 \nu_2 ; \nu_3 ; \nu \rangle \rangle_{q}^{123t} = \sum_{t_{23}t_{23}t'} U_q(\nu_1 \nu_2 \nu_3 ; \nu_1 \nu_2 \nu_3)_{t_{23}t_{23}t'} \langle \nu_1 , (\nu_2 \nu_3) \nu_2 \nu_3 ; \nu \rangle_{q}^{123t'},
\]

where four multiplicity labels appeared,

\[
t_{12} = 1, 2, \ldots , \{ \nu_1 \nu_2 \nu_{12} \}, \quad t_{23} = 1, 2, \ldots , \{ \nu_2 \nu_3 \nu_{23} \},
\]

\[
t = 1, 2, \ldots , \{ \nu_1 \nu_2 \nu_3 \}, \quad t' = 1, 2, \ldots , \{ \nu_1 \nu_2 \nu_3 \}.
\]

The \( U_q(n) \) RCs satisfy the unitarity conditions

\[
\sum_{t_{23}t_{23}t'} U_q(\nu_1 \nu_2 \nu_3 ; \nu_1 \nu_2 \nu_3)_{t_{23}t_{23}t'} U_q(\nu_1 \nu_2 \nu_3 ; \nu_1 \nu_2 \nu_3)_{t_{23}t_{23}t'} = \delta_{t_{12}p_1} \delta_{t_{12}p_2} \delta_{q_{12}p_{12}},
\]

\[
\sum_{t_{23}t_{23}t'} U_q(\nu_1 \nu_2 \nu_3 ; \nu_1 \nu_2 \nu_3)_{t_{23}t_{23}t'} U_q(\nu_1 \nu_2 \nu_3 ; \nu_1 \nu_2 \nu_3)_{t_{23}t_{23}t'} = \delta_{t_{13}p_2} \delta_{t_{13}p_3} \delta_{d_{23}p_{23}}.
\]

As is well known, the \( U_q(n) \) RCs can be expressed in terms of \( U_q(n) \) CGCs, which, however, is not suitable for practical computations. The Schur-Weyl duality relation between the quantum group \( U_q(n) \) and the Hecke algebra \( H_f(q) \) enables us to evaluate the RCs of \( U_q(n) \) from SDCs of \( H_f(q) \). The quantum version of the Schur-Weyl duality was first discussed by Jimbo, and then by many others.\(^{13, 15, 18}\) The duality relation between \( U_q(n) \) and \( H_f(q) \) is as follows: the images of the \( U_q(n) \) and \( H_f(q) \) generators vary continuously with \( q \); and the \( H_f(q) \) and \( U_q(n) \) algebras are commutants of one another in the tensor space \( (V^n)^{\otimes f} \). Hence, for generic \( q \) the structure of \( H_f(q) \) can be determined from the information about \( H_f(1) = CS_f \), which is the group algebra of the symmetric group \( S_f \); and a standard basis of \( H_f(q) \) for an irrep \( [\nu] \) is also a special Gel'fand basis of \( U_q(n) \) for the same irrep. Thus, from analytical continuation for \( q \), we can conclude that the Schur-Weyl duality between \( S_f \) and \( U(n) \) applies to \( H_f(q) \) and \( U_q(n) \) as well except that \( q \) is a root of unity. Therefore, for generic \( q \) the RC of \( U_q(n) \) can be expressed in terms of SDCs of \( H_f(q) \) as

\[
U_q(\nu_1 \nu_2 \nu_3 ; \nu_1 \nu_2 \nu_3)_{t_{23}t'} = \sum_{m_{12}m_{23}m} \langle [\nu] | \nu_{12} [\nu] \nu_{23} \rangle_q \times
\]
\[
\left\langle [\nu_{12}]_{m_{12}} \mid t_{12} [\nu_1]_{m_1} [\nu_2]_{m_2} \right\rangle_q \left\langle [\nu] \mid t' [\nu_{23}]_{m_{23}} \right\rangle_q \left\langle [\nu_{23}]_{m_{23}} \mid t_{23} [\nu_2]_{m_2} [\nu_3]_{m_3} \right\rangle_q,
\]

where the summation is carried out under fixed \( m_1, m_2, \) and \( m_3 \). The advantage of (3.4) is its being rank-\( n \) independent.

Due to (3.4) and symmetry properties of \( H_f(q) \) SDCs, the RCs have the symmetry

\[
U_q(\nu_1 \nu_2 \nu_3; \nu_{12} \nu_{23})^{t_{12} t_{23}} = \eta \cdot U_q(\nu_1 \nu_2 \nu_3; \nu_{12} \nu_{23})^{t_{12} t_{23}},
\]

where \( \eta \) is the conjugation of \( \nu \). The phase convention used is the same as that for symmetric groups set by Chen.\(^8\) As for the classical case, the multiplicity separation should be based on the \( \textit{ad hoc} \) orthogonalization procedure when the multiplicity is greater than one. In this case, some symmetry properties will be not valid. Generally, the conjugation of irreps in the coupling always involves a change from \( q \) to \( q^{-1} \). However, one can prove that the RCs of \( U_q(n) \) are independent of \( q \)-factors, namely

\[
U_q(\nu_1 \nu_2 \nu_3; \nu_{12} \nu_{23})^{t_{12} t_{23}} = U_{q^{-1}}(\nu_1 \nu_2 \nu_3; \nu_{12} \nu_{23})^{t_{12} t_{23}}.
\]

Some RCs of \( U_q(n) \) have been calculated by using (3.4) from SDCs of \( H_f(q) \), which are listed in \(^{19}\).

3.2. CGCs of \( U_q(n) \)

Suppose \([\nu_1]\) and \([\nu_2]\) are two irreps of \( U_q(n) \). The CG series is the same as the outer-product of the irreps of \( H_f(q) \). The two irreducible basis vectors \([\nu_1]W_1\)\(_q\) and \([\nu_2]W_2\)\(_q\) can be coupled to another irreducible basis vector \([\nu]W\)\(_q\) of \( U_q(n) \) by means of the CGCs of \( U_q(n) \),

\[
||[\nu]W||_q = \sum_{W_1, W_2} C^{[\nu]W}_{\nu_1 W_1, \nu_2 W_2} (q) ||[\nu_1]W_1||_q ||[\nu_2]W_2||_q,
\]

where \( W_1, W_2, \) and \( W \) designate the component indices of the Gel'fand basis of \( U_q(n) \), i.e., the Weyl tableaux. The CGCs of \( U_q(n) \) satisfy the unitarity conditions

\[
\sum_{W_1, W_2} C^{[\nu]W}_{\nu_1 W_1, \nu_2 W_2} (q) C^{[\nu]W'}_{\nu_1 W_1, \nu_2 W_2} (q) = \delta_{\nu \nu'} \delta_{W W'},
\]
\[
\sum_{\nu \tau \rho} C^{[\nu]_{\tau, \rho}}(q) C^{[\nu]_{\tau, \rho}}(q) = \delta_{\tau, \rho} \delta_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5 \nu_6}
\] (3.8)

The Schur-Weyl duality can be used again to explain why the CG series of \( U_q(n) \) and the outer-product rules of \( H_f(q) \) are the same. It has been shown that the IDCs of \( H_f(q) \) and CGCs of \( U_q(f) \) for special Gel’fand basis are the same by using the Schur-Weyl duality relation,

\[
C^{[\nu]_{\tau, \rho}} = C^{[\nu]_{\tau, \rho}}(q) \quad (3.9)
\]

with the generalized Young tableaux \( Y^{[\nu]}(\omega) \) being identified with the Weyl tableaux. In early work on evaluating CGCs of \( U(n) \) from SDCs of \( S_f \) was based on the assimilation method. In the following, we shall briefly review the assimilation method to show why it can not be applied to the quantum case. Then, we will describe the symmetrization method.

In the symmetric group case, if a Young tableau \( Y^{[\nu]} \) is understood to be a Weyl tableau \( W^{[\nu]} \), the standard basis \( \{ Y^{[\nu]}(\omega) \} \) of the symmetric group \( S_f \) becomes a special Gel’fand basis \( \{ W^{[\nu]} \} \) of the unitary group \( U(f) \). Suppose that we have \( n \leq f \) single-particle indices \( a_1, a_2, \ldots, a_n \). The following index replacement is called assimilation:

\[
1 = 2 = \cdots = f_1 \rightarrow a_1, \quad f_1 + 1 = \cdots = f_1 + f_2 \rightarrow a_2, \quad \cdots
\]

\[
f - f_n + 1 = \cdots = f \rightarrow a_n. \quad (3.10)
\]

Under the assimilation, a special Gel’fand basis vector of \( U(f) \) will go over to a general Gel’fand basis vector of \( U(n) \), up to a normalization factor,

\[
\{ Y^{[\nu]} \}_{\text{assim.}} = R^{[\nu]_{\rho}}(\{ f_1 \}) W^{[\nu]}, \quad (3.11)
\]

where \( R^{[\nu]_{\rho}}(\{ f_1 \}) \) is a normalization factor which depends on \( [\nu]_{\rho} \) and \( \{ f_1 \} = (f_1, f_2, \ldots, f_n) \), and \( W^{[\nu]} \) is a Weyl tableaux,

\[
W^{[\nu]} = \begin{array}{cccc}
  f_{11} & a_1 's & f_{12} & a_2 's & f_{13} & a_3 's & \cdots \\
  f_{22} & a_2 's & f_{23} & a_3 's & \cdots \\
  \cdots \\
  f_{nn} & a_n 's
\end{array} \quad (3.12)
\]

The Weyl tableau (3.12) corresponds uniquely to a Gel’fand symbol.
\[
\begin{pmatrix}
m_{1n} & m_{2n} & \cdots & m_{nn} \\
m_{1n-1} & m_{2n-1} & \cdots & m_{n-1n-1} \\
m_{12} & m_{22} & & \\
m_{11} & & & \\
\end{pmatrix}
\]

(3.13)

In this case, there may be more than one Young tableau \( Y^{[v]} \) which will give rise to the same Weyl tableau \( W^{[v]} \). The norm in (3.11) can be calculated by

\[
R^{[v|m]}(\{f_i\}) = \left( \sum_{p \in G} \langle [v|m|p|[v|m]\rangle \right)^{1/2},
\]

(3.14)

where \( G \) designates the direct product group \( G = S_{f_1} \times S_{f_2} \times \cdots \times S_{f_n} \), and \( S_{f_k} \) is the symmetric group operating on the indices \( F_{k-1} + 1, \ldots, F_k \) with \( F_k = \sum_{i=1}^{k} f_i \). The Gel'fand basis \( |W^{[v]}\rangle \) is defined in the tensor product space \( (a_1)^{f_1}(a_2)^{f_2}\cdots(a_n)^{f_n} \). In this space we can define the state permutation group \( S_f \) in the following way\(^6\): We assign the \( N \) distinct state labels \( (a_1, a_2, \ldots, a_n) \) to the \( f \) state indices \( i_1, i_2, \ldots, i_f \) as

\[
i_1 = \cdots = i_{f_1} = a_1, \ i_{f_1+1} = \cdots = i_{f_1+f_2} = a_2, \ \ldots, \ i_{f-f_n+1} = \cdots = i_f = a_n,
\]

(3.15)

and a state permutation operator \( P_{ab} \) is defined as an interchange of the subscripts of \( i \)'s, i.e., \( i_a \rightarrow i_b \). It is easy to realize that a single state permutation operator does not have a definite meaning due to the fact that there are more than one \( i \)'s associate with a given \( a_j \). However, it has been shown\(^6\) that complete set of commuting operators (CSCO's) of the group chain \( S_f \supset S_{f-f_n} \supset \cdots \supset S_{f_1+f_2} \supset S_{f_1} \) have a definite meaning. It has also been shown that the Gel'fand basis vector (3.13) is a simultaneous eigenvector of the CSCO's of the group chain \( S_f \supset S_{f-f_n} \supset \cdots \supset S_{f_1+f_2} \supset S_{f_1} \). Though this assimilation method\(^6\) has been used successfully for the \( SU(n) \) CGCs, it turned out to be unfeasible for calculating the norm factor \( R^{[v|m]}(\{f_i\}) \) for the Hecke algebra. Therefore, we need a new method. The symmetrization method was proposed for this purpose.

Consider a linear combination of all those Yamanouchi basis vectors \( |Y^{[v]}\rangle \) whose Young tableaux go over to the same Weyl tableau \( W^{[v]} \) in (3.12), and then require that it be symmetric under the group \( G = S_{f_1} \times \cdots \times S_{f_n} \).
such a basis can be conveniently labelled by the partition \([\nu]\) and the generalized Yanaiouchi symbol \((w)\) \(^6\),

\[
||\nu|| = \{(1^{f_1}) (1^{f_2} 2^{f_2}) \ldots (1^{f_n} 2^{f_2} \ldots n^{f_n})\} = \sum_m c_m^{[\nu]w} Y^{|\nu|}_m,
\]

(3.16)

where \(c_m^{[\nu]w}\) are called symmetrization coefficients and the \(k\)-th parenthesis \((1^{f_1} 2^{f_2} \ldots n^{f_n})\) signifies that it is totally symmetric among the indices, i.e., it is symmetric with respect to \(S_{f_k}\) with \(k = 1, 2, \ldots, n\). For example,

\[
W^{[431]} = \begin{array}{ccc}
  a & a & b \\
  b & c & c \\
  c & & \\
\end{array} \quad \longmapsto (w) = (1^2)(2)(1^2 3).
\]

(3.17)

The bases \(|W^\nu]\) in (3.11) and \(|\nu||w)\) in (3.16) have exactly the same symmetry, though the former is defined in the tensor product space \((a_1)^{f_1} (a_2)^{f_2} \ldots (a_n)^{f_n}\), and is an eigenvector of the Casimir operators of \(U(n) \supset U(n-1) \supset \cdots \supset U(1)\), while the latter is in the coordinate space where \(U(n)\) has no definition. Therefore, we can regard \(|\nu||w)\) as an image of \(|W^\nu]\) and write

\[
|W^\nu\rangle \longrightarrow |\nu||w) = \sum_m c_m^{[\nu]w} Y^{|\nu|}_m.
\]

(3.18)

The symmetrization coefficients \(c_m^{[\nu]w}\) can be found from the conditions

\[
p|\nu||w) = |\nu||w)
\]

for any adjacent permutations \(p \in G = S_{f_1} \times S_{f_2} \times \cdots \times S_{f_n}\) along with the normalization condition

\[
\sum_m \left( c_m^{[\nu]w} \right)^2 = 1.
\]

(3.20)

It is known that if the two indices \(i\) and \(i+1\) are not in the same row or column of a Young tableau \(Y^{|\nu|}_m\), up to a normalization factor, the basis vector which is symmetric in \(i\) and \(i+1\) is equal to

\[
|_{\begin{array}{c}
\underset{\text{\scriptsize \(i\)}}{i+1}
\end{array}} \rangle = \sqrt{d-1}|Y^{|\nu|}_m\rangle + \sqrt{d+1}|Y^{|\nu|}_{m'}\rangle,
\]

(3.21)
where the two dots above the boxes indicate a symmetrization in the indices $i$ and $i + 1$, while $d$ is the axial distance between $i$ and $i + 1$, $d > 0$, $Y_{m'}^{[v]} = P_{i+1} Y_{m}^{[v]}$. The symmetrization coefficients $c_{m}^{[v]}$ can be found by repeatedly using (3.21) for the adjacent indices to be symmetrized.

Next, we need to find the image in the coordinate space for the product state $|W^{[w]} W^{[v]}⟩$. Similar to the previous case, we now take a linear combination of all the Yamanouchi basis vectors $|Y_{m_1}^{[v]}(ω_1) Y_{m_2}^{[v]}(ω_2)⟩$ whose Young tableaux $Y_{m_1}^{[v]}(ω_1)$ and $Y_{m_2}^{[v]}(ω_2)$ will go over to the same Weyl tableau $W^{[v]}$ and $W^{[v]}$, respectively, after the assimilation (3.10), and require that it be symmetric under the group $G = S_{f_1} × S_{f_2} × ⋯ × S_{f_n}$. Such a basis is denoted by

$$|[v_1](w_1) [v_2](w_2)⟩ = \sum_{m_1, m_2} F_{m_1 m_2}^{[v_1] [v_2] [ω_1] [ω_2]} |Y_{m_1}^{[v_1]}(ω_1) Y_{m_2}^{[v_2]}(ω_2)⟩,$$

(3.22)

where $F_{m_1 m_2}^{[v_1] [v_2] [ω_1] [ω_2]}$ are coefficients. The symmetrization for the basis (3.22) can be accomplished in two steps. First, one symmetrizes the indices within each Young tableau to get

$$|[v_1](w_1)_{0_1}⟩ = \sum_{m_i} c_{m_i}^{[v_1] [ω_1]} |Y_{m_i}^{[v_1]}(ω_1)⟩,$$

(3.23)

where $0_1 = (1, 2, ⋯, f_1)$, $0_2 = (f_1 + 1, ⋯, f)$. Next, one takes a linear combination of $|[v_1](w_1)_{0_1} [v_2](w_2)_{0_2}⟩$ with different $ω$'s to get

$$|[v_1](w_1) [v_2](w_2)⟩ = \sum_{ω} f_{ω} Q_{ω} |[v_1](w_1)_{ω_1} [v_2](w_2)_{ω_2}⟩ =$$

$$= \sum_{ω} f_{ω} Q_{ω} |[v_1](w_1)_{ω_1} [v_2](w_2)_{ω_2}⟩,$$

(3.24)

and requires it be symmetric under the product group $G = S_{f_1} × S_{f_2} × ⋯ × S_{f_n}$. Applying the adjacent permutations $P_{i+1} ∈ G$ and using the fact that $|[v_1](w_1)_{ω_1} [v_2](w_2)_{ω_2}⟩$ has already been symmetrized,

$$P_{i+1} |[v_1](w_1)_{ω_1} [v_2](w_2)_{ω_2}⟩ = |[v_1](w_1)_{ω_1} [v_2](w_2)_{ω_2}⟩,$$

(3.25)

from (3.24) one gets
\[
\sum_{\omega} f_{\omega} (\rho_{i} Q_{\omega}) [\nu_{1}] (w_{1})_{\omega}[\nu_{2}] (w_{2})_{\omega} = \sum_{\omega} f_{\omega} Q_{\omega} [\nu_{1}] (w_{1})_{\omega}[\nu_{2}] (w_{2})_{\omega}.
\]

(2.26)

Comparing the coefficients in both sides of (2.26) we get a set of linear equations. The coefficients \( f_{\omega} \) can be determined from these equations along with the normalization

\[
\sum_{\omega} (f_{\omega})^{2} = 1.
\]

(3.27)

Now we have the analogy of (3.18)

\[
|W^{[\nu_{1}] W^{[\nu_{2}]}} \rightarrow |[\nu_{1}] (w_{1})|[\nu_{2}] (w_{2})\rangle =
\sum_{m_{1} m_{2}} f_{\omega} c_{m_{1}}^{[\nu_{1}]_{w_{1}}} c_{m_{2}}^{[\nu_{2}]_{w_{2}}} |Y_{m_{1}}^{[\nu_{1}]} (\omega_{1})\rangle |Y_{m_{2}}^{[\nu_{2}]} (\omega_{2})\rangle.
\]

(3.28)

Due to (3.18) and (3.28), the CGCs of \( U(n) \), which is an overlap in the tensor product space, can be calculated from the overlap between the generalized Yamanouchi bases in the coordinate space,

\[
\langle W^{[\nu]} |W^{[\nu_{1}] W^{[\nu_{2}]}} \rangle = \langle [\nu] (w) |[\nu_{1}] (w_{1})|[\nu_{2}] (w_{2})\rangle =
\sum_{m_{1} m_{2} m} f_{\omega} c_{m_{1}}^{[\nu_{1}]_{w_{1}}} c_{m_{2}}^{[\nu_{2}]_{w_{2}}} \langle Y_{m_{1}}^{[\nu_{1}]} (\omega_{1}) |Y_{m_{2}}^{[\nu_{2}]} (\omega_{2})\rangle.
\]

(3.29)

Equation (3.29) gives the sought for relation between the CGCs of \( U(n) \) and the IDCs of the permutation group \( S_f \). The expression (3.29) has a simple meaning: The three factors come from the symmetrization for the indices in the three Young tableaux, \( Y_{m_{1}}^{[\nu_{1}]} (\omega_{1}) \) and \( Y_{m_{2}}^{[\nu_{2}]} (\omega_{2}) \), while the coefficient \( f_{\omega} \) comes from the symmetrization of the indices in \( Y_{m_{1}}^{[\nu_{1}]} (\omega_{1}) \) and \( Y_{m_{2}}^{[\nu_{2}]} (\omega_{2}) \).

The existence of a \( q \)-analog of the Gel'fand basis for \( U_{q}(n) \) was shown by Jimbo and Ueno et al\textsuperscript{20} and \textsuperscript{21}. Therefore, the above relation between CGCs of \( U(n) \) and IDCs of \( S_f \) can be extended to the CGCs of the quantum group \( U_{q}(n) \) and the IDCs of Hecke algebra \( H_{f}(q) \) by \( q \)-continuation. In the latter case, Eq. (3.21) should be replaced by
\[
\left( \begin{array}{c}
\frac{1}{d+1} \\
\end{array} \right)_q = \sqrt{[d-1]} Y_{m}^{[\nu]} q + \sqrt{[d+1]} Y_{m'}^{[\nu]} q.
\]
(3.30)

Similar to (3.29), we have

\[
C_{[\nu_1]W_1[\nu_2]W_2}^{[\nu]}(q) = \sum_{\omega m_1 m_2 m} f_{\omega} C_{m}^{[\nu]} C_{m_1}^{[\nu_1]} C_{m_2}^{[\nu_2]} C_{[\nu_1]m_1[\nu_2]m_2}^{[\nu]}(q),
\]
(3.31)

in which the CGC of \(U_q(n)\) is expressed in terms of a linear combination of the IDCs of the Hecke algebra \(H_f(q)\). The calculations of the symmetrization coefficients \(C_{\omega}^{[\nu]}\) is quite similar to the \(q = 1\) case previously. But the calculation of \(f_{\omega}\) in (3.31) is some what different as shown below. The expression (3.24) becomes

\[
||[\nu_1](w_1)[\nu_2](w_2)||_q = \sum_{\omega} f_{\omega} Q_\omega ||[\nu_1](w_1)_\omega||_q ||[\nu_2](w_2)_\omega||_q,
\]
(3.32)

Applying the generators \(g_i \in H_{f_1}(q) \times \cdots \times H_{f_o}(q)\) to the both sides of (3.32) and using the symmetries of the basis \(||[\nu_1](w_1)_\omega||_q, ||[\nu_2](w_2)_\omega||_q||\),

\[
g_i ||[\nu_1](w_1)_\omega||_q, ||[\nu_2](w_2)_\omega||_q = q ||[\nu_1](w_1)_\omega||_q, ||[\nu_2](w_2)_\omega||_q,
\]
(3.33)

the relation \(g_i^2 = (q - q^{-1})g_i + 1\), and equating the coefficients in front of \(Q_\omega\), we can similarly obtain a set of linear equations for \(f_{\omega}\), which, however, is different from \(q = 1\) case with \(g_i^2 = 1\).

The phase convention for the IDCs fixes the absolute phase of the CGCs of \(U_q(n)\) with

\[
C_{[\nu_1]W_1[\nu_2]W_2}^{[\nu]}(q) > 0,
\]
(3.34)

where \(hw([\nu_1])\) denotes the highest weight state of the irrep \([\nu_1]\). This phase convention is identical to that used for the CGCs of \(U(n)\). Under this phase convention, the coefficients \(C_{\omega}^{[\nu]}\) and \(f_{\omega}\) are all chosen to be positive.

Using this procedure, CGCs of \(U_q(n)\) for the resulting irreps \([f_1,f_2,f_3]\) with \(f_1 + f_2 + f_3\) were shown in 22.
4. Conclusions

The Linear Equation Method (LEM) and its applications to evaluating the IDCs, SDCs, of Hecke algebras have been discussed in detail. It should be stressed that the LEM would not have been initiated without Prof. Chen's effort in designing a practical approach to the evaluation of SDCs and IDCs of Hecke algebras with one of the authors (Pan F). It is clear that the LEM is not only useful in dealing with symmetric groups and Hecke algebras, but also in dealing with Brauer and Birman-Wenzl algebras. Prof. Chen devoted a lot of effort to understanding the Schur-Weyl duality relation between the symmetric groups and unitary groups, and used this relation to derive coupling and recoupling coefficients of unitary groups from CGCs, SDCs and IDCs of the symmetric groups, which has then been extended to the Schur-Weyl-Brauer duality relation. Therefore, coupling and recoupling coefficients of both classical and quantum algebras of A, B, C, and D types can be derived from SDCs and IDCs of their centralizer algebras, the symmetric groups, Brauer algebras, and Birman-Wenzl algebras.

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