GROUP SYMMETRIES AND INFORMATION PROPAGATION

J. P. Draayer

Department of Physics and Astronomy
Louisiana State University
Baton Rouge, Louisiana 70803

INTRODUCTION

Spectroscopy concerns itself with the ways in which the Hamiltonian and other interesting operators defined in few-particle spaces are determined or determine properties of many-particle systems. But the action of the central limit theorem (CLT) filters the transmission of information between source and observer so whether propagating forward from a few-particle defining space, as is usual in theoretical studies, or projecting backward to it from measured things, each is only sensitive to averaged properties of the other. Our concern is with the propagation of spectroscopic information in the presence of good symmetries when filtering action of the CLT is effective. Specifically, we propose to address the question, "What propagates and how?"

One may question whether statistical behavior and good symmetries are mutually exclusive concepts. The answer is not necessarily so, for while the former is usually considered a relevant way of describing things at high excitation and the latter most significant in the ground state domain, even rather badly broken symmetries can be useful in roughly partitioning a spectrum and play thereby an important role in determining locally averaged results for strengths, etc. upward away from the ground state region while statistical behavior has been shown to be ergodic and stationary in energy when expressed in the local scale so its effects extend downward into the ground-state domain. The two can and do coexist. Indeed, though the methods are those of statistics, we shall come to a geometry which furnishes a natural quantitative measure for the goodness of a symmetry.
Underlying the whole subject of information propagation in spectroscopic spaces is a theory of tensor operators. Although that guides our thinking and will serve to answer most efficiently questions regarding the existence and completeness of forms for propagating averages, our requirements are more demanding because in applications it is essential that the propagation be carried by simple operators whose averages are known or are easily calculable and whose properties relate directly to the dynamics and symmetries of a system. If the number of simple operators equals the number of input averages, the representation is faithful and one has simple propagation. Often they are not equal and the challenge then is not just to determine what it is that is missing. Here, contact is made with the group theoretical notion of "integrity basis". But, the challenge is also to find the optimum representation using the restricted set and a physical interpretation of the constraint incompleteness imposes on the system.

We begin in Section 2 with some examples, using both scalar and isospin geometries to illustrate simple propagation. In Section 3 examples of matrix propagation are studied; in Section 4 contact with standard tensor algebra is established and an algorithm put forward for the expansion of any operator in terms of another set, complete or not; in Section 5 shell-model results for 20O°e using a realistic interaction and two trace-equivalent forms are presented; and in Section 6 we give a brief summary and mention some further challenges.

2. SIMPLE PROPAGATION

The simplest results are for scalar averaging so we begin with that. \( \langle \phi | \phi \rangle \propto \sum_{m} | \langle \phi | m \rangle |^{2} \), where \( \langle \phi | m \rangle \) is a vector representing a state of a system. The total number of states | \( m \rangle \rangle \) is given by \( \sum_{m} | \langle \phi | m \rangle |^{2} = \langle \phi | \phi \rangle \). For identical Fermions the sum is over all \( d(m) = \binom{n}{m} \) states of \( [2m] \), the totally antisymmetric irreducible representation (IR) of U(n), the group of unitary transformations among single-particle levels. Such a sum is a U(n) contraction so only those parts of the \( \phi \) that are proportional to U(n) scalars contribute. But in our model space the U(n) scalar operators are all polynomial functions of \( n \), the number operator. So for scalar averages, if \( \phi = \phi(k) \), a k-body operator, we have the equivalence

\[
\phi(k) \xrightarrow{\text{scalar average}} \langle \phi(k) | \phi \rangle \propto \binom{n}{k}.
\]

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Here \( \langle \phi | \phi \rangle \) is a binomial factor that is of degree \( k \) in \( n \); it vanishes for \( k < n \), reduces to unity in the k-particle defining space and for \( k = n \) it counts the number of k-particle clusters. This result for a k-body operator illustrates propagation in its simplest form,

\[
\langle \text{m-particle} \rangle = \langle k \text{-particle} \rangle \times \langle \text{number of k-particle clusters} \rangle \text{ in the m-particle space} \]

where the propagator \( \langle k \rangle \) is a polynomial of degree \( u \) in \( n \) which for \( u = n \) vanishes in all but the k-particle space where its value is unity and for \( u = m \) carries the k-particle average into the m-particle space. For example, for averages of a (0+1+2)-body Hamiltonian one finds

\[
\langle k \rangle \xrightarrow{\text{average}} \langle \binom{n}{1} (1-n)(2-n)/2 \rangle \\
+ \langle \binom{n}{2} n(2-n)/2 \rangle
\]

for the 2-particle average is replaced by the N-particle (0-hole) one,

\[
\langle k \rangle \xrightarrow{\text{average}} \langle \binom{n}{1} (n-n)/(n-1) \rangle \\
+ \langle \binom{n}{2} n(n-1)/(n-1) \rangle
\]

which illustrates the use of alternate input data, a notion that can be used to advantage in applications. The essential thing here is the existence of a complete set of polynomials in \( n \) of proper degree which act as a Green's function, allowing averages away from the input net to be calculated simply in terms of defining space quantities.

The configuration extension is immediate. The space is first partitioned by single-particle orbits which only those parts of the U(n) into a direct sum \( E + U(N) \) where \( N \) is the dimensionality of the \( z \)th orbit and \( E \). The representations are then labelled by \( \eta = (m_{1}, m_{2}, \ldots, m_{2}) \) where \( m_{i} \) is the number of particles in the \( z \)th orbit and \( \sum_{i} m_{i} = m \). Averages of interest are \( \langle \eta \rangle = U(n) \) contractions and these propagate as a polynomial function of the orbit number operators, the \( m_{i} \)'s. For example, for a two level system one has for the analogue of (5),
For the operator 
\[ \langle 0 | \mathcal{O}^T | (n,m,T) \rangle = \delta_{n}(m,T) \langle 0 | \mathcal{T}^{\dagger} | m,T \rangle \]  
(7)

where the sum is over all equivalent representations labelled \( \mathcal{T} \) and \( m,T \), and as usual the double-barred matrix element is reduced with respect to isospin couplings. Here the group structure is \( U(2) \otimes U(2) \otimes U(N/2) \), and the sum is over all states of the 1R (2\(T^2\), 1\(T\)) of U(N/2) that is conjugate to the 1R (2\(T^2\), 1\(T\)) + (m,T) of U(2). As the sum is a U(N/2) group contraction, only those parts of \( \mathcal{T} \) that are U(N/2) scalars and hence proportional to polynomial functions of the U(2) generators \( n \) and \( T \) contribute to the trace.

The only independent isoscalar objects one can construct from \( n \) and \( T \) are \( \mathcal{T} \) and \( T^2 \). So for isoscalar operators
\[ \langle 0 | \mathcal{T}^{\dagger} | 0 \rangle = \delta_{n2} \delta_{m0} \delta_{T0} \]  
(8)

It is instructive to count the number of polynomials in \( (n,T^2) \) of combined degree \( n+2k \): For \( k = 0,1,2,3,4,... \) one finds the numbers to be 1, 1, 2, 3, 4, ..., respectively, for example for \( k=2 \) the net is \( (n,n,T^2,\{n,T^2\}) \). These are precisely the number of \( \mathcal{T} \) values in the \( k \)-particle space, see Figure 1, as for the scalar case the number of polynomials is equal to the number of defining space averages and the propagation is simple. By inspection one can verify that the isoscalar analogue of (5) is

\[ \langle (1,1) | 0 \rangle = \frac{1}{3} \]  
(9)

The quantities in brackets are now fixed-isoscalar propagators of the averages, \( \langle \mathcal{H} | (k,T) \rangle \), they multiply. The propagators, \( (P,\mathcal{H},(n,T2)) \), to use a notation analogous to that introduced in (4) but now for a set of \( (k,T) \) input averages, can be easily constructed by matrix inversion of the set of equations obtained when an expansion of type (8) is assumed for each and the requirement \( \langle \mathcal{T} | (n,T^2) \rangle \langle (k,T) | \mathcal{T} \rangle \) imposed in the defining space. Unlike the scalar case, however, not all \( (m,T) \) averages carry independent information and in attempting to build propagators for alternate choices of input data this shows up as a singular matrix. For example, for any interaction \( \langle \mathcal{H} | (m,T) \rangle \langle (n-m,T2) | (n-2m) \rangle \) is a constant so one cannot substitute the set \( \{ (0,0), (1,1), (1,1), (0,2) \} \) for that used in (9).

Averages of non-isoscalar operators also propagate. The simplest operator of isoscalar rank \( \gamma \) that one can construct is \( \mathcal{V}^T \) (or \( \mathcal{V}^r \)) with \( r \) factors of \( \gamma \). Indeed the basis for propagating averages of an operator of rank \( \gamma \) is just the set \( (n,T^2,\mathcal{V}^T) \),

\[ \langle \mathcal{V}^T | (m,T) \rangle = \frac{1}{2} \langle n-T \rangle \langle \mathcal{T}^+ \rangle \delta_{n-2m} \]  
(10)

Matrix elements of \( \mathcal{V}^T \) can be determined using, for example, \( \mathcal{V}^T = \langle \mathcal{V}^T | \mathcal{T} \rangle \mathcal{V}^T \) iteratively and known reduced matrix elements of \( \mathcal{V}^T \),

\[ \langle \mathcal{V}^T | (m,T) \rangle = \frac{1}{2} \langle n-T \rangle \delta_{n-2m} \]  
(11)

Figure 1, taken from Ref. 5, shows a schematic plot of an \( (m,T) \) lattice for propagating averages of a \( n \) \( T^2 \), \( n \) \( T^2 \) operator. Averages for \( T^2 \) (shadow-zone) vanish by isospin selection and for \( T^2 \) and \( m \) (propagation zone) are determined from values on the defining net. Hence if \( m \) and \( u \) as for an isovector E1 transition or for allowed \( \delta^2 \) decay, averages for all \((m,T)\) are determined by a single average,

\[ \langle 0 | \mathcal{T}^{\dagger} | (1,1) \rangle = \frac{1}{3} \]  
(12)

Apart from the \( \mathcal{V}^T \) factor, propagators for \( \mathcal{V}^T \) from its defining net are the same as for an isoscalar operator of maximum particle rank \( n \) \( T \) from that same net. This is an important property of the structure for it means that nonisoscalar propagators are a natural part of a complete and orthogonal operator geometry.
These two examples illustrate how things go when one is dealing with a direct product partitioning of the model space, $U(p) \times U(q)$ with $pq=N$. In the scalar case $p=N$, $q=1$ while for isospin $p=N/2$, $q=2$. If the sum in a $U(p)$ [$U(q)$] group contraction, each average propagates as a polynomial function in generators of the complementing group, $U(q)[U(p)]$. For scalars, the Casimir invariants form a basis while for non-scalar operators of fixed tensorial rank one must add to those such independent tensors of that symmetry that one can construct from generators. For example, for $U(4)$ the invariants are $c_1 = C_1^2$, $c_2 = C_2^2$, $c_3 = C_3 C_4$, $c_4 = C_4^2$ and in Table 1 we show how the counting of defining space IR and polynomial invariants goes for scalar operators, the number of polynomial invariants of degree $k$ is just equal to the number of $k$-particle representations. This continues up to $kN/4$ beyond which one loses defining space IR because of Fermion antisymmetry requirements. (In the d-shell with $N=24$ the IR [7] of $U(6)$ is Pauli-forbidden.) Also, again for Fermions, since the IR of $U(p)$ and $U(q)$ are conjugate to one another, averages of a scalar operator in one space automatically reproduce scalar averages in the other.

3. MATRIX PROPAGATION

Things are not always so straightforward. Consider for example traces of scalar (L=0) operators in the nuclear p-shell. The group structure is $U(N-1)SU(3)\times U(4)$ with $U(3)$ further reduced with respect to the rotation group, $R(3)$. The sums we refer to are over all equivalent IR of $U(3)$ labelled by $m_\lambda(\mu)$ which because of the direct-product structure and the overall antisymmetry requirement for Fermions implies a sum over state labels of the conjugate IR of $U(4)$. It must be that these averages propagate as a function
of those independent R(3) scalars which one can construct from U(1) generators. A basis can be recognized by counting defining space averages as is shown in Table 2 for N large enough to support all possible U(3) IR. Besides the invariants n, C_3, C_5 of U(3) and L^2 of R(3) two others, one of degree three in the U(1) generators and one of degree four, are needed.

In seeking a resolution of the U(3) \supset R(3) state-labeling problem, Judd et al. identified this set as an "integrity basis", i.e., the minimum set of R(3) scalars constructed from U(1) generators in terms of which all others are expressible as polynomials. To resolve the (Au)-l multiplicity only X_2 = \langle x^2 \rangle^2 - Q where Q is the singlet-shell quadrupole operator or \( X_2 = \langle l_2 \rangle^2 - \langle 0 \rangle^2 \) or a linear combination of the two is needed but we see here that for propagating averages both are required. Note that for k=6 the number of (Au)L values is 2L whereas the number of polynomial scalars of degree 6 is 22. The point is that L=2 occurs twice in (Au) = (22) hence rather than 2 there are 3 independent averages, 2 that are diagonal in the L multiplicity and one that is off diagonal. For multiplicity k the counting is \( \frac{k(k-1)}{2} \). As a demonstration that these are no other scalars we found for k=50 the number of polynomials to be 35,993 which is also the number of defining space averages, including all off-diagonal ones. Note that the noninvariant scalars are needed even when the defining space representations are multiplicity free.

Another example, one where simple propagation in terms of invariants alone was recognized as being impossible early on, is for supermultiplet symmetry in U(1) \supset U(4) with U(4) further reduced with respect to spin and isospin, SU(2)xSU(2). Sums in this case are over state labels of an IR of U(4)/U(3). The invariants n, G_3, G_5, Q_4 of U(4) and \( S^a \), \( S^b \) of SU(2)xSU(2) yield, for example, only 21 independent polynomials for n=5 part of the variance of a two-body interaction \( Q^2 = \langle (N-\langle N \rangle)^2 \rangle \) so u=6, whereas there are 28 independent defining space averages. Quenched recognized that the "integrity basis" for SU(2)xSU(2) scalars from the generators of U(4) provides the 5 additional operators. Counting defining space representations shows the need for one additional (ST)-scalar of degree three, three of degree four (X_2, m_3, X_3, Y_4, Y_4) are the additional 5 scalars required to propagate the variance, one of degree five and two of degree six. Explicit forms for these have been given by Miller. Polynomial counting beyond n=7 yields numbers which grow more rapidly than the number of defining space averages, for n=7, 8, 9, 10, 11, 12, 25, 50 the numbers are (73, 139, 201, 349, 498, 813, 57125, 11766281) for the former and (73, 138, 200, 344, 691, 792, 8653, 654506) for the latter. The source of the difference cannot be that there are too many scalars, but rather that not all products and powers of the basis are independent. The redundance can also be discovered by counting (with a slightly more challenging logic) or by algebraic means or be anticipated from the form of the generating function for the number of scalars of given polynomial order. One finds that \( Z_4 = 5 \cdot E^4 \cdot T \) where \( E \) is the Gell-Mann operator and \( X_5 = X_5 \cdot E^5 \cdot T \) can only occur linearly while no products of \( X_5 = (X_5)^2 \cdot (E^5)^2 \) and \( Y^2 = (Y^2)^2 \cdot (E^5)^2 \) can occur at all. With these restrictions, and again taking into account the off-diagonal information associated with multiple occurrences of (20) values in each IR of U(4), the polynomial counting reproduces the representation counting exactly.

The new feature in these examples is the need for noninvariants in building Green's functions for propagating averages of scalar operators. If the group chain is GML and the corresponding IR labels \( \Gamma \) and \( \Lambda \), \( K \) scalars fall into three categories: 1) Invariants of \( \Gamma \); 2) Invariants of \( K \) which for fixed \( (\Gamma, \Lambda) \) act as a multiple

<table>
<thead>
<tr>
<th>Rank</th>
<th>Representations (Au)L</th>
<th>Scalars (u=k)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(000)</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>(10)1</td>
<td>n</td>
</tr>
<tr>
<td>2</td>
<td>(20)0,2; (01)1</td>
<td>n_1,L^2,C_z</td>
</tr>
<tr>
<td>3</td>
<td>(30)1,3; (11)1,2; (00)0</td>
<td>n^2,n_L^2,n_C_2,C_3,X_5</td>
</tr>
<tr>
<td>4</td>
<td>(40)0,2,4; (21)1,2,3</td>
<td>n^3,n_L^2,n_C_2,n_C_3,n_X_5</td>
</tr>
<tr>
<td></td>
<td>(02)0,2; (12)1</td>
<td>L^2,L_C_2,C_1,X_5</td>
</tr>
<tr>
<td>5</td>
<td>(50)1,1,5; (33)1,2,3</td>
<td>n^4,n_L^2,n_C_2,n_C_3,n_X_5,n_X_5</td>
</tr>
<tr>
<td></td>
<td>(12)1,2,3; (20)0,2; (01)1</td>
<td>n_L^2,n_L^2,C_2,n_C_1,n_X_5</td>
</tr>
<tr>
<td></td>
<td>(01)1</td>
<td>L^2,C_1,X_5,C_3,C_5,X_5</td>
</tr>
<tr>
<td>6</td>
<td>(60)0,2,4,6; (41)1,2,3</td>
<td>n_X (k=5 scalars)</td>
</tr>
<tr>
<td></td>
<td>(22)0,2,4,6; (03)1,3</td>
<td>L^2,L_C_2,L_C_1,L_C_1,X_5</td>
</tr>
<tr>
<td></td>
<td>(30)1,3; (12)1,2; (00)2</td>
<td>Cl,C_X_5,Cl,C_1,X_5</td>
</tr>
<tr>
<td>7</td>
<td>(70)1,3,5,7; (53)1,2,3</td>
<td>n_X (k=6 scalars)</td>
</tr>
<tr>
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<td>L^2,C_3,L_C_2,L_C_2,G_3</td>
</tr>
<tr>
<td></td>
<td>(40)0,2,4,6; (21)3,2,3</td>
<td>L^2,C_3,L_G_3,L_G_3</td>
</tr>
<tr>
<td></td>
<td>(02)0,2; (10)1</td>
<td>L^2,C_X_5,Cl,C_X_5</td>
</tr>
<tr>
<td></td>
<td></td>
<td>G_X_5,X_X_5</td>
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</tbody>
</table>
of unity; 3) others that are neither C nor K invariants. If the first two types generate a basis the propagation is simple. If not, one has matrix propagation since the noninvariant elements generate rotations in the \( (\Gamma, \Lambda) \) subspace, the dimensionality of which is equal to the number of times \( \Lambda \) is found in \( \Gamma \). Even if the basis contains operators of all three types, operators of restricted particle rank may still propagate simply. For example, for supermultiplet symmetry, the \( (27) \rightarrow (00) \) centroid of a two-body interaction is given by

\[
\sum_{\text{supermultiplet}} \left[ \begin{array}{c} \Sigma (n-1) (n-1) \end{array} \right] c \left( \begin{array}{c} 0 \end{array} \right) (00) \\
-\left( \begin{array}{c} -n \end{array} \right) (1) (11)1 \\
+ \frac{1}{8} \left( \begin{array}{c} n \end{array} \right) (2n-9+2a) \left( \begin{array}{c} 2 \end{array} \right) (11)1 \\
+ \frac{1}{8} \left( \begin{array}{c} n \end{array} \right) (2n-6+2(2^2-7) \left( \begin{array}{c} 2 \end{array} \right) (11)10 \\
+ \frac{1}{8} \left( \begin{array}{c} n \end{array} \right) (2n-12-2(2^2-7) \left( \begin{array}{c} 2 \end{array} \right) (11)01 \\
- \frac{1}{8} \left( \begin{array}{c} n \end{array} \right) (2n-12+2(2^2+7) \left( \begin{array}{c} 2 \end{array} \right) (1)100 \end{array} \right) \right) \\
\right] \right. \\
\left. \right| \text{where} \ c(nfST) \text{are the defining space averages}. \right.

4. TENSOR ALGEBRA

If \( \psi(k) \) denotes a \( k \)-particle state vector and \( \psi_{\alpha}^{\dagger}(k) \) its adjoint, then \( \psi_{\alpha}(k) \psi_{\beta}^{\dagger}(k) \) is a \( k \)-body tensor operator and the set of these for all \((\alpha, \beta')\) form a basis in terms of which any \( k \)-body operator acting in the model space may be expanded as

\[
0(k) = \sum_{\alpha, \alpha'} <k|\alpha|k\alpha'\psi_{\alpha}^{\dagger}(k)\psi_{\alpha'}(k),
\]

(14)

If the basis states are organized according to some group structure, \( \psi_{\alpha}^{\dagger}(k) \) with \( k \) a label used to distinguish among equivalent representations \( \Gamma \), the tensors can be similarly organized and instead of (14) we have

\[
0(k) = \sum_{\Gamma, \Gamma'} <k\Gamma | \left( 0^\dagger(k) \right) | k\Gamma' > \left[ \Gamma|\Lambda \right] \frac{1}{2} (-1)^{\frac{\Lambda}{2}-1} \left( \begin{array}{c} \Gamma \end{array} \right) k_1^* k_2 \end{array} \right) (k')^\dagger
\]

(15)

or for input averaged over equivalent representations

\[
0(k) \text{ symmetry average } \Gamma \sum_{\Gamma} \sum_{\Gamma'} <k\Gamma | \left( 0^\dagger(k, \Gamma) \right) | k\Gamma' > \left[ \Gamma|\Lambda \right] \frac{1}{2} (-1)^{\frac{\Lambda}{2}-1} \left( \begin{array}{c} \Gamma \end{array} \right) k_1^* k_2 \end{array} \right) (k')^\dagger
\]

(16)

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Here \( [\Gamma] = [\Lambda] \) with \( [\Gamma/\Lambda] = [\Gamma]/[\Lambda] \) and the phase is that required for \( \Phi^\dagger(k) \) to be a proper tensor of rank \( \Gamma \). If \( n \)-particle averages derive from the \( k \)-particle ones we say the averages propagate. Comparing (15) an exact expansion, with (16), an approximation, we see immediately that this will in general only be so if the \( n \)-particle sums are group contractions thereby forcing on \( 0(k) \) a vanishing result for all \( x^I x^J \) and an equal weighting of the \( x^I x^J \) terms.

So when the sums are group contractions the corresponding averages propagate. And, as we have already seen in examples, the density operators \( \rho^\dagger(k, \Gamma) \) which act as Green's functions are then expandable as polynomial functions of the generators of the symmetry group. While for \( \Lambda=0 \) there is a Casimir basis, for \( \Lambda \neq 0 \) a basis includes in addition all independent tensors of rank \( \Lambda \) one can build from the generators e.g. \((nT, U)\) for isospin. Matrix propagation comes about when we attribute to the group \( G \) a subgroup \( K \) and so organize the generators. For \( K \) scalars \( \Lambda \) need not be zero. The invariants of \( K \) and those dubbed scalar noninvariance (type 3) in Section 3, e.g. \( X_3 \) and \( X_5 \) of the \( U(3) \times SU(3) \) case, are of this type. These split but do not mix multiplets of \( G \). Here is where we encounter the group theoretical notion of an "integrity basis" and nonscalar operators there is likewise an "extended integrity basis".

The generators of a symmetry propagate simply whether the sums are group contractions or not, for example \( J^2 \) in fixed-\( J \) subspaces. Likewise, matrix propagation reduces to simple propagation if the operators of interest expand solely in terms of the Casimir elements in a basis. The following question is therefore a relevant one: "To what extent does the average of an operator propagate simply?" This is a part of a more general challenge: Given an operator, find its optimum representation in terms of another selected set. We shall respond to the latter but it is instructive to first formulate a response to the question about averages.

For operators of fixed particle ranks \( \hbar \nu \) let \( \rho^\dagger(k, \Gamma) \) be a linear combination of the corresponding density operator (16) such that in the defining space \( \left( \rho^\dagger(k, \Gamma) \right) (k(k')) = \left[ \Gamma|\Lambda \right] \delta(k(k')) \); that is, we introduce operators which project on \( k \) as well as \( \Gamma \). For \( \hbar \nu \), \( \rho^\dagger(k, \Gamma) = \rho^\dagger(k, \Gamma) \). Then, by construction,

\[
<k(k') | \rho^\dagger(k, \Gamma) | k(k')> = \left[ \Gamma|\Lambda \right] \delta(k(k'))\delta_{\Lambda\Lambda'}. \]
where \( D = \{ \mathcal{S}/6 \} \delta_{ij}(\Gamma) \) and by \( (u) \) we mean a sum over all \( ku \). When the fixed-\( \Gamma \) sums (16) are group contractions this orthogonality automatically extends to all other operators as well. For any \( \phi_{\alpha}^d \):

\[
\phi_{\alpha}^d \text{ symmetry: } \sum_{\Gamma} \frac{1}{\xi_{\alpha}^d(k,\Gamma)} \phi_{\alpha}^d(k,\Gamma) = \sum_{\Gamma} \frac{1}{\xi_{\alpha}^d(k,\Gamma)} \phi_{\alpha}^d(k,\Gamma).
\]

This is a generalization of (4), see also (8) and (10).

Excluding an operator from \( \{ \phi_{\alpha}^d \} \) leads to a constrained propagation since each such operator has itself an exact expansion of type (19). The constraint is equivalent to requiring that \( \phi_{\alpha}^d \) be orthogonal, in the sense of (18), to those combinations of the \( \phi_{\alpha}^d \)'s. Explicitly, if for \( \phi_{\alpha}^d \) we let \( \phi_{\alpha}^d \) be the right hand side of (19), then for any two operators we have an inner product

\[
\left< \phi_{\alpha}^d, \phi_{\beta}^d \right> = \sum_{k} \xi_{\alpha}^d(k,\Gamma) \phi_{\alpha}^d(k,\Gamma) \phi_{\beta}^d(k,\Gamma) = 0
\]

and for the norm

\[
\| \phi_{\alpha}^d \| = \left( \langle \phi_{\alpha}^d, \phi_{\alpha}^d \rangle \right)^{1/2}.
\]

A constraint implies \( \phi_{\alpha}^d = \phi_{\alpha}^d + \phi_{\alpha}^d \) where if \( \phi_{\alpha}^d \) is the excluded operator, \( \phi_{\alpha}^d = \phi_{\alpha}^d - \phi_{\alpha}^d \). For several \( \phi_{\alpha}^d \) this extends once the set is orthogonalized. ||\( \phi_{\alpha}^d \)|| ||\( \phi_{\alpha}^d \)|| provides a quantitative measure of the impact of the deletion. For example, to propagate f- and shell SU(3) centroids of a two-body interaction (\( \phi_{\alpha}^d \phi_{\alpha}^d \), U(10)xSU(3)) requires four scalars, one of which is of the non-Casimir type. For the Brown-Xan interaction, dropping \( \phi_{\alpha}^d \) leads to only a 4% reduction in the norm of that part of \( \phi_{\alpha}^d \) that generates the SU(3) centroids. So although the propagation of SU(3) centroids in this case is simple, it is very nearly so.

In the defining space there is a complete set of tensor operators and the notion of an inner product and orthogonality extends to all of them. What we have is a linear algebras among operators defined in the model space and, as a consequence, a very simple algorithm for expanding one in terms of others, (12), (13)

\[
\phi_{\alpha}^d = \sum_{\alpha} c_{\alpha}^d \phi_{\alpha}^d + \phi_{\alpha}^d = \phi_{\alpha}^d + \phi_{\alpha}^d
\]

\[
\langle \phi_{\alpha}^d, \phi_{\alpha}^d \rangle = \sum_{\alpha} c_{\alpha}^d \phi_{\alpha}^d \phi_{\alpha}^d
\]

The \( C_0 \) can be determined by matrix inversion and \( x^d \) is that part of \( \phi_{\alpha}^d \) that lies outside the space spanned by the \( \phi_{\alpha}^d \). The norm of \( \phi_{\alpha}^d \) compared to the norm of \( \phi_{\alpha}^d \) provides a measure for completeness of the expansion. If the set \( \{ \phi_{\alpha}^d \} \) is built from an integrality basis \( \phi_{\alpha}^d \) vanishes. Note that for two operators, \( \phi_{\alpha}^d \phi_{\alpha}^d \) measures the projection of \( \phi_{\alpha}^d \) along \( \phi_{\alpha}^d \) and \( \phi_{\alpha}^d \). For \( \phi_{\alpha}^d \) (12) is the correlation between them. So there is an elementary geometry associated with the algorithm. Also, in the present form one can easily see that when the partitioning is by direct product or a direct sum so tensors carry more than a single IR label, simultaneous propagation of averages is possible if the \( \{ \phi_{\alpha}^d \} \) are built from integrity bases for the parts. (14)

5. APPROXIMATIONS TO MODEL INTERACTIONS

To illustrate the techniques in greater detail and show how they can be used to address questions regarding the goodness of symmetries, we have made a comparison of shell-model results for excitation spectra, eigenstate overlaps and E2 transition strengths of the (da)^x nucleus 246He(T=0) using a Brown-Yaxo (BY) interaction and its SU(3) trace-equivalent approximation. (23) For \( H \) we write

\[
H = H_{TE} + X
\]

where the trace-equivalent (TE) part is required to reproduce the trace of \( H \) in all subspaces of the model space. This separation is only possible when the partitioning is by group symmetries. Then for the centroid and variance, which is the square of the spectral width and a measure of the total strength of an interaction, we have

\[
\epsilon(m) = \left< H X \right|m \right> = \left< \left( d(m,\Gamma)/d(m) \right) c(m,\Gamma) \right>
\]

\[
\phi_{\alpha}^d = \left< \left( d(m,\Gamma)/d(m) \right) c(m,\Gamma) \right> c(m,\Gamma) \]

So although \( H_{TE} \) reproduces the model-space centroid it only reproduces that part of the variance associated with the displacement of subspace centroids, \( c(m,\Gamma) \), from the model-space centroid, \( \epsilon(m) \). To construct \( H_{TE} \) we know there is a basis of \( G \) scalars. One can extend it by including projection of \( H \) along additional traceless operators. If the additional operators preserve the symmetry basis they generate internal spreading, for example \( L^2 \) if \( G=SU(3) \). Others (for example for \( G=SU(3) \) the one-body operator \( \sum \phi_{\alpha}^d \), which look after orbital separation) are symmetry breaking and generate external spreading. The ratio of the spectral width of an approximation to the full width can in this way be used to provide a measure for
the goodness of a symmetry. If the ratio is near unity the symmetry is an important one and if not but some symmetry breaking effect raises it to near unity one has learned as well why it is not.

For SU(3) in the ds-shell the space is first partitioned into its space and spin-isospin parts, U(24) \text{SU}(6) \times U(4), and the space part U(6), whose IR are labelled by \( \{f\} \), is then further reduced with respect to SU(3) whose IR are labelled by \( \{\lambda\} \). For a \( (0+1) \text{-body} \) interaction there are five input averages and the invariants \( n \) and \( c_2 \) of U(4) and \( c_2 \) of SU(3) provide a basis,

\[
\begin{align*}
H^T &= \frac{1}{2} \left( \left(1-n \right)(2-n) \right) c_0(0)(1) + n(2-n) c_1(1 \bar{1})(1) + \frac{1}{2} c_2(2 \bar{2})(1) \left[ 2 n (n-15) - 6 c_2 c_2 \right] (2)(2) + \frac{1}{2} c_1(1 \bar{1}) \left[ 4 n(n+21) - 12 c_2 c_2 \right] (2)(2) + \frac{1}{2} (n(n-5) + 6 c_2 c_2) (2 \bar{2})(2) \right. \\
&= \left. \frac{1}{2} \left( n(n-15) - 6 c_2 c_2 \right) \right] (2)(2) + \frac{1}{2} \left. (n(n+21) - 12 c_2 c_2 \right] (2)(2) + \frac{1}{2} (n(n-5) + 6 c_2 c_2) (2 \bar{2})(2) \right). \\
\end{align*}
\]

Here \( c(m) \) is the centroid energy of the \( \{f\} \) \( \{\lambda\} \) subspace.

\( c_2 \) has eigenvalues \( 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 \), where \( f \) is conjugate to \( \bar{f} \) and so \( c_2 \) has values 6 and 10 for \( \{f\} = [2] \) and \( \{\bar{f}\} = [2] \), respectively. \( c_2 \) has eigenvalues \( 4 \left( 1^2 + 3^2 \right) \left( 1^2 + 5^2 \right) - 4 \). For the BK interaction with 1/0 single-particle energies this approximation \(-2.302, -8.523, -6.130, -4.056 \text{ MeV} \) for 1 and 2 particle averages in (23), respectively, generates 67% of the full width.

This approximation does not take account of spin and isospin splittings for which the set \( n, c_2, s^2, t^2 \) forms a basis and moreover, within a given \( \{\lambda\} \) all \( L \)-values are degenerate. This can be corrected and the approximation extended (22) with the symmetry preserving operators \( S^2 \), \( T^2 \), and \( L^2 \),

\[
H' = H^{TE} + 0.03 S^2 + 0.03 T^2 + 0.03 L^2
\]

where SU(3) centroids have been subtracted for the separation to be an orthogonal one. This approximation generates about 72% of the total width of \( H' \). Though other symmetry preserving operators exist, none have simple eigenvalues and furthermore, on this approximation, one expects the main missing ingredient to be a splitting of the single-particle orbitals. If we include the \( n_s \) linearly in our basis the width ratio jumps to 96%, which is a simple demonstration of the coexistence (competition) of shell and collective effects (Q-Q + C2 - 3 L^2) in nuclei.

In Figure 2 we show the excitation spectra for low-lying states of \( ^{20}\text{Na} \). The column labelled SU is for the \( H' \) of (24) and the one labelled SU+SP in addition optimum (22) single-particle energies for the orbital number operators. In the SU(3) limit the degeneracy \( s^2, t^2 \) and \( s^2, t^2, \text{states} \) are from \( [f] \{\lambda\} = [2] \{42 \} \) while the 14.5 MeV \( 0^\pm \), \( 2^\pm \) doublet is from \( \{\lambda\} = [61] \) with \( L=1, S=1 \). In Table 3 overlap percentages between eigenstates of the approximations and the BK results are given. In Table 4 the BE values are compared. The yrast states are well represented by the approximations but for others the ordering is not always the same and the identification fuzzy as overlap percentages drop below 50%. But as sums over the first five levels indicate, even when a one-to-one
Table 3. Overlap percentages between eigenstates for \(^{20}\text{He}(T=0)\) generated with the BK interaction and its e) SU3 SPE and b) SU3 trace-equivalent approximations. The sum measures the projection of the BK state into the subspace spanned by the first five eigenstates of the trace-equivalent interaction.

<table>
<thead>
<tr>
<th>J^p</th>
<th>(0^+)</th>
<th>(1^+)</th>
<th>(2^+)</th>
<th>(3^+)</th>
<th>(4^+)</th>
</tr>
</thead>
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<tr>
<td>(0^+_1)</td>
<td>88(^a))</td>
<td>5</td>
<td>0</td>
<td>99</td>
<td></td>
</tr>
<tr>
<td>(0^+_2)</td>
<td>77(^b))</td>
<td>10</td>
<td>1</td>
<td>92</td>
<td></td>
</tr>
<tr>
<td>(0^+_2)</td>
<td>114</td>
<td>45</td>
<td>0</td>
<td>93</td>
<td></td>
</tr>
<tr>
<td>(0^+_3)</td>
<td>2</td>
<td>29</td>
<td>7</td>
<td>86</td>
<td></td>
</tr>
<tr>
<td>(1^+_1)</td>
<td>93(^a))</td>
<td>1</td>
<td>0</td>
<td>96</td>
<td></td>
</tr>
<tr>
<td>(1^+_2)</td>
<td>82(^b))</td>
<td>3</td>
<td>7</td>
<td>96</td>
<td></td>
</tr>
<tr>
<td>(1^+_2)</td>
<td>9</td>
<td>70</td>
<td>9</td>
<td>90</td>
<td></td>
</tr>
<tr>
<td>(1^+_3)</td>
<td>3</td>
<td>51</td>
<td>0</td>
<td>58</td>
<td></td>
</tr>
<tr>
<td>(1^+_3)</td>
<td>3</td>
<td>15</td>
<td>70</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>(2^+_1)</td>
<td>89(^a))</td>
<td>1</td>
<td>0</td>
<td>96</td>
<td></td>
</tr>
<tr>
<td>(2^+_2)</td>
<td>76(^b))</td>
<td>3</td>
<td>11</td>
<td>99</td>
<td></td>
</tr>
<tr>
<td>(2^+_2)</td>
<td>9</td>
<td>71</td>
<td>9</td>
<td>95</td>
<td></td>
</tr>
<tr>
<td>(2^+_3)</td>
<td>0</td>
<td>51</td>
<td>16</td>
<td>95</td>
<td></td>
</tr>
<tr>
<td>(3^+_1)</td>
<td>4</td>
<td>12</td>
<td>0</td>
<td>67</td>
<td></td>
</tr>
</tbody>
</table>

Table 4. Calculate BE2 values for \(^{20}\text{He}(T=0)\). Each set of three is for a) BK and its trace-equivalent approximations, b) SU3 SPE, and c) SU3. The sum measures the average strength to the first five final states.

<table>
<thead>
<tr>
<th>(J_f)</th>
<th>(0^+_1)</th>
<th>(0^+_2)</th>
<th>(0^+_3)</th>
<th>(1^+_2)</th>
<th>(2^+_1)</th>
<th>(2^+_2)</th>
<th>(2^+_3)</th>
<th>(3^+_1)</th>
<th>(3^+_2)</th>
<th>(3^+_3)</th>
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<tbody>
<tr>
<td>(2^+_1)</td>
<td>49.3(^a))</td>
<td>2.02</td>
<td>0.03</td>
<td>10.3</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2^+_2)</td>
<td>52.9(^b))</td>
<td>0.13</td>
<td>0.01</td>
<td>10.6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2^+_3)</td>
<td>55.9(^c))</td>
<td>0</td>
<td>0</td>
<td>11.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2^+_1)</td>
<td>0.09</td>
<td>16.5</td>
<td>0.08</td>
<td>3.37</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2^+_2)</td>
<td>0</td>
<td>0.01</td>
<td>7.2</td>
<td>0.06</td>
<td>1.47</td>
<td></td>
<td></td>
<td></td>
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<td>(2^+_3)</td>
<td>16.7</td>
<td>0</td>
<td>0</td>
<td>3.34</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>(2^+_1)</td>
<td>0.28</td>
<td>8.8</td>
<td>2.7</td>
<td>2.50</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>(2^+_2)</td>
<td>0.13</td>
<td>18.1</td>
<td>0.8</td>
<td>3.84</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>(2^+_3)</td>
<td>0</td>
<td>11.8</td>
<td>0</td>
<td>2.36</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3^+_1)</td>
<td>61.1(^d))</td>
<td>0.38</td>
<td>0.37</td>
<td>12.4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3^+_2)</td>
<td>66.1(^e))</td>
<td>0.02</td>
<td>0.15</td>
<td>13.3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3^+_3)</td>
<td>70.9(^f))</td>
<td>0</td>
<td>0</td>
<td>14.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>(3^+_1)</td>
<td>2.9</td>
<td>0.2</td>
<td>13.4</td>
<td>3.46</td>
<td></td>
<td></td>
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<tr>
<td>(3^+_2)</td>
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<td>17.1</td>
<td>4.5</td>
<td>4.50</td>
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<tr>
<td>(3^+_3)</td>
<td>0</td>
<td>22.8</td>
<td>5.6</td>
<td>5.79</td>
<td></td>
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<td></td>
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<tr>
<td>(3^+_1)</td>
<td>0</td>
<td>0.2</td>
<td>6.8</td>
<td>6.09</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>(3^+_2)</td>
<td>0.13</td>
<td>0.1</td>
<td>13.9</td>
<td>3.14</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>(3^+_3)</td>
<td>0</td>
<td>0.1</td>
<td>12.7</td>
<td>2.54</td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>
correspondence is missing the dominant features are present in neighboring eigenstates.

For SU(3) coupled states the full set of tensors are labelled
\[ U(6) \times U(4) \times SU(2)_L \times SU(2)_R \]

For the scalar (L=0) parts of the interactions we have used, these are given in Table 5. The four U(6) scalars \((\{f\}^0)\) can be

Table 5. U(6)xSU(4) tensor coefficients for two-body scalar interactions in the ds-shell. Subscripts A and S refer to symmetric and antisymmetric two-particle space (\(K=1, L=0\)) and spin-isospin (\(S=A, S=0, T=0\)) symmetries.

\[
\begin{array}{cccccccc}
\theta & (\{f\}^0\{n\}^0) & \{f\} & n & (\{g\}^2) & C_2 & s^2 & t^2 & C_4^2/4 & L^2 \\
1 & [0\{0\}^0\{0\}^0] & 17.40 & -5.61 & -33.67 & 2.81 & 2.81 & -16.03 & -8.02 & \\
2 & [2^2] & 3.68 & -5.61 & 5.61 & -5.61 & 22.82 & \\
3 & [4^2] & 5.58 & 5.58 & 5.58 & 5.58 & 22.82 & \\
4 & [5^2] & 0.92 & 0.92 & 0.92 & 0.92 & 3.67 & \\
5 & [0\{0\}^0\{0\}^0] & -3.75 & -6.12 & -61.24 & -1.84 & -1.84 & 24.49 & 12.23 & \\
7 & [4^2] & -1.24 & -1.24 & 2.21 & & & \\
8 & [2^2] & -0.30 & -0.30 & -0.30 & -0.30 & & \\
9 & [4^2] & -1.24 & -1.24 & 2.21 & & & \\
10 & [2^2] & -0.30 & -0.30 & -0.30 & -0.30 & & \\
11 & [4^2] & -1.20 & -1.20 & -1.20 & -1.20 & & \\
12 & [2^2] & -0.10 & -0.10 & -0.10 & -0.10 & & \\
13 & [4^2] & 0.92 & 0.92 & 0.92 & 0.92 & & \\
14 & [2^2] & 0.92 & 0.92 & 0.92 & 0.92 & & \\
15 & [2^2] & -1.11 & -1.11 & -1.11 & -1.11 & & \\
16 & [4^2] & -0.21 & -0.21 & -0.21 & -0.21 & & \\
17 & [2^2] & -0.22 & -0.22 & -0.22 & -0.22 & & \\
18 & [4^2] & -0.50 & -0.50 & -0.50 & -0.50 & & \\
19 & [4^2] & 0.83 & 0.83 & 0.83 & 0.83 & & \\
20 & [2^2] & 0.16 & 0.16 & 0.16 & 0.16 & & \\
21 & [2^2] & 0.21 & 0.21 & 0.21 & 0.21 & & \\
22 & [4^2] & -0.06 & -0.06 & -0.06 & -0.06 & &
\end{array}
\]

written in terms of powers and products of generators of U(4). For these the set \((\{f\}, \{g\}, s^2, t^2)\) form a basis. All eleven U(4) scalars \((\{f\}^0, \{g\}^0)\) are combinations of U(6) generators. Of the latter only three are SU(3) scalars and for these \((\{f\}, \{g\}, \{c\})\) provides a basis. L is built from generators of SU(3) so does not break the symmetry but is neither a U(6) or SU(3) scalar. In II there are also other generator elements associated with the vector \((L=1)\) and tensor \((L=2)\) parts of the interaction. For these both \(\{f\}\) and \(\{g\}\) are nonzero as well as \(\{a\}\). The ratio \(\sigma^2/\epsilon = 74\%\) is therefore a pessimistic indication of the goodness of SU(3), a conclusion which the results for eigenvalues, overlaps and BE2 values support.

CONCLUSION

Averages propagate when the sums are group contractions and Green's functions can be built from generators of the symmetry. The propagation is simple if Casimir elements suffice but if non-invariant elements are needed it is in general a matrix of numbers that is carried forward. Special operators propagate simply regardless of circumstances and this motivates the notion of expanding one operator approximately in terms of others. In the defining space there is an algebra complete with inner product and norm which allows this to be done in a unique way. The norm of an approximation compared to that of the operator itself provides a completeness measure which if the approximation is built from group generators furnishes as well a measure for the goodness of the symmetry.

The technical challenge of calculating fixed-symmetry averages of products of operators remains. Though it would be desirable to have special purpose programs for this, scalar averaging suffices when an operator basis is known for then propagators can be built and the defining space average of one of these with any other operator produces the corresponding fixed-symmetry average. (In the defining space the propagators are fixed-symmetry projectors.) With such inner-product sums one can double of particles but for scalar averaging these are compensating (particle-hole) simplifications. So there is a need for versatile programs that can calculate scalar traces of products of several operators or at least which are of particle rank two or less. Better yet, in keeping with the spirit of statistical spectroscopy, there is a need for approximate methods for trace evaluation. Though some results are known, much remains unknown about this.

In our studies we have found that some group theoretical notions acquire new significance. Further work is suggested for it seems quite possible that in addition to shedding new light on the so-called missing-label problem (inner multiplicity) one should
might into the physics of outer multiplicities, relationships to symmetry breaking in physical situations where the space is partitioned to situations where the space is containing, other direct products are disturbing examples illustrating certainly a need for another operator in terms of others. At the same time for expanding a restricted basis for representations (Q-Q and Y for R, for example) on firm spec- tor appears here, for example, to be the potential shell link between microscopic and phenomenological shelling.

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J. P. Draayer

BIBLIOGRAPHY


