FERMION SYSTEMS WITH FUZZY SYMMETRIES
(LEVERAGING THE KNOWN TO UNDERSTAND THE UNKNOWN)

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Microscopic models, which embody the simplicity and significance of a dynamical
symmetry approach to nuclear structure, are reviewed. They can reveal striking
features of atomic nuclei when a symmetry dominates and solutions in domains
that may otherwise be unreachable.

1. Overview of algebraic fermion models. A theory that invokes
group symmetries is driven by an expectation that the wave functions of
the quantum mechanical system under consideration can be characterized
by their invariance properties under the corresponding symmetry transfor-
mations. But even when the symmetries are not exact, if one can find
near invariant operators, the associated symmetries can be used to help
reduce the dimensionality of a model space to a tractable size. Throughout
the years, group-theoretical approaches have identified fundamental sym-
metries in light to heavy nuclei and achieved a reasonable reproduction of
experimental data (for a review of fermion models, see ¹). In addition, they
provide theoretical predictions for nuclear systems including heavy unsta-
ble nuclei not yet explored, and 'exotic' nuclei, such as neutron-deficient or
$N \approx Z$ nuclei on the path of the nucleosynthesis $rp$-processes.

It is well-known that effective two-body interactions in nuclei are dom-
inated by pairing and quadrupole terms. The former gives rise to a pairing
gap in nuclear spectra, and the latter is responsible for enhanced electric
quadrupole transitions in collective rotational bands. Indeed, within the
framework of the harmonic oscillator shell-model, both limits have a clear
algebraic structure in the sense that the spectra exhibit a dynamical symme-
try. In the pairing limit the symplectic $\text{Sp}(4)$ ($\sim \text{SO}(5)^{2,3}$) group together
with its dual $Sp(2\Omega)$, for $2\Omega$ shell degeneracy, use the seniority quantum number\(^4\)\(^5\) to classify the spectra. On the other hand, in the quadrupole limit the SU(3) (sub-) structure\(^6\) governs a shape-determined dynamics.

In light deformed nuclei, $A \lesssim 28$, the Elliot's SU(3) model\(^6\), which incorporates the particle quadrupole and angular momentum operators, proved successful for a microscopic description of collective modes. Indeed, SU(3) is the exact symmetry group of the spherical oscillator, which is a reasonable approximation for the average potential experienced by nucleons in nuclei. Also, SU(3) is the dynamical symmetry group of the deformed oscillator, when, as is usually the case, the deformation is generated by quadrupole interactions. In many cases, a single-irrep or few-irreps calculations suffice to achieve good agreement with experimental rotational energy spectra and electromagnetic transitions (e.g., see \(^7\)\(^8\)).

Limitations due to the fact that the SU(3) model is applied within a shell and in turn requires effective charges for transition strengths are overcome in the Sp(3, R) symplectic shell model for light nuclei (for a review see \(^9\)). It embeds the SU(3) symmetry and in addition introduces important intershell excitations, including high-$\hbar \omega$ correlations and core excitations. The symplectic shell model is a microscopic formulation of the Bohr-Mottelson collective geometric model with a direct relation between a second- and a third-order scalar products of the quadrupole operator and the ($\beta$, $\gamma$)-shape variables. The Sp(3, R) symplectic model provides a microscopic description of monopole and quadrupole collective modes in deformed nuclei and a reproduction of experimental rotational energy spectra and electromagnetic transitions without effective charges (e.g., see \(^10\)\(^11\)).

Furthermore, in the domain of light nuclei one can combine the Sp(3, R) symplectic shell model and the no-core shell model (NCSM)\(^12\) to push forward the present frontiers in nuclear structure physics. The NCSM+Sp(3, R) allows us to use modern realistic interactions without any approximation (for the interaction and the size of the model space) for low-$\hbar \omega$ configurations and hence to fully account for important short- and intermediate-range correlations, while selecting only dominant high-$\hbar \omega$ basis states responsible for multi-shell development of collective motion.

In the region of medium mass nuclei around the $N = Z$ line (currently explored by radioactive beam experiments) protons and neutrons occupy the same major shell and hence their mutual interactions are expected to strongly influence the structure and decay modes of such nuclei. In addition to like-particle ($pp$ and $nn$) pairing correlations the close interplay of $pp$, $nn$ and proton-neutron ($pn$) pairs and the isospin-symmetry influence are microscopically described by the Sp(4) pairing model\(^13\).

For heavy nuclei ($A \gtrsim 100$), the discovery of the pseudo-spin symmetry\(^14\)\(^15\) and its fundamental nature\(^16\)\(^17\)\(^18\) establishes the pseudo-SU(3) model\(^19\). The pseudo-spin scheme is an excellent starting point for a many-particle description of heavy nuclei, whether or not they are deformed. As for the SU(3) shell model, in many cases leading-irrep calculations (e.g., see \(^20\)) or mixed-irrep calculations (e.g., see \(^21\)) achieve good agreement with experimental data. The pseudo-SU(3) shell model provides a further understanding of the $M1$ transitions in nuclei such as the even-even $160$–$164$ Dy and $156$–$160$ Gd isotopes, specifically it reflects on the scissors and twist modes as well as the observed fragmentation, that is, the break-up of the $M1$ strength among several levels closely clustered around a few strong transition peaks in the 2–4 MeV energy region\(^22\).

In medium-mass and heavy nuclei, where the pseudo-spin scheme can be applied to the normal parity orbitals and valence spaces are intruded by a unique parity highest-$f$ orbit from the shell above, a major step towards understanding the significance of the intruder level is achieved by a pseudo-SU(3) plus intruder level shell model\(^23\) that is currently under development\(^24\).

Furthermore, the advantages of the symplectic Sp(3, R) extension of the SU(3) model can be employed beyond the light nuclei domain towards a description of heavy nuclei in the framework of the pseudo-Sp(3, R) shell model (e.g., see \(^25\)). While early developments demonstrate the potential of such a model for studying the structure of heavy nuclear systems, it has not been fully explored.

In what follows, we present some recent results for three algebraic fermion models where the symmetries are fuzzy – meaning not exact – but nevertheless extremely useful in gaining a deeper understanding of the structure of real nuclei.

2. Symplectic Sp(4) pairing model. An algebraic approach, with Sp(4) the underpinning symmetry and with only six parameters, can be used to provide a reasonable microscopic description of pairing-governed $0^+$ states in a total of 306 even-even and odd-odd nuclei with mass $40 \leq A \leq 100$ where protons and neutrons are filling the same major shell\(^26\)\(^12\). We employ the most general Hamiltonian with Sp(4) dynamical symmetry,

\[ H_{sp(4)} = -G \sum_{i=1}^{4} \mathbf{A}_i \mathbf{A}_i - F \mathbf{A}_0 \mathbf{A}_0 - \frac{\hbar^2}{2\Omega} \left( \mathbf{T}^2 - \frac{3N}{4} \right) - D \left( \mathbf{T}^2 - \frac{N}{4} \right) - G \mathbf{N} \mathbf{N} \left( \frac{N-1}{2} \right) - \epsilon \mathbf{N}, \]

(1)
where $G, F, E, D, C$ and $\epsilon > 0$ are parameters (refer to Table I in 13 for their estimates). In (1), $\bar{N}$ counts the total number of particles, $T^2$ is the isospin operator and the $A_{0, \pm 1, \pm 1}$ group generators, which build the basis states by a consequent action on the vacuum state (a core like $^{40}$Ca or $^{56}$Ni), create, respectively, a proton-neutron (pn) pair, a proton-proton (pp) pair or a neutron-neutron (nn) pair of total angular momentum $J^z = 0^+$ and isospin $T = 1$. The model Hamiltonian (1) represents an effective microscopic interaction that conserves the $T_0$ third projection of the isospin and includes proton-neutron and like-particle isovector ($T = 1$) pairing plus symmetry terms (the latter is related to a proton-neutron isoscalar ($T = 0$) force). The model interaction (1) is found to correlate strongly with realistic interactions like CD-Bonn+3 terms 27 in the $1f_{7/2}$ region 28 and reflects a large portion of the GXPFI realistic interaction 29 in the upper-$fp$ shell. In addition, the $D$-term in (1) introduces isospin symmetry breaking and the $F$-term accounts for a plausible, but weak, isospin mixing 30. Both terms are significant in non-analog $\beta$-decays studies and also yield quantitative results that are better (e.g., by 85% in $1f_{7/2}$) than the ones with $F = D = 0$.

Good agreement with experiment (small $\chi$-statistics) is observed (Fig. 1) when theoretical eigenvalues of (1) are compared to Coulomb-corrected 31 experimental energies 32. The theory predicts the lowest isobaric analog $0^+$ state energy with a deviation ($\chi/\Delta E_{0,\text{exp}} \times 100\%$) of 0.5% for $1f_{7/2}$ and $1f_{5/2}2p_{3/2}2p_{3/2}1g_{9/2}$ nuclei in the corresponding energy range considered, $\Delta E_{0,\text{exp}}$. The model estimates the binding energy of the proton-rich $^{68}$Ni nucleus to be 348.1 MeV, which is 0.07% greater than the sophisticated semi-empirical estimates 33 and only 4% away from the experimental value reported later-on 34. The $^{68}$Se waiting-point nucleus along the rp-path is estimated to be 574.3 MeV, only 2.7% away from the 2004 precise mass measurement 35. Likewise, for the odd-odd nuclei with energy spectra not yet measured the theory predicts the energy of their lowest isobaric analog $0^+$ state: 358.62 MeV ($^{44}$V), 359.34 MeV ($^{48}$Mn), 357.49 MeV ($^{48}$Co), 394.20 MeV ($^{56}$Co) (Fig. 1, right). The $Sp(4)$ model predicts the relevant $0^+$ state energies for an additional 165 even-A nuclei in the medium mass region (Fig. 1, left). The binding energies for 25 of them are also calculated in 33. For these even-even nuclei, we predict binding energies that on average are 0.05% less than the semi-empirical approximation 33.

Furthermore, without any parameter variation, the theoretical energy spectra of the isobaric analog $0^+$ states are found to agree remarkably well with the experimental values where data is available 13. This agreement represents a valuable result because the higher-lying $0^+$ states under consideration constitute an experimental set independent of the data that enters the statistics to determine the model parameters in (1).

In addition, we examine the detailed features of nuclei by discrete derivatives of the energy function (1) filtering out the strong mean-field influence 36. This investigation reveals a remarkable reproduction of the two-proton $S_{2p}$ and two-neutron $S_{2n}$ separation energies, the irregularities found around the $N = Z$ region, the like-particle and pn isovector pairing gaps, and a prominent staggering behavior observed between groups of even-even and odd-odd nuclei 36. The zero point of $S_{2p}$ along an isotope sequence determines the two-proton-drip line, which according to the $Sp(4)$ model for the $1f_{5/2}2p_{3/2}2p_{3/2}1g_{9/2}$ shell lies near the following even-even nuclei: $^{66}$Ge, $^{64}$Se, $^{62}$Kr, $^{72}$Sr, $^{76}$Zr, $^{78}$Zr, $^{82}$Mo, $^{86}$Ru, $^{90}$Pd, $^{96}$Cd, beyond which the higher-$Z$ isotones are unstable with respect to proton emissions in close agreement with other estimates 37,33,38 despite the lack of experimental data. In addition, we find a small quadratic mean of the difference in $S_{2p}$ between our model and the other theoretical predictions where data is available, namely, 0.32 MeV in comparison with 37, 0.78 MeV with 33 and 0.43 MeV with 39.

While the model describes only isobaric analog $0^+$ states of even- $A$ medium mass nuclei with protons and neutrons in the same shell, it reveals a fundamental feature of the nuclear interaction, which governs these states. Namely, the latter possesses a clear $Sp(4)$ dynamical symmetry.
3. Pseudo-SU(3) plus intruder level model. The role of intruder levels that penetrate down into lower-lying shells in atomic nuclei has been the focus of many studies and debates. These levels are found in heavy deformed nuclei where the strong spin-orbit interaction destroys the underlying harmonic oscillator symmetry of the nuclear mean-field potential.

We carry out m-scheme shell-model calculations for the $^{58}$Cu and $^{64}$Ge nuclei in the $1f_{7/2}2p_{1/2}2p_{3/2}1g_{9/2}$ model space assuming the occupancy of the $f_{7/2}$ orbital to be ‘frozen’. This choice was motivated by the $f_{7/2}$ orbital’s high occupation as reported elsewhere\(^{39}\). The Hamiltonian we use is a $G$-matrix with a phenomenologically adjusted monopole part\(^{40}\). A renormalized version of this interaction in the $pf_{5/2}$ space has been introduced for describing beta decays\(^{41}\).

In addition, after rescaling occupancies of the states obtained in the upper $fp$ subspace, a pattern that is very similar to that of the occupancies in the larger $1f_{7/2}2p_{1/2}2p_{3/2}1g_{9/2}$ space is obtained.

In short, novel shell-model calculations for $^{58}$Cu and $^{64}$Ge in the $1f_{7/2}2p_{1/2}2p_{3/2}1g_{9/2}$ model space using a realistic interaction are compared to those generated by an appropriately renormalized counterpart of the interaction in the truncated upper $fp$ subspace. The results suggest that reliable computations can be performed in a space that does not explicitly include the intruder level as long as the interaction and the transition operators are renormalized appropriately.

4. No-core shell model plus Sp(3, R) extension. The symplectic shell model is based on the noncompact symplectic $sp(3, R)$ algebra with a subalgebraic structure that gives rise to rich underlying physics for a microscopic description of multiple collective modes in nuclei. This follows from the fact that the mass quadrupole and monopole moments operators, the many-particle kinetic energy, the angular and vibrational momenta are all elements of the $sp(3, R) \subset su(3)$ algebraic structure\(^{9}\).

The symplectic basis states, $|\Gamma_\sigma, \Gamma_\nu, \kappa; L(\omega)/M (\nu)\rangle$, are constructed by acting with polynomials in the symplectic raising operator, $A^{(20)}$, on a set of basis states of a symplectic bandhead, $|\Gamma_\nu\rangle$. They are labeled according to the reduction chain

$$\text{Sp}(3, R) \supset U(3) \supset SO(3)$$

$$\Gamma_\sigma \quad \Gamma_\nu \rho \quad \Gamma_\omega \quad \kappa \quad L$$

where $\Gamma_\sigma = N_\sigma (\lambda_\sigma, \mu_\sigma)$ labels Sp(3, R) irreducible representations. The $\Gamma_\nu = n (\lambda_\nu, \mu_\nu)$ set of quantum numbers gives the overall SU(3) coupling of $n/2$ raising operators acting on $|\Gamma_\sigma\rangle$. $\Gamma_\omega = N_\omega (\lambda_\omega, \mu_\omega)$ specifies the SU(3) symmetry of a symplectic state and $N_\omega = N_\sigma + n$ is the total number of oscillator quanta related to the eigenvalue, $N_\omega\hbar\omega$, of a center-of-mass motion free harmonic-oscillator (HO) Hamiltonian. The basis states of a Sp(3, R) irreducible representation can be expanded in HO ($m$-scheme) basis, which is the basis utilized by the no-core shell-model (NCSM)\(^{12}\).

In the case of $^{12}$C one can construct 13 unique so-called $0\hbar\Omega$-Sp(3, R) irreducible representations. $0\hbar\Omega$ means that the symplectic bandhead basis states, $|\Gamma_\sigma\rangle$, lie within $0\hbar\Omega$ many-particle harmonic-oscillator space. For each of the $0\hbar\Omega$-Sp(3, R) irreducible representations we generate basis states up to $N_{\text{max}} = 6 (6\hbar\Omega)$, which is the current limit for NCSM calculations. Typical the dimension of a symplectic representation is of order $10^{3}$.
The lowest-lying eigenstates of $^{12}$C are calculated by NCSM approach using the Many Fermion Dynamics (MFD) code$^{13}$ with the effective interaction derived from the realistic JISP16 $NN$ potential for oscillator strengths of $\hbar\Omega = 15$ MeV. The large overlaps of the symplectic states with the NCSM wave functions for 0, 2, 4 and 6$\hbar\Omega$ subspaces of the $m$-scheme basis (Tables 1) reveal that around 80% of the latter are symmetric under Sp(3,$\mathbb{R}$) transformations. Apparently, for all three wave functions, the highest contribution comes from the leading, most deformed, (0$4\rangle S = 0$ Sp(3,$\mathbb{R}$) irreducible representation. This contribution gets higher towards $J = 4^+_1$, where mixing due to other, less deformed, configurations decreases.

Clearly, the $0^+_1$, $2^+_1$ and $4^+_1$ states, which are constructed in terms of the three Sp(3,$\mathbb{R}$) irreps with probability amplitudes defined by the overlaps with the NCSM wave functions, can be used as a quite good approximation for a microscopic description of the $0^+_1$, $2^+_1$ and $4^+_1$ states in $^{12}$C. Within this assumption, the $B(E2: 2^+_1 \rightarrow 0^+_0,\text{stat.})$ value turns out to be as much as 81% of the NCSM estimate.

In short, the low-lying states in $^{12}$C are quite well explained by only three Sp(3,$\mathbb{R}$) irreps of 1998 symplectic states, that is only 0.003% of the NCSM space dimension, with a dominance of the most deformed (0$4\rangle S = 0$ collective configuration. Our findings, as a 'proof-of-principle', suggests that a NCSM+Sp(3,$\mathbb{R}$) structure could allow one to extend no-core calculations to higher $\hbar\Omega$ and heavier nuclei.

In summary, models based on exact or just good (broken but dominant) symmetries in fermion systems can play a significant role in our understanding low-lying nuclear structure; specifically, as shown here, in the development of collective rotational motion and the formation of correlated pairs in nuclei. They also allow us to truncate a model space to typically only a fraction the size encountered in models that do not exploit what we have dubbed here as fuzzy symmetries.

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References
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