QUANTUM ALGEBRAS AND QUANTUM GROUPS
NONCOMMUTATIVE GEOMETRY

Extension of the Draayer–Akiyama Procedure for the $u_q(3)$ Algebra

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Abstract—The well-known Draayer–Akiyama approach to calculating $u(3)$ Wigner coefficients is extended to the $u_q(3)$ quantum algebra. The analytic formula for special "seed" $u_q(3)$ Wigner coefficients that was presented by the authors elsewhere is simplified by reducing a triple sum to a single sum. The resulting formulation can be used as a starting point for a new generation Draayer–Akiyama-like code for $u(3)$ and $u_q(3)$ Wigner coefficients.

1. INTRODUCTION

The problem of decomposing a tensor product of $u(3)$ irreducible representations (irreps) into irreducible components and of giving a canonical definition for the outer multiplicity, which distinguishes multiple occurrences of an irrep in the product of $u(3)$ irreps, has been thoroughly investigated by many authors [1]. On the basis of Biedenharn, Louck, and Hecht (BLH) studies cited in [1], Draayer and Akiyama (DA) [2] developed a practical algorithm for evaluating $u(3)\otimes u(2)\times u(1)$ Wigner coefficients. The starting point of the DA prescription is a set of special "seed" Wigner coefficients

$$\langle \lambda_1 \mu_1 | H_1, (\lambda_2 \mu_2) e_2 A_2 m_2 | (\lambda_3 \mu_3) H_3 \rangle_{p_{\text{max}}}^q, \quad (1)$$

which are generated by using an empirically deduced result (see equation (20) from [2]). An analytic expression for these special seed coefficients of the DA algorithm was introduced in [3]. The derivation made use of the projection-operator (PO) method developed by the Moscow group [4].

In this study, we simplify the expression for the coefficients in (1) that was obtained in [3] in a form of triple sum to a single sum. This is done by using the q-analog [5] of the Jucys–Bandzaitis resummation procedure [6]. It should be noted that the possibility of expressing these coefficients in terms of the hypergeometric series $\mathbb{F}_2$ was indicated in [7].

This article is organized as follows. In Section 2, we review key features of $u_q(3)$ representation theory and reproduce the results from [3]. A simplification of these results is described in Section 3. Formulas that can be obtained for some finite sums of the ratios of $q$-factorials and which are needed to achieve a reduction of the expression for the seed coefficient from a triple sum to a single sum are given in the Appendix.

2. SEED WIGNER COEFFICIENTS FOR THE $u_q(3)$ QUANTUM ALGEBRA

Let $A_{ik}$ with $i, k = 1, 2, 3$ be a Cartan–Weyl basis for the $u_q(\mathfrak{u}(3))$ algebra [8]. The generators $A_{ik}$ obey the commutation relations

$$[A_{ij}, A_{kk}] = 0, \quad [A_{ip}, A_{kk}] = 0, \quad [A_{ip}, A_{jk}] = \delta_{ij} A_{ik} - \delta_{ik} A_{ji}, \quad (2)$$

$$[A_{ik}, A_{kj}] = A_{ii} - A_{kk}. \quad (3)$$

Composite generators can be expressed in terms of the $q$-deformed commutators

$$A_{13} = [A_{12}, A_{23}]_q = A_{12} A_{23} - q A_{23} A_{12}, \quad (5)$$

$$A_{31} = [A_{32} A_{21}]_q = A_{32} A_{21} - q^{-1} A_{21} A_{32} \quad (6)$$

and satisfy the identities

$$[A_{12}, A_{13}]_q = 0, \quad [A_{21}, A_{31}]_q = 0, \quad (7)$$

$$[A_{23}, A_{13}]_q = 0, \quad [A_{32}, A_{31}]_q = 0,$$

$$[A_{21}, A_{13}] = A_{23} A_{11} - A_{12}, \quad [A_{31}, A_{12}] = q^{-A_{11}} A_{23} A_{12}, \quad (8)$$

$$[A_{13}, A_{32}] = q^{-A_{23}} A_{12}, \quad [A_{23}, A_{31}] = A_{21} q^{-A_{32}} A_{31}.$$

The generators $A_{12}$ and $A_{31}$ are chosen for the canonical $u_q(2)$ subalgebra, which is used for labeling a basis state in the $U$ basis.

Here and below, $[x]$ and $[x]!$ are, respectively, $q$-numbers and $q$-factorials defined as

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad [x]! = [x][x-1][x-2][1]. \quad (8)$$
where \([1] = [0] = 1\) and \([-n]\) = \(-\infty\). These expressions are all symmetric under the interchange \(q \rightarrow q^{-1}\). In what follows, it is also assumed that \(q\) is a nonnegative real number.

Let \(D^{(q)}\) be an irrep of the \(u_q(3)\) algebra with highest weight \([f] = (f_1, f_2, f_3)\). The symbol \(D^{(\lambda, \mu)}\) will also be used to denote an irrep of \(su(3)\); here, \(\lambda = f_1 - f_2\) and \(\mu = f_2 - f_3\). The basis states \([f] (\alpha) = [\lambda, \mu] e M A \Lambda\) of the irrep \(D^{(q)}\) are labeled by the hypercharge \(Y\) (or \(e\)), the isospin \(\Lambda\), and its projection \(M_{\Lambda}\).

\[
\varepsilon = -3Y = - (\lambda + 2\mu) + 3(l + k), \quad (9)
\]

\[
\Lambda = \frac{1}{2}(\lambda + l - k), \quad (10)
\]

\[
M_{\Lambda} = r - \Lambda, \quad (11)
\]

where the integers \(l, k, r\) satisfy the inequalities \(0 \leq k \leq \lambda, 0 \leq l \leq \mu,\) and \(0 \leq r \leq 2\Lambda\).

A decomposition of the tensor product of irreps into irreducible components can be represented as

\[
D^{(\lambda_1, \mu_1)} \times D^{(\lambda_2, \mu_2)} = \sum_{(\lambda, \mu)} V_{(\lambda, \mu)} D^{(\lambda, \mu)} \quad (12)
\]

An outer multiplicity label \(\rho = 1, 2, \ldots, \), \(V_{(\lambda, \mu)}\) is used to distinguish multiply occurring irreps \(D^{(\lambda, \mu)}\). The right-hand side of (12) takes the same form for \(su(3)\) and for \(su_q(3)\).

By definition, the Wigner coefficients \(\langle (\lambda_1, \mu_1), (\lambda_2, \mu_2) : \rho (\lambda, \mu) \alpha \rangle_q\) are the elements of a unitary transformation between the coupled and uncoupled representations of \(su_q(3)\) in the \(\alpha\) scheme:

\[
\langle (\lambda_1, \mu_1), (\lambda_2, \mu_2) : \rho (\lambda, \mu) \alpha \rangle_q = \sum_{\alpha_1, \alpha_2} \langle (\lambda_1, \mu_1) \alpha_1, (\lambda_2, \mu_2) \alpha_2 \rangle (\lambda, \mu) \alpha \rangle_q \quad (13)
\]

\[
\times [\langle \lambda_1 \mu_1 \rangle \alpha_1 (\lambda_1 \mu_2 \alpha_2) \rangle_q \quad (13) \right. \]

\[
\times [\langle \lambda_1 \mu_1 \rangle \alpha_1 (\lambda_2 \mu_2 \alpha_2) \rangle_q
\]

The importance of Wigner coefficients for physics applications is well known. On the basis of the BLH scheme for the outer multiplicity, DA developed a practical algorithm for evaluating all \(su(3) \supset su(2) \times u(1)\) Wigner coefficients. The DA prescription includes a straightforward Schmidt orthogonalization procedure for producing independent orthogonal vectors in the multiplicity subspace.

As was noted above, the DA scheme requires special seed coefficients as given by (1). If these seeds are known, all the remaining ("nonseed") Wigner coefficients can be generated recursively. The seed coefficients are associated with an identity tensor operator that is characterized by a maximal null space and which includes the highest weight states \([H] = (f_1, f_2, f_3)\) and \([H]\) of the irreps \((\lambda_1, \mu_1)\) and \((\lambda_2, \mu_2)\), respectively.

The original DA prescription used a result determined empirically for the seed coefficients.

The analytic expressions are now available from (3). The analytic result for the coefficient in (1) has the form

\[
\langle (\lambda_1, \mu_1) H_{1q}, (\lambda_2, \mu_2) e_2 A_2 m_2 (\lambda_1, \mu_1) H \rangle_q \quad (14)
\]

where

\[
A = (-1)^{N_1 - N_2 + \lambda_1 - \lambda_2 - m_2 q} \quad (15)
\]

\[
\times \left[ \frac{[\lambda + 1]! [\mu + 1]! [\lambda + \mu + 2]! [2\lambda_2 + 1]!}{[\lambda + \mu + 2 + N_1]!} \right]^{1/2}
\]

\[
\times \left( \frac{[\lambda_1]! [\mu_1]! [\lambda_1 + \mu_1 + 1]! [N_1]!}{[N_2]! [\lambda_1 - N_1]!} \right)^{1/2}
\]

\[
\times \left( \frac{[\lambda_1]! [\mu_1]! [\lambda_1 - \lambda_2 - \mu_2 + 1]! [\lambda_1 + N_1 + 1]!}{[\lambda_1 + N_1 + 1]! [\lambda_1 + N_2 + 1]!} \right)^{1/2}
\]

\[
\times \left( \frac{[\lambda_2]! [\mu_2]! [\lambda_2 + \mu_2 + 1]! [\lambda_2 - m_2]!}{[\lambda_2 - m_2]! [\lambda_2 + m_2 - N_2]!} \right)^{1/2}
\]

\[
\times \left( \frac{[N_2 + \mu_2 - \mu + \Lambda_2 - m_2]! [\mu - \lambda_2 + m_2 - N_1]!}{[\lambda_2 + \lambda_2 - m_2 + N_1]!} \right)^{1/2}
\]

\[
\times \left( \frac{[N_2 + \mu_2 - \mu + \Lambda_2 - m_2 - N_1]!}{[\lambda_2 + \mu_2 + \lambda_2 - m_2 - N_2 + 1]!} \right)^{1/2}
\]

\[
\times \left( \frac{1}{[\lambda_2 + \mu_2 + \lambda_2 - m_2 - N_1 + 1]!} \right)
\]

with

\[
\alpha = \frac{1}{2}(N_1 (\lambda_2 - \lambda_1 - \mu_1) + N_2 (\mu_1 + \mu_2) - \mu_1 - (\lambda_2 - m_2)) (\Lambda_2 + m_2 + 1) \quad (16)
\]

We also have

\[
V = \sum_{u, r, s} B_{ur, s} \quad (17)
\]

where

\[
B_{ur, s} = (-1)^{r + s} q^r \left[ \Lambda_2 + m_2 + r \right] [\lambda_2 - m_2 - r] [\lambda_2 + m_2 + r] [\lambda + N_1 - u + 1] [\mu - \lambda_2 + N_1 + 1] [\mu_2 - \mu + N_2 + r] [\lambda_1 + \lambda_2 + \mu_2 + s - u + 2] [\mu - N_2 + s - r] [\lambda - N_2 + s + 1] [\lambda + \mu - N_2 + s + 2] [\mu_1 - s] [s - N_2 - u] \quad (18)
\]
with
\[ \beta = u + r(\lambda_2 + \mu_2 - \mu + \mu_1 + N_2 + 1) + s(\mu - \mu_2 - N_2 - r), \]  
(19)
\[-\lambda_2 \leq m_2 \leq \lambda_2, \quad m_2 = \frac{1}{2}(\lambda - \lambda_1), \]  
(20)
\[ \epsilon_2 = - (\lambda + 2\mu) + (\lambda_1 + 2\mu_1), \]  
(21)
\[ N_1 = 2n + \mu - \mu_1 - \mu_2, \quad N_2 = \mu_1 - n, \]  
(22)
\[ n = \frac{1}{3}(-\lambda - 2\mu + \lambda_1 + 2\mu_1 + \lambda_2 + 2\mu_2). \]  
(23)

The normalization factor has the form
\[ N^2 = \langle \langle \lambda_1 \mu_1 \rangle \epsilon_1 \Lambda_1 \Lambda_1, (\lambda_2 \mu_2) M_2 \rangle (\lambda \mu) M \rangle_{\text{max}}^2 \]  
(24)
\[ = \sum_{n_1, n_2} G_{n_1, n_2}, \]
where
\[ G_{n_1, n_2} = \frac{(-1)^{n_1 + n_2} q^{n_1 + n_2 - n_2 \mu_2}}{[n_1!][n_2!][\lambda + 1 + n_1][\mu + 1 + n_2]} \]
\[ \times \frac{[\lambda + 1 + n_2][\lambda + 2 + n_2][\lambda + \mu + 2 + n_2 + 1]}{[\lambda + 1 + n_1 + n_2 + 1][\lambda + 1 + n_1 + n_2 + 1][\lambda + 1 + n_1 + n_2 + 1]} \]
\[ \times \frac{[\lambda + 1 + n_1 + n_2 + 1][\lambda + 1 + n_1 + n_2 + 1][\lambda + 1 + n_1 + n_2 + 1]}{[\lambda + 1 + n_1 + n_2 + 1][\lambda + 1 + n_1 + n_2 + 1][\lambda + 1 + n_1 + n_2 + 1]} \]  
(25)

with
\[ k_1 = \frac{1}{3}(-2\lambda - \mu + 2\lambda_1 + \mu_1 + 2\lambda_2 + \mu_2) \]
\[ = \mu - \mu_1 - \mu_2 + 2n, \]  
(26)
\[ l_1 = \frac{1}{3}(-2\lambda - \mu - \lambda_1 + \mu_1 - \lambda_2 - 2\mu_2) = \mu_1 - n. \]  
(27)

The integers \( k_1, l_1, n_1, \) and \( n_2 \) satisfy the restrictions \( 0 \leq k_1 \leq \lambda_1, 0 \leq l_1 \leq \mu, 0 \leq n_1 \leq k_1, \) and \( 0 \leq n_2 \leq \mu - l_1 - l. \)

The Wigner coefficient in (24) corresponds to the basis state,
\[ \langle (\lambda \mu) M \rangle_q = \langle (\lambda \mu) \epsilon_M \Lambda_M \Lambda_M \rangle_q, \]  
(28)
which is characterized by the dominant weight \( M = (f_3 f_3 f_2) \). The quantum numbers \( \Lambda_M \) and \( \epsilon_M \) of this state are given by
\[ \Lambda_M = \frac{1}{2}(\lambda + \mu), \quad \epsilon_M = -(\lambda - \mu). \]  
(29)

Similarly, \( M_2 \) is the dominant weight in the irrep \( (\lambda_2 \mu_2) \); that is,
\[ \Lambda_{M_2} = \frac{1}{2}(\lambda_2 + \mu_2), \quad \epsilon_{M_2} = -(\lambda_2 - \mu_2). \]  
(30)

The values of the quantum numbers \( \Lambda_1 \) and \( \epsilon_1 = \epsilon - \epsilon_2 \) are
\[ \Lambda_1 = \frac{1}{2}(\lambda + \mu) - \frac{1}{2}(\lambda_2 + \mu_2) \geq 0, \]  
(31)
\[ \epsilon_1 = -(\lambda - \mu) + (\lambda_2 - \mu_2). \]

The Wigner coefficient in (24) corresponds to the maximum possible value of \( \Lambda_{M_2} \) and the maximum change,
\[ \Delta \Lambda = |\Lambda - \Lambda_1|, \]  
(32)
in the angular momentum, \( \Delta \Lambda = \frac{1}{2}(\lambda_2 + \mu_2). \)

The results presented in (15)–(32) are valid for \( \lambda + \mu \geq \lambda_2 + \mu_2 \). For \( \lambda + \mu < \lambda_2 + \mu_2 \), it is necessary to use the symmetry properties of the Wigner coefficients with respect to a permutation of the irreps \( (\lambda_1 \mu_1) \) and \( (\lambda \mu) \) or to determine an analytic continuation of the relations found above to the region \( \lambda + \mu < \lambda_2 + \mu_2 \) (see [9]).

3. SIMPLIFICATION OF THE FORMULA FOR THE STARTING WIGNER COEFFICIENTS

The goal of this section is to simplify the expression for the starting Wigner coefficients (14). A specific objective is to reduce the triple factorial sum for \( V \) in (17) to a single sum by means of a resummation of the \( q \)-factorial expressions in \( B_{\text{ws}} \) (18). The procedure that we will use relies on the summation formulas presented in [6] for finite \( q \)-factorial sums. The resummation procedure is the \( q \)-analog of the well-known classical scheme of Jacobi and Bandzaitis [6]. We begin by rewriting expressions (17)–(19) in the more convenient form
\[ V = \sum_{u, r, x} B_{\text{ws}} \]
\[ = \sum_{r} (-1)^{r} q^{(\lambda_2 + \mu_2 + \mu_1 - \mu + N_2 + 1)} \frac{[\lambda_2 + m_2 + r]}{[\lambda_2 - m_2 - r][\mu_2 - \mu + N_2 + r]} V_1, \]  
(33)
where
\[ V_1 = \sum_{u, x} (-1)^{x} q^{(\mu - N_2 + s - r)} \]
\[ \times \frac{[\mu - N_2 + s + 1]}{[\mu - N_2 + s - r][\mu_1 - s]} \]
\[ \times q^{s} [\lambda + \mu + N_1 - N_2 + s + u + 2] \]
\[ \times \frac{1}{[\lambda + N_1 - u + 1]}, \]  
(34)
First, we transform the sum over the index \( u \) in \( V_1 \) (denoted by \( V_2 \)) into a triple sum by introducing the \( \delta(u, v) \) symbol given by (A.3). We then have:

\[
V_2 = \sum_{u \neq v} q^{(\lambda + \mu + N_1 - N_2 - s - u + 2)!} \frac{(s - N_2 - u)!(\lambda + N_1 - u + 1)!}{(u - z)!(z - v)!(v)!(r - v)!(\mu + r + v)!}.
\]

Finally, the triple-sum form for \( V \) as given by (33) reduces to a single sum over the index \( r \). Specifically, we have

\[
V = \sum_{u, r, s} B_{ur} = G \sum_r B'_r,
\]

where

\[
G' = \frac{(-1)^{N_2}}{[\lambda + \mu + 1 - N_2 + 2]!(\mu + 1 - N_2 + 1)!}.
\]

The resulting expressions (41)–(43) can be recast into a form that can be associated with a Wigner coefficient for the \( u_4(1, 1) \) algebra or presented in terms of the basis hypergeometric function \( F_1 \) [7, 11].

In conclusion, we note that the simplified expression that we have obtained for the starting \( u_4(3) \) Wigner coefficients can form a basis for developing a new generation DA-like algebraic computer code for calculating \( u(3) \) and \( u_4(3) \) Wigner coefficients. In other words, there is now a new opportunity to build an even simpler and more user-friendly \( u(3) \) toolkit.

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APPENDIX

Finite \( q \)-Factorial Sums [10]

Using the \( q \)-analog of the Vandermonde formula (see [10]),

\[
\sum_{\substack{a \geq 0 \\ b \geq 0}} \frac{(-1)^a [\beta]_q}{[s]!} \frac{q^{d(a + b + 1)}}{[c + d + 1]!(b + d + 1)!} = \frac{[\gamma - \beta]_n}{[\gamma]_n} q^{\beta n}, \tag{A.1}
\]

where

\[
[a]_n = [a][a + 1]...[a + n - 1]. \tag{A.2}
\]
we can obtain the following summation formulas for finite q-factorial sums:

\[ \sum_s \frac{(-1)^{a-r}}{(a-s)!(b-s)!} \frac{x^{a(b-1)}}{q} = \delta(a, b), \]  

\[ \sum_s \frac{(-1)^{a-r}}{(a-s)!(b-s)!} \frac{x^{a(b-1)+a}}{q} = \delta(a, b), \]  

\[ \sum_s \frac{1}{(a-s)!(b-s)!} x^{a} \]  

\[ = \frac{[a]!}{[b]![c]!(a-b)!(a-c)!} x^{bc}, \]  

\[ \sum_s \frac{(-1)^{a-s}}{(a-s)!} x^{2a} \]  

\[ = \frac{1}{[b]![c]!(b-a+1)!(b-c)!} x^{2bc}, \]  

for \( b > a \geq c; \)

\[ \sum_s \frac{(-1)^{a-s}}{(a-s)!} x^{2a} \]  

\[ = \frac{[a-b]![a-c]!}{[b-a]![c]!} x^{2bc}, \]  

for \( a \geq b, c; \) and

\[ \sum_s \frac{(-1)^{a+s}}{(a+s)!} x^{2a} \]  

\[ = \frac{[a]![b+c-a]!}{[c]![b+c]!(b-a)!} x^{2(a+1)c}, \]  

for \( b \geq a. \)

REFERENCES


