Exactly Solvable Pairing
Plus Mean Field Model

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Abstract. A mean-field plus extended pairing interaction Hamiltonian that includes multi-
pair scattering terms is proposed for describing well-deformed nuclei. Eigenvalues of the
model are easily obtained. The investigation shows that even through the extended pairing
includes many-body terms, the one- and two-body interactions continue to dominate the
dynamics for relatively small values of the pairing strength. As the strength of the pairing
interaction grows, however, the higher many-body interaction terms grow in importance. A
numerical study of even-odd mass differences in the 144-171Yb isotopes demonstrates the
applicability of the theory.

Pairing is an important interaction that is widely used in nuclear physics and
other fields. And it is one that Stuart Pittel, whose 60th birthday we celebrate,
along with his collaborators have added so much by articulating its microscopic
underpinnings as well as macroscopic outpinnings [1]. Through this contribution,
my colleagues and I wish to step into that world and offer what we believe is
a heretofore unknown solution of an extended pairing model — so, Stu, in what
follows we present to you and your colleagues an 'algebraic solution' to an extended
version of the pairing problem that has been so central to your work in nuclear
physics [2].

The Bardeen-Cooper-Schrieffer (BCS) [3] and Hartree-Fock-Bogolyubov (HFB)
[4] methods for finding approximate solutions are well known. However, the limita-
tions of BCS methods, when applied in nuclear physics, are also well known.
First of all, not only is the number of nucleons in a nucleus typically small, the
number of valence particles (n ~ 10) which dominate the behavior of low-lying
states is far too few to support the underlying assumptions of the approxima-
tions, that is, particle number fluctuations are non-negligible. As a result, particle
number-conservation effects can lead to serious problems such as spurious
states, nonorthogonal solutions, and so on. Furthermore, an essential feature of
pairing correlations are differences between neighboring even and odd mass nu-
clei, which are driven mainly by Pauli blocking effects, and it is difficult to treat
even-odd differences with either the BCS or HFB theories because different quasi-

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particle bases must be introduced for different blocked levels. Another problem with approximate treatments of the pairing Hamiltonian is related to the fact that both the BCS and the HF approximation break down for an important class of physical situations. A remedy that uses particle number projection techniques complicates the algorithms considerably and does not help to achieve a better description of the higher-excited part of the spectrum of the pairing Hamiltonian. The importance of having exact solutions of the pairing Hamiltonian has driven a great deal of work in recent years. In particular, building on Richardson's early work [5, 6, 7] and extensions to it based on the Bethe ansatz, several authors have introduced novel approaches [8, 9, 10, 11, 12, 13, 14]. For the algebraic approaches based on the Bethe ansatz, the solutions are provided by a set of highly non-linear Bethe Ansatz Equations (BAEs). Though these applications demonstrate that the pairing problem is exactly solvable, solutions are not easily obtained and normally require extensive numerical work, especially when the number of levels and valence pairs are large. This limits the applicability of the methodology to relatively small systems; for example, it cannot be applied to large systems such as well-deformed nuclei.

The standard pairing Hamiltonian for well-deformed nuclei is given by

$$\hat{H} = \sum_{j=1}^{p} \epsilon_j n_j - G \sum_{i,j=1}^{p} a_i^+ a_j,$$  \hspace{1cm} (1)

where $p$ is the total number of Nilsson levels, $G > 0$ is the pairing strength, $\epsilon_j$ is single-particle energies taken from the Nilsson model, $n_j = c_{j1}^d c_{j1}^d + c_{j2}^d c_{j2}^d$ is the fermion number operator for the $j$-th Nilsson level, and $a_i^+ = c_{i1}^d c_{i1}^d \ (a_i = (a_i^+)^\dagger = c_{i1}^d c_{i1}^\dagger)$ are pair creation (annihilation) operators. The up and down arrows in these expressions denote time-reversed states. Since each level can only be occupied by one pair due to the Pauli Principle, the Hamiltonian (1) is also equivalent to a finite site hard-core Bose-Hubbard model with infinite range one-pair hopping and infinite on-site repulsion. Specifically, the operators $a_i^+, a_i$, and $n_i^\dagger = n_i/2$ satisfy the following hard-core boson algebra:

$$ (a_i^+)^2 = 0, \quad [a_i, a_j^+] = \delta_{ij}(1 - 2n_i^\dagger), \quad [a_{i+}, a_{j+}] = [a_i, a_j] = 0. $$ \hspace{1cm} (2)

As an extension of the usual approach (1), we construct the following new (extended) pairing Hamiltonian:

$$\hat{H} = \sum_{j=1}^{p} \epsilon_j n_j - G \sum_{i,j=1}^{p} a_i^+ a_j$$

$$-G \left( \sum_{\mu=1}^{\infty} \frac{1}{(\mu!)^2} \sum_{i_1 \neq i_2 \neq \ldots \neq i_{2\mu}} a_{i_1}^+ a_{i_2}^+ \cdots a_{i_\mu}^+ a_{i_{\mu+1}} a_{i_{\mu+2}} \cdots a_{i_{2\mu}} \right),$$ \hspace{1cm} (3)

where no pair of indices among the $\{i_1, i_2, \ldots, i_{2\mu}\}$ are the same for any $\mu$. Besides the usual mean-field and the standard pairing interaction (1), this form includes
many-pair hopping terms that allow nucleon pairs to simultaneously scatter (hop) between and among different levels. With this extension, we will show that the model is exactly solvable.

Because of infinite on-site repulsion, the sum in (3) truncates for \( \mu \leq [p/2] \), where \([x]\) is the integer part of \( x \). It is also easy to see that each term of the form \( a_i^+ a_j^z \) that enters into eigenstates of (3) should come with different indices \( i \neq \cdots \neq j \).

Let \( |j_1, \cdots, j_m\rangle \) be the pairing vacuum state that satisfies

\[
a_i |j_1, \cdots, j_m\rangle = 0
\]

for \( 1 \leq i \leq p \), where \( j_1, j_2, \cdots, j_m \) indicates those \( m \) levels that are occupied by single nucleons. Any singly-occupied state is blocked by the Pauli principle.

The algebraic ansatz introduced in [15] suggests that the \( k \)-pair eigenstates of (3) can be written as

\[
|k; \zeta; j_1, \cdots, j_m\rangle = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq p} C_{i_1 i_2 \cdots i_k}^{(k)} a_{i_1}^+ a_{i_2}^+ \cdots a_{i_k}^+ |j_1, \cdots, j_m\rangle,
\]

where \( C_{i_1 i_2 \cdots i_k}^{(k)} \) is an expansion coefficient that must be determined, and the strict ordering to the indices \( i_1, i_2, \cdots, i_k \) is a reminder that double occupation is not allowed. It is always assumed that the level indices \( j_1, j_2, \cdots, j_m \) should be excluded from the summation in (5). Since the formalism for even-odd systems is similar, in what follows we focus on the even-even seniority zero case.

The expansion coefficient \( C_{i_1 i_2 \cdots i_k}^{(k)} \) can be expressed very simply as

\[
C_{i_1 i_2 \cdots i_k}^{(k)} = \frac{1}{1 - \frac{1}{\pi(k)} \sum_{\mu=1}^{k} \epsilon_{i_{\mu}}},
\]

where, similar to the results given in the Bethe ansatz approach, \( \pi(k) \) is a c-number that is to be determined. To prove that the algebraic ansatz given in (5) and (6) are consistent, one may directly apply Hamiltonian (3) on the \( k \)-pair state (5). Using the hard-core boson algebraic relation given by (2) and a procedure that is similar to that used in Ref. [10] for finding exact solution of a Heisenberg algebra Hamiltonian, one can determine that for the mean-field part of the Hamiltonian (3)

\[
\sum_j \epsilon_j n_j |k; \zeta; 0\rangle = \frac{2}{\pi(k)} \left( |k; \zeta; 0\rangle - \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq p} a_{i_1}^+ a_{i_2}^+ \cdots a_{i_k}^+ |0\rangle \right),
\]

and for the re-arranged extended pairing part of the Hamiltonian (3)

\[
\left( \sum_i a_i^+ a_i + \sum_{\mu=1}^{\infty} \frac{1}{(\mu)!^2} \sum_{i_1 \neq i_2 \cdots \neq i_{2\mu}} a_{i_1}^+ a_{i_2}^+ \cdots a_{i_{2\mu}}^+ \cdots a_{i_{2\mu-1}}^+ a_{i_{2\mu+2}}^+ \cdots a_{i_{2p}}^+ \right) |k; \zeta; 0\rangle =
\]

\[
\left( \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq p} C_{i_1 i_2 \cdots i_k}^{(k)} \right) \left( \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq p} a_{i_1}^+ a_{i_2}^+ \cdots a_{i_k}^+ |0\rangle + (k-1)|k; \zeta; 0\rangle \right)
\]
By combining Eqs. (7) and (8), the $k$-pair excitation energies of (3) are given by:

$$E_k^{(c)} = \frac{2}{x^{(c)}} - G(k - 1),$$

(9)

where the undetermined variable $x^{(c)}$ is given by

$$\frac{2}{x^{(c)}} + \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq p} \frac{G}{(1 - x^{(c)}) \sum_{\mu=1}^{k} \epsilon_{i_{\mu}}} = 0.$$  

(10)

The additional quantum number $\zeta$ now can be understood as the $\zeta$-th solution of (10). Similar results for even-odd systems can also be derived by using this approach except that the index $j$ of the level occupied by the single nucleon should be excluded from the summation in (5) and the single-particle energy term $\epsilon_j$ contributing to the eigenenergy from the first term of (3) should be included. Extensions to many broken-pair cases are straightforward.

Comparing Eqs. (9) and (10) to exact solutions of the Heisenberg algebraic Hamiltonian with a one-body interaction [10], one can consider the operator product $a_{i_1}^+ a_{i_2}^+ \cdots a_{i_k}^+$ in (5) as a ‘grand’ boson. The corresponding ‘single-particle energy’ of this ‘grand’ boson is $E_{i_1 i_2 \ldots i_k} = \sum_{\mu=1}^{k} 2 \epsilon_{i_{\mu}}$, since (10) and the eigenstates (5) are similar to those for a multi-boson system with a one-body interaction as shown in [10], even though the Hamiltonians are totally different. It should be noted that even through all of the eigenstates (5) with distinct roots given by (10) are orthogonal [10], they are not normalized (5) but can be made so using standard procedures.

Eigenenergies of the standard pairing model can be expressed in terms of $k$ variables that satisfy $k$ coupled nonlinear equations. Such a system is very difficult to solve numerically, especially when the number of pairs $k$ and number of levels $p$ are large. However, in contrast to a BAE solution of the standard pairing model, for the extended pairing model there is but a single variable $x^{(c)}$. It should be noted that the solution (10) requires the single-particle energies $\sum_{\mu=1}^{k} \epsilon_{i_{\mu}}$ to be all different. Fortunately, this is the case when the single-particle energies are generated from typical mean fields such as, for example, the Nilsson potential. When this holds, (10) has $\frac{p!}{(p - k)!}$ distinct roots, which could be a large number, for example, for an entire deformed major shell.

If (10) is rewritten in terms of a new variable $z^{(c)} = \frac{2}{(Gx^{(c)})}$ and the dimensionless energy of the ‘grand’ boson $\bar{E}_{i_1 i_2 \ldots i_k} = \sum_{\mu=1}^{k} \frac{2 \epsilon_{i_{\mu}}}{G}$, (10) takes the form:

$$1 = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq p} \frac{1}{(\bar{E}_{i_1 i_2 \ldots i_k} - z^{(c)})}$$

(11)

Since there is only a single variable $z^{(c)}$ in (11), the zero points of the function can be determined graphically, in a manner that is similar to the one-pair solution of the TDA and RPA approximations with separable potentials[4]. From expression (11), it is clear that any solution $z^{(c)}$ of (11) is located between two nearby values of the dimensionless ‘grand’ boson energy $\bar{E}_{i_1 i_2 \ldots i_k} = \sum_{\mu=1}^{k} \frac{2 \epsilon_{i_{\mu}}}{G}$ and the smallest solution $z^{(c)}$ would be smaller than $\min\{\bar{E}_{i_1 i_2 \ldots i_k}\}$.
FIGURE 1. (a) Spectral structure of the standard pairing interaction given by Eq. (1), and 
(b) spectral structure of the extended pairing interaction given by Eq. (3), as functions of the 
pairing interaction strength $G$ for $k = 5$ pairs for a system with $p = 10$ levels and single-particle 
energies $\epsilon_1 = 1.179$, $\epsilon_2 = 2.65$, $\epsilon_3 = 3.162$, $\epsilon_4 = 4.588$, $\epsilon_5 = 5.006$, $\epsilon_6 = 6.969$, $\epsilon_7 = 7.262$, $\epsilon_8 = 8.687$, $\epsilon_9 = 9.899$, $\epsilon_{10} = 10.201$, where the single-particle energies and $G$ are given in arbitrary units. 
The straight dashed line is the expectation value of the Hamiltonian in the pure pairing ($\epsilon_i = 0$) 
ground state.

It is important to understand the differences between the extended pairing intro-
duced by (3) and the standard pairing given in (1). For this purpose, we consider 
a simple example in which there are $p = 10$ levels and the single-particle energies 
are given by $\epsilon_i = i + \chi_i$ for $i = 1, 2, \ldots, 10$, where $\chi_i$ are random numbers within the 
interval $(0, 1)$ to avoid accidental degeneracy required for exact solvability, and the 
pairing strength $G$ is allowed to vary from 0.01 to 0.10. Fig. 1 shows the lowest 
few energies of the standard and extended pairing models. From this graph it is 
very clear that there are essential differences in the spectral structure of these two 
models. As shown by Inset (b), the extended pairing model very rapidly develops 
a paired ground-state configuration which is strongly dependent on the pairing 
strength $G$. In this case the transition from mean-field eigenstates to pairing eigen-
states is very sharp and fast, while the standard pairing model, Inset (a), exhibits 
a much slower and smoother transition. The quantitative difference in the two 
spectra, with the extended pairing case showing a much stronger dependence on $G$ 
than for standard pairing, is a very clear distinguishing characteristic and can be 
used to explore cases where the extended pairing concept might be more relevant 
and appropriate than the standard pairing model.

It is important to know whether the dynamics is still dominated by the one- and 
two-body interactions, and, if not, under what conditions the higher order terms
can be treated perturbatively. To explore this, we calculated as a function of $G$ the expectation value of each higher order term $\langle V_\mu \rangle$ defined by

$$V_1 = \sum_{i,j} a_i^\dagger a_j, \quad V_\mu = \frac{1}{(\mu!)}^2 \sum_{\ell_1 \neq \ell_2 \neq \ell_p} a_{\ell_1}^\dagger a_{\ell_2}^\dagger \cdots a_{\ell_p}^\dagger a_{\ell_p+1} a_{\ell_p+2} \cdots a_{\ell_1}$$

(12)

with $\mu = 2, 3, \cdots$, for $k$-pair ground states. Then, we calculate the ratio $R_\mu = \langle V_\mu \rangle / \langle V_{\text{total}} \rangle$, where $\langle V_{\text{total}} \rangle$ is the sum of all terms given in (12). The results up to the half-filled case are shown in Fig. 2. It can be seen that the two-body pairing interaction ($V_1$) dominates the dynamics of the system as long as the interaction strength $G$ is small. With increasing interaction strength, the system is driven mainly by $V_2$, less by $V_3$ (but it may be comparable with $V_2$), and much less by the higher order terms. As one would expect, increasing the number of pairs $k$ drives the critical point where $V_2$ becomes dominant towards smaller $G$ values. The situation and the graphs beyond the half-filled case are qualitatively similar because of the particle-hole symmetry. The critical point where $V_2$ becomes dominant is actually at higher $G$ values for $p - k$ pairs than for the $k < p/2$ pairs. Even though higher order terms appear beyond the half-filled case, these terms are always less important than $V_1$ for small $G$ and $V_2$ for big values of $G$. It is important to note that for large values of $G$ when the dynamics is dominated by the pairing interaction, and thus independent of $G$, the $V_2$ term dominates followed in importance by the $V_3$ term.

As an example of an application of the theory to well-deformed nuclei, we fit even-odd mass differences of the $^{164-171}$Yb isotopes. The single-particle energies of each nucleus were calculated using the Nilsson deformed shell model with experimentally determined deformation parameters [16]. Fig. 3 shows the theoretical fit in comparison with the corresponding experimental values[17]. Except for small deviations for $^{167-161}$Yb, the experimental results are well reproduced. The deviations
FIGURE 3. Even-odd mass difference $P(A) = E(A) + E(A-2) - 2E(A-1)$ for $^{154-171}$Yb, where $E(A)$ is the total binding energy, and the dots correspond to the experimental numbers. The theoretical values for even-odd mass $P(A)$ are connected by the lines.

TABLE 1. Pairing interaction strength $|G|$ (keV) used in Fig. 3 for the $^{154-171}$Yb isotopes.

<table>
<thead>
<tr>
<th>$^{154}$Yb</th>
<th>$^{156}$Yb</th>
<th>$^{158}$Yb</th>
<th>$^{160}$Yb</th>
<th>$^{162}$Yb</th>
<th>$^{164}$Yb</th>
<th>$^{166}$Yb</th>
<th>$^{168}$Yb</th>
<th>$^{170}$Yb</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>$G$</td>
<td>245</td>
<td>41.1</td>
<td>6.5</td>
<td>3.02</td>
<td>1.11145</td>
<td>0.4675</td>
<td>0.2337</td>
<td>0.1376</td>
</tr>
<tr>
<td>$^{158}$Yb</td>
<td>$^{157}$Yb</td>
<td>$^{159}$Yb</td>
<td>$^{161}$Yb</td>
<td>$^{163}$Yb</td>
<td>$^{165}$Yb</td>
<td>$^{167}$Yb</td>
<td>$^{169}$Yb</td>
<td>$^{171}$Yb</td>
</tr>
<tr>
<td>$k$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>$G$</td>
<td>270.7</td>
<td>42.0</td>
<td>10.9</td>
<td>1.0</td>
<td>1.471</td>
<td>0.649</td>
<td>0.3477</td>
<td>0.2185</td>
</tr>
</tbody>
</table>

for $^{157-161}$Yb can be traced to the fact that there are singular points in Eq. (10) when $z^{(c)}$ approaches the value $1/\sum_{\mu} e^{-\mu z}$. So although the results are quite good, we do not get a perfect fit for these nuclei with the Nilsson single-particle-energies provided by [17]. One could also consider an empirical expression for the pairing strength $G$ as a function of the number of valence pairs, but that goes beyond the purpose of the present paper and such enhancements will therefore be considered elsewhere.

The corresponding $G$ values are given in Table 1, from which one can see that the pairing interaction strength decreases with increasing number of pairs $k$ from 245 keV for 1 pair to 0.0948 keV for 9 pairs, while the single-particle energy gaps are always about a few hundreds keV. This situation is characteristic of the extended pairing model. However, since the strengths of the many-pair terms in (3) are the same as the one-pair term, the model has little flexibility. The results suggest that the extended pairing model may be applicable to well-deformed nuclei and in other physical systems where pairing plays an important role.
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