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EXACT SOLUTIONS OF THE ISOVECTOR PAIRING INTERACTION

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Exact solutions for low-lying $J = 0$ states of $2k$ nucleons interacting with one another through an isovector charge-independent pairing interaction are derived by using the Bethe ansatz method. The results show that a set of highly nonlinear equations must be solved for $k \geq 3$.

1. Introduction

Pairing has long been considered to be an important interaction in nuclei. The concept was first introduced by Racah within the context of a seniority coupling scheme. Various applications to realistic nuclear systems have been carried out following suggestions from Bohr, Mottelson, and Pines. A lot of effort has been dedicated to the pure neutron or pure proton pairing interactions using various techniques. Extensions to neutron-neutron, neutron-proton, and proton-proton pairing interactions have been formulated. It is well-known that the isovector charge-independent pairing Hamiltonian can be built by using generators of the quasi-spin group $Sp_j(4)$, where $j$ labels the orbits considered in the model space, and from this it also follows that the pairing Hamiltonian can be diagonalized within a given irreducible representation (irrep) of the direct

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product group \( S_{p_1}(4) \times \cdots \times S_{p_p}(4) \), where \( p \) is the number of orbits. In this case exact solutions – even if only generated numerically – can be given.\(^{[8]}\) It is also well-known that approximate numerical solutions can be obtained by using the BCS formalism.\(^{[9–11]}\)

A lot of effort has been devoted to finding exact analytic solutions of the nuclear pairing Hamiltonian.\(^{[12–18]}\) Extensions to a consideration of generalized and orbit-dependent pairing interactions have been the focus of recent work based on the algebraic Bethe ansatz and infinite dimensional Lie algebraic methods.\(^{[16–19]}\) A method for finding roots of the Bethe ansatz equations for the equal strength pairing model that was solved earlier by Richardson has also been proposed.\(^{[20]}\) However, these exact solutions are for proton-proton or neutron-neutron pairing interactions only. In this paper, exact solutions for the mean-field plus isovector charge independent equal strength pairing interaction are revisited using the Bethe ansatz method. We find that the solutions offered by Richardson\(^{[21]}\) and Chen and Richardson\(^{[22–23]}\) are only valid when the number of pairs is less than or equal to two.

In this paper we introduce a new formalism for solving the problem. In Section 2, the mean-field plus isovector pairing Hamiltonian and its \( S_{p}(4) \) quasi-spin structure are reviewed. In Section 3, a general procedure for solving the isovector charge independent pairing Hamiltonian is outlined and detailed results for seniority-zero states. The results show that a set of highly nonlinear equations will enter whenever the number of nucleons is greater than or equal to six. Section 4 is reserved for a short discussion regarding implications of our findings.

2. The isovector pairing Hamiltonian and the \( S_{p}(4) \) quasi-spin structure

The mean-field plus isovector charge independent pairing Hamiltonian can be expressed in terms of generators of quasi-spin groups \( S_{p_j}(4) \), where \( j \) labels the total spin of the corresponding orbits. Generators of \( S_{p_j}(4) \) are the pair creation, \( A_{j}^{\dagger}(\mu) \), and the pair annihilation, \( A_{j}(\mu) \), operators with \( \mu = +, -, 0 \); the total nucleon number operator \( N_{j} \) for orbit \( j \); and the isospin operators \( T_{\mu}(j) \):

\[
A_{j}^{\dagger}(\mu) = \sum_{m > 0} (-)^{j-m} a_{j+m,\mu} a_{j-m,\mu}^\dagger \quad \text{for} \quad \mu = +, -
\]  \hspace{1cm} (2.1a)
\[ A_j^+(\mu) = \sqrt{\frac{1}{2}} \left\{ \sum_{m > 0} (-)^{j-m} a_{jm,+}^+ a_{j-m,-}^+ + \sum_{m > 0} (-)^{j-m} a_{jm,-}^+ a_{j-m,+}^+ \right\}, \] (2.1b)

\[ A_j(\mu) = \sum_{m > 0} (-)^{j-m} a_{j-m,\mu} a_{jm,\mu} \text{ for } \mu = +, -, \] (2.2a)

\[ A_j(0) = \sqrt{\frac{1}{2}} \left\{ \sum_{m > 0} (-)^{j-m} a_{j-m,-} a_{jm,+} + \sum_{m > 0} (-)^{j-m} a_{j-m,+} a_{jm,-} \right\}, \] (2.2b)

\[ N_j = \sum_{m_{mt}} a_{jm,m_t}^+ a_{jm,m_t}, T_+(j) = \sum_{m} a_{jm,+}^+ a_{jm,-}, T_-(j) = \sum_{m} a_{jm,-}^+ a_{jm,+}, \]

\[ T_0(j) = \frac{1}{2} \sum_{m} (a_{jm,+}^+ a_{jm,+} - a_{jm,-}^+ a_{jm,-}), \] (2.3)

where \( a_{jm,m_t}^+ \) (\( a_{jm,m_t} \)) is the creation (annihilation) operator for a nucleon in the state with angular momentum \( j \), angular momentum projection \( m \), and isospin projection \( m_t \) with \( m_t = +\frac{1}{2}, -\frac{1}{2} \). According to the Wigner-Eckart theorem, the pair creation operators \( A_j^+(\mu) \) with \( \{ A_j^+(+) = -A_j^+(-), A_j^+(0) = A_j^+(0), A_j^-(+)) = A_j^+(--) \} \) and the pair annihilation operators \( A_j(\mu) \) with \( \{ A_j(+) = A_j(-), A_j(0) = -A_j(0), A_j(-) = -A_j(+) \} \) are \( T = 1 \) irreducible tensor operators, that satisfy the following conjugation relation:

\[ A_j(\mu) = (-1)^{1-\mu} (A_j^+(\mu))^\dagger. \] (2.4)

The mean-field, with single-particle energies \( \epsilon_j \) from the spherical shell model, plus isovector charge independent pairing interaction Hamiltonian can be expressed as

\[ \hat{H} = \sum_j \epsilon_j N_j - G \sum_{jj',\mu} A_j^+(\mu) A_{j'}(\mu), \] (2.5)

where \( G > 0 \) is the overall pairing interaction strength. Since \( \hat{H} \) is invariant under isospin rotation, both the isospin quantum number \( T \) and its
third component $T_0$ with eigenvalue $M_T$ are good quantum numbers of the system.

3. Exact solutions

In this paper, we only consider seniority-zero states. Hence, the lowest weight state is an isospin scalar. A $k$-pair excitation eigenstate can be written as

$$|\xi; [\lambda]_k M, T T_0 \rangle = \sum_{P \in S_k} Q^{[\lambda]}(x_{P(1)}, x_{P(2)}, \cdots, x_{P(k)}) \times$$

$$\left( A^+(x_{P(1)}) \times A^+(x_{P(2)}) \times \cdots \times A^+(x_{P(k)})T_{T_0} \right) |0\rangle,$$

(3.1)

where $|0\rangle$ is the seniority-zero and isospin scalar state satisfying

$$A^*_j(\mu)|0\rangle = 0 \quad \text{for} \quad \mu = +, -, 0,$$

(3.2)

and $[\lambda]$ is an irrep of $S_k$. It has been confirmed in exact solutions of the equal strength pairing problem with only neutron-neutron or proton-proton pairing interaction[13–19] that the building blocks $A^+_j(x)$ can be expressed as elements of the non-linear Gaudin algebra $G(SU(2))$ with

$$A^+_\mu(x) = \sum_j \frac{A^+_j(\mu)}{1 - \varepsilon_j x} \quad \text{for} \quad \mu = +, -, \cdots.$$

(3.3)

It suffices to use the non-linear Gaudin algebra $G(Sp(4))$ to construct the eigenstates, which is generated by

$$g_\mu(x) = \sum_{j=1}^p \frac{1}{1 - \varepsilon_j x} g_j(\mu),$$

(3.4)

where $p$ is the total number of orbits, $g_j(\mu)$ are the $Sp_\mu(4)$ generators, and $\varepsilon_j$ is the single-particle energy of the $j$-th orbit.

It should be noted that the possible irreps $[\lambda]$ occurring in (3.1) should be determined by properties of the $A^+_j(\mu)$ operators. Because $\mu$ can only take on three different values, $\mu = +, -, 0$, Young diagrams constructed from those $A^+_j(\mu)$ operators can have at most three rows. Furthermore, because the Schur-Weyl duality relation between the permutation group $S_k$
and the unitary group $U(N)$, the irrep $[\lambda]$ with exact $k$ boxes of $S_k$ can be regarded as the same irrep of $U(N)$. Since the irreps $[\lambda]$ contain at most three rows, in this case they can be considered to be equivalent to the same irreps of $U(3)$. Therefore, the possible isospin quantum number $T$ for a given irrep $[\lambda]$ of $S_k$ can be obtained by the reduction[23] of $U(3) \supset SO(3)$. The remaining problem is to find the expansion coefficients $Q^{[\lambda]}(x_1, x_2, \cdots, x_k)$ and to establish the Bethe ansatz equations based on the corresponding eigenvalue equation. The following elements of the Gaudin algebra $G(Sp(4))$ will be useful:

$$A^+_{\mu}(x) = \sum_j \frac{A^+_{\mu}(j)}{1 - \varepsilon_j x}, \quad A_{\mu}(x) = \sum_j \frac{A_{\mu}(j)}{1 - \varepsilon_j x}, \quad T_{\mu}(x) = \sum_j \frac{T_{\mu}(j)}{1 - \varepsilon_j x} \quad (3.5)$$

for $\mu = +, -, 0$, and

$$N(x) = \sum_j \frac{N_j}{1 - \varepsilon_j x}. \quad (3.6)$$

Then, the Hamiltonian (2.6) can be rewritten as

$$\hat{H} = \frac{\partial N(x)}{\partial x} \bigg|_{x=0} + G A^+(0) \cdot A(0). \quad (3.7)$$

Solving the eigenvalue equation

$$\hat{H} |\zeta; [\lambda]_k M, T T_0 \rangle = E^{[\lambda]_k T}_\zeta |\zeta; [\lambda]_k M, T T_0 \rangle \quad (3.8)$$

with the Bethe ansatz wavefunction (3.1), implies that one simultaneously determines the expansion coefficients $Q^{[\lambda]}(x_1, x_2, \cdots, x_k)$ and the Bethe ansatz equations that the spectral parameters $x_1, x_2, \cdots, x_k$, should satisfy.

Our results confirm that solutions reported in [21-23] are valid for $k \leq 2$ and for any $k$ with $T = k$. In other cases, however, the solutions are not correct. For $k = 3$, using the building blocks $A^+(x)$, one can construct the Bethe ansatz wavefunction for different irreps of the permutation group $S_3$ by using the induced representation method.[24-25] For example, wavefunctions for the symmetric $k = 3$ and $T = 1$ case should be written as

$$|\zeta; [3, 0, 0], T = 1, M_T = 1 \rangle =$$

$$(1 + g_2 + g_1 g_2) Q^{[3]}(x_1 x_2; x_3) B^0(x_1 x_2) A^+_1(x_3) |0\rangle, \quad (3.9)$$
where \( \hat{g}_i \) \((i = 1, 2)\) are generators of \( S_3 \), which are nothing but nearest neighbor permutations defined by \( g_i = (i, i + 1) \) for \( i = 1, 2, \ldots, k - 1 \), \( Q_i(x_1, x_2, x_3) \) is the expansion coefficient, and

\[
B^{T=0}(x_1, x_2) = A^+_1(x_1)A^+_2(x_2) + A^+_1(x_1)A^+_2(x_2) + A^+_2(x_1)A^+_2(x_2). \tag{3.10}
\]

It is obvious that \( x_1 \) and \( x_2 \) in the primitive vector \( B^0(x_1, x_2) \) are symmetric with respect to \( x_1 \leftrightarrow x_2 \) permutation. Up to a constant, the coefficients in \((1 + g_2 + g_1 g_2)\) are taken from the Induction Coefficients \(^{24-25}\) (IDCs) of \( S_2 \times S_1 / S_3 \) for the coupling \([2] \otimes [1] \rightarrow [3] \). It should be emphasized that, generally, \( Q_i(x_1 x_2; x_3) \neq Q_i(x_1 x_3; x_2) \neq Q_i(x_2 x_3; x_1) \), where

\[
Q_i(x_1 x_2; x_3) = g_2 Q_i(x_1 x_2; x_3), \quad Q_i(x_2 x_3; x_1) = g_1 g_2 Q_i(x_1 x_2; x_3). \tag{3.11}
\]

For the \([3, 0, 0]\) and \( T = 1 \) case, we need the following commutation relation:

\[
[[H, B^0(x_1, x_2)], A^+_3(x_3)]|0\rangle = \left\{-G A^+_1(x_1) B^0(0, x_2) \left( \frac{x_2}{x_2 - x_3} - \frac{x_1}{x_1 - x_3} \right) - G A^+_1(x_2) B^0(x_1, 0) \left( \frac{x_1}{x_1 - x_3} - \frac{x_2}{x_2 - x_3} \right) + G A^+_1(0) B^0(x_1, x_2) \left( \frac{x_1}{x_1 - x_3} + \frac{x_2}{x_2 - x_3} \right) + G A^+_1(0) B^0(x_1, x_3) \frac{x_3}{x_3 - x_2} + G A^+_1(x_2) B^0(0, x_3) \frac{x_3}{x_3 - x_1} - G A^+_1(x_1) B^0(0, x_3) \frac{x_3}{x_3 - x_2} - G A^+_1(x_2) B^0(0, x_3) \frac{x_3}{x_3 - x_1} + G A^+_1(x_3) B^0(0, x_1) \frac{x_3}{x_3 - x_2} + G A^+_1(x_3) B^0(0, x_2) \frac{x_3}{x_3 - x_1}\right\}|0\rangle. \tag{3.12}
\]

Using (3.12) with (3.8), one can prove that the eigen-energies are given by

\[
E^{[30]}_{\xi} T=1 = \sum_{i=1}^{3} \frac{2}{x_i(\xi)}. \tag{3.13}
\]

However, in this case, there are nine independent basis vectors in the final expression. Except for the original eigenstate, (3.9), all other coefficients
in front of these basis vectors should vanish. Therefore, \( \frac{2}{x_i} + \Omega(x_i) \) should be chosen to satisfy the same condition,

\[
\frac{2}{x_i} + G \Omega(x_i) = G F_i(x_1, x_2, x_3, \alpha^{[3]}, \beta^{[3]}, \gamma^{[3]}) \tag{3.14}
\]

for \( i = 1, 2, 3 \), where

\[
\alpha^{[3]} = Q^{[3]}(x_1, x_2, x_3), \quad \beta^{[3]} = Q^{[3]}(x_1, x_3, x_2), \quad \gamma^{[3]} = Q^{[3]}(x_2, x_3, x_1) \tag{3.15}
\]

are functions of \( x_i \) (\( i = 1, 2, 3 \)) satisfying conditions (3.14), and

\[
F_i(x_1, x_2, x_3; \alpha^{[3]}, \beta^{[3]}, \gamma^{[3]}) \]

for \( i = 1, 2, 3 \), is a function of \( x_i \). After symmetrization, we get

\[
F_1^{[3]} = \frac{x_2}{x_2 - x_1} \frac{\beta^{[3]} + \gamma^{[3]} - \alpha^{[3]}}{\alpha^{[3]} + \beta^{[3]} + \gamma^{[3]}} + \frac{x_3}{x_3 - x_1} \frac{\alpha^{[3]} + \gamma^{[3]} - \beta^{[3]}}{\alpha^{[3]} + \beta^{[3]} + \gamma^{[3]}},
\]

\[
F_2^{[3]} = \frac{x_1}{x_1 - x_2} \frac{\beta^{[3]} + \gamma^{[3]} - \alpha^{[3]}}{\alpha^{[3]} + \beta^{[3]} + \gamma^{[3]}} + \frac{x_3}{x_3 - x_2} \frac{\alpha^{[3]} + \beta^{[3]} - \gamma^{[3]}}{\alpha^{[3]} + \beta^{[3]} + \gamma^{[3]}},
\]

\[
F_3^{[3]} = \frac{x_2}{x_2 - x_3} \frac{\alpha^{[3]} + \beta^{[3]} - \gamma^{[3]}}{\alpha^{[3]} + \beta^{[3]} + \gamma^{[3]}} + \frac{x_1}{x_1 - x_3} \frac{\alpha^{[3]} + \gamma^{[3]} - \beta^{[3]}}{\alpha^{[3]} + \beta^{[3]} + \gamma^{[3]}}, \tag{3.16}
\]

The cancellation of unwanted terms requires that \( \alpha^{[3]}, \beta^{[3]}, \) and \( \gamma^{[3]} \) satisfy the following equations:

\[
\begin{align*}
\alpha^{[3]} F_1^{[3]} + \frac{x_2}{x_2-x_1}(\gamma^{[3]} + \alpha^{[3]} + \beta^{[3]}) - \frac{x_3}{x_3-x_1}(\gamma^{[3]} + \alpha^{[3]}) &= 0, \\
\alpha^{[3]} F_2^{[3]} + \frac{x_1}{x_1-x_2}(\gamma^{[3]} + \beta^{[3]} + \alpha^{[3]}) - \frac{x_3}{x_3-x_2}(\beta^{[3]} + \alpha^{[3]}) &= 0, \\
\alpha^{[3]} F_3^{[3]} - \frac{x_2}{x_2-x_3}(\alpha^{[3]} + \beta^{[3]}) - \frac{x_1}{x_1-x_3}(\gamma^{[3]} + \alpha^{[3]}) &= 0, \tag{3.17a}
\end{align*}
\]

\[
\begin{align*}
\beta^{[3]} F_1^{[3]} + \frac{x_3}{x_3-x_1}(\gamma^{[3]} + \beta^{[3]} + \alpha^{[3]}) - \frac{x_2}{x_2-x_1}(\beta^{[3]} + \gamma^{[3]}) &= 0, \\
\beta^{[3]} F_2^{[3]} + \frac{x_1}{x_1-x_2}(\gamma^{[3]} + \alpha^{[3]} + \beta^{[3]}) - \frac{x_2}{x_2-x_1}(\beta^{[3]} + \alpha^{[3]}) &= 0, \\
\beta^{[3]} F_3^{[3]} - \frac{x_1}{x_1-x_3}(\alpha^{[3]} + \beta^{[3]}) - \frac{x_3}{x_3-x_2}(\gamma^{[3]} + \beta^{[3]}) &= 0, \tag{3.17b}
\end{align*}
\]

\[
\begin{align*}
\gamma^{[3]} F_1^{[3]} + \frac{x_3}{x_3-x_2}(\beta^{[3]} + \gamma^{[3]} + \alpha^{[3]}) - \frac{x_1}{x_1-x_2}(\beta^{[3]} + \gamma^{[3]}) &= 0, \\
\gamma^{[3]} F_2^{[3]} + \frac{x_3}{x_3-x_1}(\beta^{[3]} + \alpha^{[3]} + \gamma^{[3]}) - \frac{x_1}{x_1-x_3}(\beta^{[3]} + \gamma^{[3]}) &= 0, \\
\gamma^{[3]} F_3^{[3]} - \frac{x_3}{x_3-x_1}(\alpha^{[3]} + \gamma^{[3]}) - \frac{x_2}{x_2-x_1}(\beta^{[3]} + \gamma^{[3]}) &= 0. \tag{3.17c}
\end{align*}
\]
Due to relations (3.15), the three sets of equations (3.17a), (3.17b), and (3.17c) can be changed into one another through the permutation $g_2$ and $g_1g_2$, and therefore, they are not independent. Substituting (3.16) into (3.17), one can get relations among $\alpha^{[3]}$, $\beta^{[3]}$ and $\gamma^{[3]}$, which lead to three sets of complicated polynomial solutions depending on the spectral parameters $x_1$, $x_2$ and $x_3$. Thus, we get three set of solutions of $\alpha_i^{[3]}$ and $\beta_i^{[3]}$ in terms of $\gamma^{[3]}$. By substituting them into Eq. (3.16), the final expressions for $F_i^{[3]}$ will be $\gamma^{[3]}$ independent; and the corresponding Eq. (3.14) provides solutions for the spectral parameters $x_1$, $x_2$ and $x_3$ of the problem. Similar results were also derived for [21] irrep.

4. Discussion

A general procedure, based on the Bethe ansatz, for building algebraic solutions for seniority-zero $J = 0$ states of $2k$ nucleons interacting through an isovector charge independent pairing interaction has been introduced. We used the procedure to generate explicit results for seniority-zero $J = 0$ states.

The results derived for $k \leq 2$ as well as for $2k$ nucleons for symmetric irreps of $S_k$ with $T = k$ agree with those given by Richardson [21] and by Chen and Richardson [22-23]. However, it is showed that the results given in [21-23] are not valid for six or more nucleons in non-symmetric irreps of the permutation group. The main difference lies in the fact that in the present work the expansion coefficients $Q^{[\lambda]}$ are considered to be functions of the spectral parameter $x_i$ and different from one another for non-symmetric irreps of the permutation groups, while the expansion coefficients in the work of Richardson and Chen and Richardson were assumed to be spectral parameter independent. In fact, for $2k$ nucleon configurations, the present calculation shows that the expansion coefficients $Q^{[\lambda]}$ can be taken to be the same only for totally symmetric irreps $[k]$ of the permutation groups $S_k$ with $T = k$ or totally anti-symmetric irreps $[1^k]$ with $k = 1, 2, 3$. But for other cases, general solutions of the type introduced here are required; those offered in [21-23] as solutions for general irreps are not possible.

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References