Electron Scattering in the Symplectic Shell Model

P. ROCHFORD and J. P. DRAAYER
Department of Physics and Astronomy
Louisiana State University, Baton Rouge, LA 70803

ABSTRACT

The charge density multipoles used in calculating electron scattering form factors are given in a form suitable for use in second-quantized SU(3) shell-model calculations as pioneered by Ted Hecht and collaborators. The expression is an exact representation of the charge density multipoles, extends over all major oscillator shells, and is tailored for symplectic shell-model applications. Through a SU(3) tensor decomposition of these multipoles the importance of considering multiple irreducible representations in symplectic model calculations for ds-shell nuclei is explored, as well as the extent to which higher shell correlations will influence form factor predictions. Results for a SU(3) model application to $^{24}$Mg is given and the significance of the theory for symplectic shell model applications is discussed.

1. Introduction

Electron scattering provides important information on the radial charge and current distributions in nuclei and hence serves as a sensitive test of nuclear models. This information, as given by longitudinal and transverse form factors displayed as a function of momentum transfer, indicates that a microscopic description of nuclei is necessary if one wishes to describe the structural features of nuclei like the nuclear charge and current distributions. From previous ds and fp shell model studies it is well known that the coupling of $0
^{\text{th}}$ valence shell states to configurations in higher shells is important to achieve a good description of the scattering data.

A microscopic theory which takes into full account the collective coherence between oscillator shells is the one which unites the U(3) group symmetry of the harmonic oscillator with that for linear collective motion, namely the symplectic shell model based on the Sp(3,R) group. This fermionic model, which takes the Pauli principle into proper account, incorporates multi-hto shell couplings that are the source of core polarization. Applications, which for the most part have been performed within model spaces defined by single irreducible representations (irreps) of Sp(3,R), show that this model can describe a diversity of collective phenomena. When multiple Sp(3,R) irreps and the interaction between them are taken into account, one achieves a multi-shell realization of the nuclear shell model. Symplectic model predictions of elastic and inelastic form factors, therefore, can be used to determine the extent to which multi-shell correlations are responsible for the observed differences between experiment and the traditional shell-model descriptions.

Since multiple irrep symplectic model calculations are difficult to carry out, it is important to determine whether electron scattering form factor predictions obtained with wavefunctions from a single Sp(3,R) irrep calculation are a good approximation to those acquired on considering several irreps and the mixing between them. Given that the Sp(3,R) irreps are built upon the valence shell SU(3) irreps, which belong to the Elliott SU(3) submodel of the symplectic model, a good indication of the validity of a single irrep approximation can be gained from exploring the single and multiple irrep predictions in the Elliott SU(3) model. The results of such a study will be presented for the ds-shell nucleus $^{24}$Mg, as the SU(3) and symplectic models have both successfully reproduced its energy spectra and E2 transition strengths. The focus in this paper will be on the longitudinal form factors for which there already exists some experimental data. The expansion of the charge density multipoles in terms of SU(3) tensors needed to perform these calculations will be presented, and have a very simple form thanks to the relation for the expansion of $J_L(q\rho)Y_{LM}(\Omega)$ in terms of SU(3) unit tensors derived by Ted Hecht and his coworkers. The results of the calculations suggest that single leading irrep symplectic model predictions for the longitudinal form factors of ds-shell nuclei will yield a good approximation to a complete mixed symmetry calculation.

Since Sp(3,R) irreps are infinite-dimensional and must be truncated at some finite number of shells when performing numerical calculations, it is also important to explore how much multi-shell correlations influence form factor predictions, as this sets limits for the truncation of the Sp(3,R) model space. To gain insight into this matter the strength of the nhto excitation tensors that occur for the charge density quadrupole operator will be shown. It will be clear from the results that correlations up to at least $0
^{\text{th}}$ are needed.

2. Charge Density Multipoles

The longitudinal electron scattering form factor provides information on the charge density distribution of nuclei. For transitions from an initial state of angular momentum $J_i$ to a final state of angular momentum $J_f$ in a nucleus of charge Z, it is defined in a first-order Plane Wave Born Approximation (PWBA) as

$$F_{\ell, J_i}^{J_f} (q) \propto \frac{4\pi}{Z^2 (2\ell + 1)} \sum_{L=0}^{\infty} \langle J_f \mid \tilde{\rho}_L(q) \mid J_i \rangle^2. \quad (1)$$

The charge density multipole operator $\tilde{\rho}_L(q)$ which occurs in this form factor is given as a function of momentum transfer q by

$$\tilde{\rho}_L(q) = \sum_{n=1}^{N} \frac{1 - r_n^{2a}}{2} J_L(qr_n)Y_{LM}(r_n), \quad (2)$$

where the nucleon coordinate $r_n = r_n - R$ is relative to the nuclear center-of-mass R.
Since it is a one-body operator, for the expression, the charge density multipole operator in terms of second quantized fermion creation and annihilation operators \(a^\dagger\) and \(a\) is:

\[
\mathbf{\hat{p}}_{LM}(q) = \sum_{\alpha, \beta} \frac{1}{2} (\mathbf{\hat{r}}_\alpha)^\dagger \mathbf{r}_\alpha \mathbf{Y}^M_L (r_{\alpha})^\dagger \mathbf{r}_{\beta} \mathbf{Y}^M_L (r_{\beta})^\dagger \mathbf{a}_{\alpha}^\dagger \mathbf{a}_{\beta},
\]

(3)

where the sum runs over the single-particle shell-model states in all major shells, and \(\alpha, \beta = (\eta \ell 1/2 j m, 1/2 m')\) are the spatial-spin-isospin labels for the single-particle states, with \(\eta\) representing the major oscillator shell number. Through some simple angular momentum coupling and straightforward SU(3) coupling algebra, the tensor expansion for the charge density multipole operator can be derived in terms of the SU(3) \(\supsetneq\) SO(3) tensors \(\{a^\dagger\mathbf{l}\}^\dagger \mathbf{r}_M^\dagger \mathbf{r}_M\} T\), where \((\alpha \mu)\) is the SU(3) irrep label, \(S\) the total spin, \(K\) the multiplicity label for the angular momentum \(L\) in \((\lambda \mu)\), and \((J, T)\) the total angular momentum and isospin coupling. The expression has been shown to be given by:

\[
\mathbf{\hat{p}}_{LM}(q) = \sum_{\eta \ell \eta_1 \eta_2 (\lambda \mu)} \mathcal{C}_{\lambda \mu \eta_1 \eta_2}(b q) \frac{1}{2} \left\{ \left[ a^\dagger \mathbf{l}_M \right]^\dagger \mathbf{r}_M^\dagger \mathbf{r}_M \left[ a^\dagger \mathbf{l}_M \right]^\dagger \mathbf{r}_M^\dagger \mathbf{r}_M \right\},
\]

(4)

where the coefficient

\[
\mathcal{C}_{\lambda \mu \eta_1 \eta_2}(b q) = \sum_{\lambda \mu \lambda_1 \lambda_2} \Theta ((\lambda \mu) \lambda \lambda_1 \lambda_1 \eta \eta_1 \eta_2 \eta_2) \Theta ((\lambda \mu) \lambda \lambda_1 \lambda_1 \eta \eta_1 \eta_2 \eta_2) \Theta ((\lambda \mu) \lambda \lambda_1 \lambda_1 \eta \eta_1 \eta_2 \eta_2)
\]

(5)

involves the dimension of a SU(3) irrep, \(\dim((\lambda \mu) = (\lambda + 1)(\mu + 1)(\lambda + \mu + 2)/2\), and q-dependent expansion coefficients \(\Theta\) that occur when expressing the \(\mathbf{r}_M^\dagger \mathbf{r}_M\) product in SU(3) tensorial form. The complete expression for the \(\Theta\) coefficient is given by:

\[
\Theta ((\lambda \mu) \lambda \lambda_1 \lambda_1 \eta \eta_1 \eta_2 \eta_2) = \exp (-\frac{1}{4} (b q)^2) \left[ \frac{\dim((\lambda \mu) \lambda \lambda_1 \lambda_1 \eta \eta_1 \eta_2 \eta_2)}{4\pi(2L + 1) \dim((\eta \eta_1 \eta_2))} \right]^{1/2}
\]

(6)

\[
\sum_{n=0}^{n \eta_2} \frac{\binom{n + \eta_2 - 2}{2}}{2^{n + \eta_2 - 2}} \left[ \frac{(n + \eta_2 - 1)!}{n!(\eta_1 - \eta_2 - 1)!} \right]^{1/2}
\]

\[
\sum_{l=0}^{\eta_1 n} \frac{\binom{n + l - 2}{2}}{2^{n + l} (2l + 1) 2^{2l + 1} \binom{n + l + 1}{2} \binom{n + l + 2}{2} \binom{n + l + 3}{2} \binom{n + l + 4}{2}} \left[ \frac{(2l + 1)! (2l + 2)!}{2^{2l + 1} (2l + 1)! (2l + 2)!} \right]^{1/2}
\]

3. Tensor Analysis of Multipole Operators

Almost all symplectic model applications to date have been performed in the model space defined by the leading Sp(3,R) irrep, which is identified by the \(\mathbf{o}^{(0)}\) valence shell irrep of SU(3) with the largest value for the SU(3) quadratic casimair invariant. From several applications it is known that the \(\mathbf{o}^{(0)}\) component of the leading symplectic irrep carries 60–80% of the symplectic eigenstate strength for members of the ground state rotational band. This means that by and large the \(\mathbf{o}^{(0)}\) component of the symplectic irreps govern the dynamics of collective rotations in nuclei. Moreover, the importance of expanding the model space to include several such Sp(3,R) irreps, and not just the leading one, for electron scattering predictions, will be determined by the extent to which the interactions mix and fermion tensors couple the different \(\mathbf{o}^{(0)}\) irreps. For the charge density multipoles, it is therefore relevant to examine the relative strength of the momentum dependent coefficients for the \(\mathbf{o}^{(0)}\) valence-shell fermion tensors, as these will determine the extent to which Sp(3,R) irrep mixing influences the resulting longitudinal form factors, and thus its importance for describing the radial structure of nuclei.

When assessing the relative strength of the fermion tensors, one need only examine their SU(3) tensor character, as it is the latter which determine the number and size of non-zero matrix elements between different \(\mathbf{o}^{(0)}\) irreps. This is clear when one observes that the physical dependence of the fermion tensor matrix elements is contained in their SU(3) reduced matrix elements, which depend on only \((\lambda \mu)\), spin, and isospin. Since the spin-isospin dependence is fixed, it is the different SU(3) characters of the tensors that will determine the influence \(\mathbf{o}^{(0)}\) irrep mixing will have on form factor predictions. Given that symplectic model applications for ds-shell nuclei have been very successful, a good case to investigate are those tensors occurring in the expansion of the charge density multipoles for the ds-shell. Shown in Figure 1 are the non-negligible momentum dependent coefficients \(\Omega\) that occur for the tensor decomposition of these operators. The left-hand three frames in Figure 1 show the contributions for the three possible charge density multipoles.

The important feature to note from the curves in Figure 1 is the strong dominance of the expansion coefficients for the \(\mathbf{o}^{(0)}\) and \(\mathbf{o}^{(1)}\) tensors relative to those for the \(\mathbf{o}^{(2)}\) tensor. This feature implies that the former tensors provide a greater contribution to the multipole matrix elements than the \(\mathbf{o}^{(2)}\). Given that the \(\mathbf{o}^{(0)}\) tensor is an invariant and the isospin scalar \(\mathbf{o}^{(1)}\) tensor for the one-body operator is the generator of SU(3), these two tensors will have nonzero matrix elements only between states of the same SU(3) irrep. Their respective contributions to the form factors will thus be a weighted sum of the individual contributions from each of the SU(3) representations in the wavefunctions. Since many realistic wavefunctions for deformed ds-shell nuclei are found to be strongly dominated by a single SU(3) irrep, the contributions from these tensors will be fairly insensitive to the
Figure 1
Values of the momentum dependent coefficients $C_{(\alpha \mu)\ell \mu L \mu \ell_2 n_2}$ in the range $0 \leq q \leq 5$ for SU(3) tensors which occur in the expansion of the charge density monopole, quadrupole, and hexadecapole for the ds-shell. The three left-hand figures show contributions for tensors within the ds-shell, while the three on the right-hand side are for the quadrupole tensors producing 2h0, 4h0 and 6h0 excitations from the ds-shell.

details of irrep mixing in the wavefunctions. The (22) tensor is the only term which couples different SU(3) irreps and whose contribution is expected to be sensitive to the variation in SU(3) composition of the wavefunctions. One therefore expects the influence of SU(3) irrep mixing to be important only in form factors for which the (22) tensor provides an important contribution. This is clearly true for $F_{\ell=4+}$ which involves the so-called stretched transition as the (22) is the only contributing tensor.

From previous symplectic model applications it is well-known that symplectic wavefunctions for low-lying rotational bands have significant multi-shell components up to at least 6h0, and that the dominant contribution from each nho shell are from those basis states belonging to the $(\lambda_\phi n_\phi \mu_\phi)$ SU(3) irrep contained within it. This leads one to speculate that multi-shell excitations may have considerable influence on shell model predictions for form factors, and that the largest contribution to the matrix element of any multipole operator from the nho shell will be for multi-shell excitation tensors which couple $(\lambda_\phi n_\phi \mu_\phi)$ to the irrep $(\lambda_0 \mu_0)$ defining the 0ho shell of the Sp(3, R) irrep, as the nho strengths in the wavefunctions decrease with increasing shell number n. These $[a^\dagger \mu_\phi]_n^{0\mu_0}$ tensors are the ones which excite nucleons from the 0ho shell $1_{\ell_0}$ to the shell $\ell=\ell_0+n$, and vice versa. By examining the relative strength of the momentum dependent expansion coefficients for these tensors, it is therefore possible to obtain an indication of how much multi-shell correlations influence form factor predictions.

To investigate multi-shell effects for the current case of ds-shell nuclei, the expansion coefficients for the $[a^\dagger \mu_\phi]_n^{0\mu_0}$ tensors that can excite nucleons up to 6h0 above the ds-shell were calculated, i.e. those coefficients with $\ell_1=\ell_2=n$, where $\ell_1=2$ and n=2, 4, and 6. The reverse situation of deexcitation was not considered as they are simply hermitian conjugates of the former. The possible $(\lambda \mu)$ tensors are found from the couplings $(n+2,0)\otimes(0,2)$. In order to restrict the number of curves, only $\rho_2$ was considered, as the easiest inelastic transitions to study were SU(3) rotational bands at $\Omega=2$ transitions. Shown in the right-hand side frames of Figure 1 are the curves obtained for the expansion coefficients.

The first feature to be noted from the curves is the strength of the expansion coefficients progressively increase to higher momentum transfer values, and decrease at low $b q$, with increasing $(\ell_1=\ell_2)\ell_0$ for $\rho_2$. The second feature is the curves for SU(3) tensors $(\lambda \mu)$ with the same $\mu$ values alternate in sign. The first can be attributed to a shift in momentum transfer range the nucleons can acquire when being excited up to specific higher shells, while the second can be traced to the $(-1)^{\ell_1+\ell_2+\ell_0}$ dependence in the $\Theta$ coefficients (c.f. Eq. 6). With regard to the peak amplitudes of the various curves, they are found to decrease moderately with increasing shell excitations, with the largest contribution for any given shell excitation coming from the $\mu=0$ tensors. The latter are also seen to have significant amplitudes for all nho excitations considered.

Since the multi-shell content of symplectic model wavefunctions for low-lying rotational states is typically 10-25%, 5-15%, and 2-7% for the 2h0, 4h0, and 6h0 components, respectively, it is clear that multi-shell correlations up to at least 6h0 will be important for reproducing electron scattering form factors. Given that all previous attempts to include multi-shell correlations have been with the RPA, and have only included excitations up to at most 2h0, it is clear that the additional contributions from the 4h0 and
6f½ shells which are most important for describing the high momentum dependence of the form factors have not yet been included in any model calculation. To describe the high momentum dependence in form factors it is therefore important that multi-shell excitations up to at least 6f½ be incorporated in shell model calculations.

4. SU(3) Application to 24Mg

As an example that calculations can be done using the SU(3) fermion representation of the charge density multipole, an Elliott SU(3) model calculation was undertaken for the low-lying states of the ds-shell nucleus 24Mg. Since the valence shell (0h0) SU(3) irrep (84) comprises a large fraction of the single irrep symmetric model eigenstates for the low-lying rotational bands, the form factors obtained with only the valence-shell SU(3) wavefunctions were expected to provide a reasonable description of the electron scattering data, and thereby give a good indication of the importance of such factors as the degree of irrep mixing that is required. Moreover, since such a full valence shell calculation in a SU(3) scheme had not been previously performed, it will also serve as a study of Elliott SU(3) model predictions for the nuclear structure form factors of 24Mg.

The Hamiltonian employed to generate the 24Mg wavefunctions used in the calculation of the form factors is comprised of a harmonic oscillator Hamiltonian (H₀), which generates the average nuclear field, plus five other effective interactions. These are a deformation generating SU(3) quadrupole-quadrupole force (Q²Q, Q²P), where Q² = 14m/5 (b²P(source) + r²Y(target)) is the quadrupole operator of the SU(3) algebra, a spatial symmetry splitting Majorana interaction M=M, the single-particle level splitting spin-orbit (Σl-1/2) and orbit-orbit (Σl, l') interactions, a rotor term J², and an operator K² to account for the K-band splitting of the excited rotational band.

\[
H = H₀ - \frac{1}{2} \sum_{n=1}^{A-1} \left( C_{l,n} s_n + D_{l,n} s_n^2 + A l^2 + B K^2 \right) \tag{9}
\]

The five effective interactions are in dimensionless units, i.e. \( \chi, a_M, \ldots \) are in MeV.

Due to the large dimensions, a model space comprised of all the Pauli allowed SU(3) irreps for isotopon T=0 for just the four leading space symmetries [f] = [44], [43], [42], and [32] was included when generating the wavefunctions for 24Mg. A set of Hamiltonian parameter values which are found to yield a good fit to the experimental spectrum are \( \chi=0.180, a_M=-0.296, A=-0.0446, B=0.847, C=1.52 \) and D=0.146. The strength of the H₀ term (ho) did not need to be specified, as the model space is confined to a single major oscillator shell. The oscillator length parameter value employed (b=1.813 fm⁻¹) is typical of that used in shell-model calculations.²

Using the mixed-symmetry SU(3) wavefunctions generated as described above, and the expansion given by Eq. 6, the longitudinal PWBA form factors for the elastic and inelastic scattering transitions for the low-lying 0⁺, 2⁺, and 4⁺ states of 24Mg were calculated. The form factors include not only corrections for center-of-mass motion but also those for the finite-size effect of the nucleons.³ No effective charges occur as no attempt was made to include within the valence model space the core polarization effects due to coupling to higher oscillator shells.² The results for the longitudinal form factors as compared with experiment are shown in Figure 2.

For the elastic 0⁺→0⁺ form factor, the mixed-SU(3) prediction is seen to be in good agreement with experiment up to q<1.4 fm⁻¹, and then poorer beyond until q<2 fm⁻¹ due to the second maximum being suppressed in magnitude. From previous studies it is well-known that the first minimum at finite q in shell-model form factors is filled in by Disorted Born Wave Approximation (DBWA) corrections, with this being the only major difference from PWBA results.³ This implies that one can expect some improvement with respect to experiment around 1.5 fm⁻¹ when calculating the present form factor in DWBA, leaving only a modest suppression to be accounted for in the 1.5<q<2 range. For the other inelastic transitions of the ground state band, the F₁(6.4→6.4) exhibits a suppressed first maximum with respect to the measured results, while the F₁(6.4→8.1) prediction is too large in magnitude and does not reproduce the observed shape. This latter feature may be due to an incorrect degree of irrep mixing in the wavefunctions. In contrast, however, one finds that the longitudinal form factors for inelastic transitions to the K=2 band are in reasonable accord with that measured experimentally, indicating that the extent of irrep mixing is fairly realistic for these states.

To explore the extent to which irrep mixing influenced these predictions, the form factors obtained when restricting to only the leading (84) SU(3) irrep were calculated and are also shown in Figure 2. The eigenstates employed contained mixtures of multiply occurring orbital angular momentum states of the same L within the SU(3) irrep. The single irrep result for the elastic 0⁺→0⁺ form factor is seen to be almost indistinguishable from the mixed irrep prediction, and that for the 2⁺ member of the ground state band is only marginally different. While the differences for the remaining inelastic transitions are somewhat larger, they are still found to be quite modest. Overall, one concludes from this single to mixed irrep comparison that single irrep calculations for the longitudinal form factors of 24Mg are a good approximation. This implies that single Sp(3,R) irrep calculations can be expected to yield a good approximation for the full multi-shell model space when making predictions for the longitudinal form factors in 24Mg.

5. Summary

An expansion for the charge density multipole operator in a SU(3) tensorial basis has been presented. The expression is given explicitly in fermion second quantized form, is an exact shell-model representation of the charge density multipole operator, extends over all possible oscillator shells, and can be easily coded into a computer algorithm. With this expression the nuclear structure longitudinal form factors, which contain charge-density information, can be computed with symplectic model wavefunctions due to the bandhead reduction technology developed by Hecht and Suzuki.¹ This is the case for both single as well as mixed Sp(3,R) irrep calculations.

By conducting a tensor analysis of the ds-shell terms in these expansions, it has been shown that valence shell irrep mixing will only moderately influence Elliott SU(3) model predictions for all the elastic and most of the inelastic longitudinal form factors of ds-shell..
nuclei. This was confirmed by calculations for $^{24}$Mg, which compared single irrep versus mixed irrep calculations. These results lead one to believe that single irrep symplectic model predictions for $d$-shell nuclei will be a good approximation to complete mixed irrep calculations for the longitudinal form factors. This is fortuitous as the single irrep calculations are much more tractable to carry out.

It has also been shown that multi-shell correlations up to at least 6$\hbar$, and perhaps higher, could provide significant contributions to the longitudinal form factors in $d$-shell nuclei. This discovery underscores the importance of obtaining symplectic model predictions in order to determine the exact extent to which multi-shell correlations can describe the charge density distributions of collective rotational states in nuclei. From the agreement these results achieve with experiment, a better limit will be provided on the importance of the non-nucleon degrees of freedom in describing nuclear structure, which are already known to be significant in few-body systems like $^4$He. Once these predictions are available, it is hoped they will provide the impetus for experiments probing the charge distributions of collective states in nuclei, which will be possible at facilities such as the Continuous Electron Beam Accelerator Facility (CEBAF) in Virginia.

References