Quantum Algebras and Quantum Groups
Noncommutative Geometry

Braid-Group Approach to Deriving Universal $\tilde{R}$ Matrices*

Feng Pan¹, L. Dai¹, and J. P. Draayer²

Abstract—A new method for deriving universal $\tilde{R}$ matrices from braid-group representation is discussed. In this case, universal $\tilde{R}$ operators can be defined and expressed in terms of products of braid-group generators. An advantage of this method is that matrix elements of $\tilde{R}$ are rank-independent, leaving multiplicity problem concerning coproducts of the corresponding quantum groups untouched.

1. INTRODUCTION

Universal $\tilde{R}$ matrices are solutions to spectral parameter-free Yang–Baxter equations. Yang–Baxter equations are of importance in both mathematics and physics; in particular, they appear in statistical models [1], in scattering matrices [2], in knot theory [3], and in conformal field theory [4]. Once parameter-free $\tilde{R}$ matrices are known, a parameter-dependent matrix $\tilde{R}(x)$ can be obtained by using the so-called Baxterization procedure [5–7]. Up to now, standard $\tilde{R}$ matrices have been derived through the representation theory of quantum groups by many authors (including Drinfeld [8], Jimbo [9, 10], and Reshetikhin [11]) and by taking limit of statistical models [12–14] or by using Witten’s approach of link polynomials [15, 16]. There are also many other methods to construct $\tilde{R}$ matrices [17–19]. Based on these methods, various classes of $\tilde{R}$ matrices have been obtained, which can easily be found in the current mathematical-physics literature. From these methods, it can easily be seen that $\tilde{R}$ matrices are related either to the tensor products of generators or to the Clebsch–Gordan (CG) coefficients of the corresponding quantum groups. Thus, knowledge of the representation theory of quantum groups, such as coupling coefficients and projection operators, is very important in these methods for constructing standard $\tilde{R}$ matrices. In some cases, the multiplicity problem will be involved in the coproducts of the corresponding quantum groups, which is very complicated to solve. Second, an $\tilde{R}$ matrix usually expressed in terms of the relevant CG matrix with summation over all possible resultant irreducible representations (irrep) of the corresponding quantum groups can only be derived for specific $n$, for example, of $A_n$ or $B_n$ at a time. When $n$ increases, the CG matrix will become very large. It will soon become intractable for higher $n$ because of a drastic increase in the number of CG coefficients involved. It is also well known that matrix representations of braid-group generators can be constructed by using the $\tilde{R}$ matrix via

$$g_i = 1 \otimes 1 \otimes \ldots \otimes 1 \otimes \tilde{R} \otimes 1 \otimes \ldots \otimes 1,$$

(1)

where $\tilde{R}$ is in the $i$th and $(i+1)$th spaces. It can easily be proven that, in general, braid-group representations constructed in this way are not irreducible. On the other hand, however, the $\tilde{R}$ operator can be regarded as a deformed permutation operator that permutes two representations of the corresponding quantum groups. From this point of view, an $\tilde{R}$ matrix realizes representations of $\tilde{R}$ operators in the uncoupled basis of the corresponding quantum groups.

2. $\tilde{R}$ OPERATORS

Let $V^{[\lambda_1]}$ and $V^{[\lambda_2]}$ be spaces spanned by basis vectors of $[\lambda_1]$ and $[\lambda_2]$ irreducible representations of any quantum group. The action of $\tilde{R}$ is then given by

$$\tilde{R}(V^{[\lambda_1]} \otimes V^{[\lambda_2]}) \rightarrow V^{[\lambda_1]} \otimes V^{[\lambda_2]}.$$  

(2)

We assume that the maximum rank of $[\lambda_1]$ and $[\lambda_2]$ is $f$—that is, $[\lambda_1]$ and $[\lambda_2]$ can be formed by at most $f$-fold coproducts of rank-1 tensor operators of the corresponding quantum group. If, for example, $[\lambda_1]$ is of a maximum rank, we can write

$$T^i \otimes T^2 \otimes \ldots \otimes T^f \rightarrow T^{[\lambda_1]}.$$  

(3)

where $T^i$, with $i = 1, 2, \ldots, f$, are vector operators in the $i$th space. In the case of $A$-type quantum group, the
basis of $T^{[\lambda_1]}$ can be labeled by assigning a Weyl tableau $w^{[\lambda_1]}(\omega^0_1)$ to $T^{[\lambda_1]}$, where $(\omega^0_1) = (1, 2, ..., f)$ is used to indicate that the $f$ vectors are coupled to $[\lambda_1]$. We now assume that the rank of $[\lambda_2^0](\omega^0_1, 2f - k + 1, 2f - k + 2, ..., 2f)$, where $(\omega^0_1) = (f + 1, f + 2, ..., 2f - k)$, while the indices $(2f - k + 1, 2f - k + 2, ..., 2f)$ are used to label the remaining scalars; that is,

$$T^1 \otimes T^2 \otimes \ldots \otimes T^{2f-k} \otimes \otimes 1^{2f-k+1} \otimes \ldots \otimes 1^{2f} \rightarrow T^{[\lambda_2]}.$$  \hspace{1cm} (4)

Hence, the uncoupled basis vectors of $w^{[\lambda_1]}_1$, $w^{[\lambda_2]}_1$ can be written explicitly as

$$w^{[\lambda_1]}_1(\omega^0_1), w^{[\lambda_2]}_1(\omega^0_2, 2f - k + 1, ..., 2f)$$  \hspace{1cm} (5)

according to the braiding group action; that is, (5) can now be understood as uncoupled basis vectors of the braiding group $B_{2f}$ under the operation $\hat{R}$ with

$$\hat{R}(\{(\omega^0_1), (\omega^0_2, 2f - k + 1, 2f - k + 2, ..., 2f)\} \rightarrow \{(\omega^0_2, 2f - k + 1, 2f - k + 2, ..., 2f), (\omega^0_1)\}).$$

Within this labeling scheme, we find that the operator $\hat{R}$ can be expressed in terms of the generators of the braiding group $B_{2f}$ as

$$\hat{R}_{f=1} = g_1,$$  \hspace{1cm} (6)

$$\hat{R}_{f=2} = g_2 g_1 g_3 g_2 = g_2 \hat{R} g_3 g_2.$$  \hspace{1cm} (7)

By induction, we finally obtain

$$\hat{R} = g_2 g_1 g_3 g_2 ... g_{2f} g_{2f-1} g_3 g_2.$$

Equation (8) gives the universal $\hat{R}$ operator for fixed $f$ in braiding-group-generator product form, where $f$ is the maximum rank of irreps of the corresponding quantum groups. The universality means that the operator given in (8) satisfies (2) for any irrep of any type of the corresponding quantum groups, which is just what Jimbo and Drinfeld meant in [8-10]. The differences between the present study and those of Jimbo and Drinfeld are the following (i) the universal $\hat{R}$ operator is now written in terms of a braiding-group-generator product form; and (ii) the $\hat{R}$ operator defined in (8) is rank-$f$ dependent. It seems that the $\hat{R}$ operator given in (8) loses some universality. However, we can use it to compute all universal $\hat{R}$ matrices because equation (8) is valid for an arbitrary rank $f$ of irreps in the tensor-product space of the corresponding quantum groups given in (2). In the studies of Jimbo and Drinfeld, the $\hat{R}$ operator is written in terms of quantum double basis of a Hopf algebra, which is independent of the rank $f$. At the same time, expression (8) can be regarded as a braid-group form of the universal $\hat{R}$ operators formerly defined by Jimbo and Drinfeld.

The problem concerning the braiding-group realization for a fixed type of the corresponding quantum group was studied by many authors [9, 10, 20]. Among other things, they found that the braiding-group realization is a Hecke algebra for $A$-type quantum groups; a Birman-Wenzl algebra for the $B$, $C$, and $D$ types; and a Kalka-Annari algebra [22] for $G_2$. However, the problem is still open for $F_4$ and the $E$-type quantum groups. In the next section, we will outline a procedure for evaluating the $\hat{R}$ matrices for $A$-type quantum groups and will also give a simple example for $B$-, $C$- and $D$-type cases.

3. EVALUATION OF $\hat{R}$ MATRICES

In this section, we will outline a procedure for evaluating $\hat{R}$ matrices. We will consider a Hecke algebra for $A$-type quantum groups and give an example of a Birman-Wenzl algebra for $B$-, $C$- and $D$-type cases separately.

The $\hat{R}$ operator is a braid-group element that can be expressed in terms of the braiding-group generators by using (8). This operator acts on the coordinate indices $\{1, 2, ..., 2f\}$. Any uncoupled basis vectors $w^{[\lambda_1]}_1(\omega^0_1), w^{[\lambda_2]}_1(\omega^0_2, 2f - k + 1, 2f - k + 2, ..., 2f)$ of any quantum group can be further expanded in terms of uncoupled basis vectors of $(2f - k)$-fold basic representations, namely,

$$w^{[\lambda_1]}_1(\omega^0_1), w^{[\lambda_2]}_1(\omega^0_2, 2f - k + 1, ..., 2f) = \sum a_w Q_w(a_1, a_2, ..., a_{2f-k}, 1^{2f-k+1}, ..., 1^2),$$

where $a_w$ can be obtained by using the CG coefficients for the coupling $((1 \otimes)^{f_1}[\lambda_1])((1 \otimes)^{f_2}[\lambda_2])$ of the corresponding quantum group; $\{a_1, a_2, ..., a_{2f-k}\}$ are the vector components of the quantum group satisfying the normal ordering $a_1 \leq a_2 \leq ... \leq a_{2f-k}$; and $Q_w$ is the left coset representatives in the decomposition

$$B_{2f} = \sum Q_w(B_1 \times B_1 \times ... \times B_1).$$  \hspace{1cm} (10)
BRAID-GROUP APPROACH

By way of example, we indicate that the use of the CG coefficients of $U_q(2)$, which are tabulated in [21], yields

$$|aa, ab\rangle = \frac{1}{\sqrt{3}}|a, a, b\rangle + \frac{1}{\sqrt{2}}|b, a, b\rangle$$

(11a)

$$|ab, aa\rangle = \frac{1}{\sqrt{2}}|b, a, a\rangle + \frac{1}{\sqrt{2}}|b, a, a\rangle$$

(11b)

$$\left(\frac{q}{q^2}\right)^{g_2} = \left(\frac{g_1}{g_2^2} + \frac{g_2}{g_1^2}\right)|a, a, b\rangle,$$

where $[x]$ is the $q$-number of $x$. The vector-space indices are arranged in natural order, for example,

$$[a, b, c] = T_a^b T_b^c T_c^a,$$

(12)

and the uncoupled basis vectors $Q_q|a_1, a_2, ..., a_{2f-1}, 1^{2f-k+1}, 1^{2f-k+2}, ..., 1^{2f}\rangle$ with different $\omega$ and $\omega_1$ are orthonormal:

$$\langle a_1, a_2, ..., a_{2f-1}, 1^{2f-k+1}, ..., 1^{2f} | Q_q^* Q_q | a_1, a_2, ..., a_{2f-1}, 1^{2f-k+1}, ..., 1^{2f}\rangle = \delta_{\omega,\omega_1} \prod_{j=1}^{2f} \delta_{a_j, a_j}.$$  

(13)

In other words, we use the orthogonal uncoupled basis of the quantum group. In this case, it can be proven that the braid-group parameters—for example, $q$—must be real; otherwise, equation (13) will be no longer valid. Equivalently, we used the following star operation:

$$g_i^* = g_i$$ for $i = 1, 2, ..., 2f-1$.  

(14)

However, results for generic braid-group parameters can be obtained by means of analytic continuation; that is, the final results are valid for generic parameters as well.

The action of $g_i$ on the basis vectors $|a_1, a_2, ..., a_{2f-1}, 1^{2f-k+1}, ..., 1^{2f}\rangle$ is given by the rules

$$g_i|a_1, a_2, ..., a_{2f-1}, 1^{2f-k+1}, ..., 1^{2f}\rangle = q|a_1, a_2, ..., a_{2f-1}, 1^{2f-k+1}, ..., 1^{2f}\rangle$$

(15a)

and if the components $a_i$ and $a_{i+1}$ are identical. This rule can be proven by using the symmetrization method outlined in [21]. At the same time, we have

$$g_i|a_1, a_2, ..., a_{2f-1}, 1^{2f-k+1}, ..., 1^{2f}\rangle = |a_1, ..., a_{i+1}, a_p, ..., a_{2f-1}, 1^{2f-k+1}, ..., 1^{2f}\rangle$$

(15b)

if the components $a_i$ and $a_{i+1}$ are different. Because of the property of braid groups, we must always write uncoupled basis vectors in the operator form. By way of example, we indicate that, in the case of a Hecke algebra,

$$g_i|b, a\rangle = g_i^*|b, a\rangle = (q^{-1} - q)|b, a\rangle + |b, a\rangle.$$  

(16a)

otherwise, the notation $\hat{R}$ is rather confusing in practical computations.

Hence, the action of the operator $\hat{R}$ on the uncoupled basis vectors of quantum groups is well defined. Using equations (15) and defining relations among braid-group generators, we can derive the $\hat{R}$ matrix elements. In the following, we will demonstrate the application of this procedure to deriving $\hat{R}$ matrix elements only for the case of a Hecke algebra.

The Hecke algebra $H_q(x)$ is generated by $f - 1$ elements $g_1, g_2, ..., g_{f-1}$, which satisfy the well-known braid relations

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1},$$

(17a)

$$g_i g_j = g_j g_i, \text{ for } |i - j| \geq 2,$$  

(17b)

$$(g_i)^2 = g_i(q^{-q^{-1}}) + 1.$$  

(17c)

The braid-group elements $\hat{R}$, which can be expressed in terms of the braid-group generators by using (8), act on the vector-space indices $[1, 2, ..., 2f]$. Any uncoupled irreducible basis vectors $w_{[k]}^{[\omega]}(\omega_0^0)$, $w_{[k]}^{[\omega]}(\omega_2^0, 2f-k+1, 2f-k+2, ..., 2f)$ of $U_q(n)$ can be further expanded in terms of uncoupled basis vectors of $(2f-k)$-fold basic representations as given by (9). In the following, we restrict ourselves to $[\lambda_1] = [\lambda_2]$ and use the case of $[2] \times [2]$ as an example.

**Step 1.** It is necessary to write all the uncoupled basis vectors of the corresponding quantum group.

In the $[2] \times [2] U_q(n)$ case, these are

$$|ii, ii\rangle = |i, i, i, i\rangle,$$

(18a)

$$|ij, ij\rangle = \left(\frac{q^{-1}}{q}\right)_{[1]} |i, j, 1, 1\rangle + \frac{q^{1/2}}{q^{1/2}} |\overline{1}, i, 1, 1\rangle$$

(18b)

$$\times \left(\frac{q^{-1}}{q}\right)_{[1]} |1, i, j, 1\rangle + \frac{q^{1/2}}{q^{1/2}} |\overline{1}, 1, j, 1\rangle$$

$$= A_{12} A_{34} b_{ii, ij},$$

where

$$A_{12} = \frac{1}{q^{1/2}}(q^{-1} + q^{1/2} g_{12}),$$

(19a)

$$A_{34} = \frac{1}{q^{1/2}}(q^{-1} + q^{1/2} g_{34}).$$

(19b)

Here, we used the CG coefficients of $[1] \times [1] \downarrow [2]$ of $U_q(n)$. Similarly, we arrive at

$$|ij, kl\rangle = A_{12} A_{34} |i, j, k, l\rangle \text{ for } i < j < k < l,$$

(19c)
\[ |kl, ij\rangle = \tilde{R}|ij, kl\rangle = g_2 g_3 g_1 g_2 |i, j, k, l\rangle. \]  

(19d)

All other basis vectors can be written in a similar way.

**Step 2.** This step consists in deriving algebraic relations between $\tilde{R}$, $A_{12}$, and $A_{34}$.

It can be proven that

\[ \tilde{R} A_{12} = A_{34} \tilde{R}, \]  

(20a)

\[ \tilde{R} A_{34} = A_{12} \tilde{R}. \]  

(20b)

Thus, we obtain

\[ \tilde{R} A_{12} A_{34} = A_{12} A_{34} \tilde{R}. \]  

(21)

Relation (21) is very useful in practical computations.

We also need the following quadratic equation of $\tilde{R}$:

\[ R^2 = (q - q^{-1})g_1 g_3 \tilde{R} + (q - q^{-1})^2 \tilde{R} \]

\[ + (q - q^{-1}) g_3 g_1 g_3 g_1 + (q - q^{-1}) g_3 g_2 + \]

\[ + (q - q^{-1}) g_2 g_3 g_2 + (q - q^{-1}) g_2 + 1. \]

(22)

**Step 3.** Applying $\tilde{R}$ to all uncoupled basis vectors obtained at step 1 and using the algebraic relations derived at step 2 and equation (15), we obtain all $\tilde{R}$ matrix elements at this step. By way of example, we indicate that, in the $[2] \times [2] U_q(n)$ case, we have

\[ \tilde{R}|ii, ii\rangle = g_2 g_3 g_2 |i, i, i, i\rangle = q^4 |ii, ii\rangle. \]

(23)

\[ \tilde{R}|ij, ij\rangle = \tilde{R} A_{12} A_{34} |i, j, i, j\rangle = \tilde{R} A_{12} A_{34} g_3 |i, i, j, j\rangle \]

\[ = A_{12} A_{34} g_3 g_3 g_2 |i, i, j, j\rangle \]

\[ = A_{12} A_{34} (q - q^{-1}) |i, j, i, j\rangle + q^2 A_{12} A_{34} |i, i, j, j\rangle. \]

(24)

Using the relation

\[ A_{12} |i, i\rangle = q^{-1/2} [2^{1/2}|i, i\rangle, \]

(25)

we obtain

\[ \tilde{R}|ij, ij\rangle = (q^2 - q^{-2}) |ij, ij\rangle + q^2 |ij, ij\rangle \text{ for } i < j. \]  

(26)

Similarly, we have

\[ \tilde{R}|ij, kl\rangle = |kl, ij\rangle \text{ for } i < j < k < l, \]  

(27a)

\[ \tilde{R}|kl, ij\rangle = \tilde{R}^2 |ij, kl\rangle \]  

(27b)

The procedures for evaluating $\tilde{R}$ matrix elements of other quantum groups are very similar. For example, we can evaluate $\tilde{R}$ matrix elements for the $G_2$ case with the aid of a Kalfagianner algebra [22]. Hence, the problem of constructing the $\tilde{R}$ matrix amounts to finding all possible algebraic realizations of a braid group. It should be noted that this method becomes very tedious when the rank of the irrep increases. In this case, the $\tilde{R}$ operator is expressed in terms of a lengthy braid-group-generator product. Although this method is not simpler than other methods for higher dimensional irreps of quantum groups, it shows a new way to calculate $\tilde{R}$ matrix elements, a new perspective of $\tilde{R}$ matrices, and a transparent view of its braid-group structure.

**ACKNOWLEDGMENTS**

The project is supported by the Natural Science Foundation of China and by the Excellent Young Teacher’s Foundation of the State Education Commission of China.

**REFERENCES**