ALGEBRAIC REALIZATION OF THE ROTOR

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INTRODUCTION

The rotor enjoys a prominent place in physics. In classical mechanics it is usually offered as the most challenging example of rigid-body motion. Applications extend from the simple symmetrical top to the dynamics of mechanical gyros, satellite behavior, and even planetary motion. All physicists learn early in their career that "the polhode rolls on the herpolhode..."

One of the first applications of the new quantum mechanics was to the rotor. The symmetric case was considered first. Its eigenstates are known to have a relatively simple representation in terms of Euler angles and rotation matrices. Because of the symmetry, the projection of the angular momentum on the body-fixed symmetry axis is a good quantum number. The asymmetric case is a more challenging problem. Its eigenstates are Lamé functions. These can be expressed as a linear combination of the eigenfunction of the symmetric rotor. A major application is found in molecular physics. The simplest structure possessing an asymmetric rotor geometry is H₂O, the ordinary water molecule. Some excellent reviews of rotor dynamics as it applies in molecular physics are available.

It is for the rotor that algebraic methods, as understood and used today, were first introduced in physics. The thesis of H.G.B. Casimir is a treatise on the "Rotation of a Rigid Body in Quantum Mechanics". Following Klein, Casimir used Heisenberg's matrix mechanics in an analysis of the dynamics of the rotor. This is in contrast, for example, with the work of Kramers and Ittman which was based on Schrödinger's wave mechanics. Casimir established the connection between eigenfunction of the rotor and irreducible representations of the rotation group in three dimensions. Indeed, in his thesis one finds an outline of the general theory of continuous groups. The rotor example clearly illustrates the advantages of using algebraic over analytic methods when the system Hamiltonian possesses a high degree of symmetry.

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Two decades after these initial efforts, a major new application of rotor dynamics emerged for the spectra of many nuclei were found to display rotational characteristics. In contrast with molecular applications where the electronic (-10^4 eV), vibrational (-10^{-2} eV), and rotational (-10^{-3} eV) modes are well-separated in energy, in nuclei the intrinsic nucleonic modes are energetically comparable to vibrational excitations (-1 MeV) while rotations, when they occur, are again smaller (-0.1 MeV) but only by about one order of magnitude. Since rotational energetics depend quadratically on the angular momentum \( |E_r| \sim l(l+1) \), except for the very lowest spins \( |l=0,2,4,6| \) rotational excitations in nuclei are not well-separated from the other modes. Nonetheless, the success of the collective model in describing the properties of nuclei is now well-established and in this rotations play an important, often dominant role.

Although the idea of collective rotations seemed to be in direct conflict with parallel developments supporting a shell-model interpretation of nuclear structure, as evidence was gathered it became clear that both pictures had validity. The challenge of understanding this duality continues to be a driving force stimulating new work in nuclear theory; the question, How does a strongly interacting many-body system support rotational behavior?, lives on. Elliott made a giant step forward towards resolving the paradox when he demonstrated that a quadrupole-quadrupole interaction operating in a space partitioned into irreducible representation of SU(3), the symmetry group of the oscillator, gives rise in a very natural and simple way to \( l(l+1) \) rotational spectra.

In the last decade other algebraic theories have been introduced that provide an even simpler shell-model interpretation of nuclear structure. The most popular of these is the Interacting Boson Model (IBM), The building blocks of the theory are \( s(L=0) \) and \( d(L=2) \) bosons which represent coupled pairs of fermions. The largest symmetry group of the model is \( U(6) \) with its thirty-six dimensional Lie algebra generated by bilinear combinations of the boson creation and annihilation operators. It supports three subgroup chains which have been associated with vibrational, rotational, and so-called gamma unstable nuclear configurations. A complementary theory is the Microscopic Collective Model (MCM). The largest symmetry group in this case is \( Sp(3,R) \), the dynamical group of the oscillator with a two-one dimensional Lie algebra built out of bilinear combinations of the fermion coordinate and momentum operators. The theory is a multishell generalization of the simplest Elliott model and contains the rotor angular momentum and quadrupole operators as a subalgebra.

Both the IBM and MCM models, as well as Elliott's SU(3) model which works for light nuclei, and its extension, the so-called pseudo SU(3) model which applies for heavier deformed systems, contain a root SU(3) \( \sim SO(3) \) group structure, see Table 1. In each case that root structure has been associated with rotational motion. And for each there is a prescription for determining the SU(3) content of the model space and the representations of SU(3) that are expected to be the dominant ones in a description of the low-lying states of a system. From this, and certain complementary developments in group theory and statistical spectroscopy, we were convinced that the SU(3) \( \sim SO(3) \) algebra truly embodies the dynamics of rotational motion. Could it be, even though its been a quarter of a century since the pioneering work of Elliott first appeared, that there is yet more to be learned about SU(3)? We believe the answer to that question is an emphatic yes! What follows is a report on our findings.
Table 1. The SU(3) → SO(3) root structure in shell-model theories.

<table>
<thead>
<tr>
<th>BOSON</th>
<th>FERMIAN</th>
</tr>
</thead>
<tbody>
<tr>
<td>IBM</td>
<td>S_{p}(3,R)</td>
</tr>
<tr>
<td>↓</td>
<td>↓</td>
</tr>
<tr>
<td>U(6)</td>
<td>SU(N_p) × SU(N_n)</td>
</tr>
<tr>
<td>SU(3)</td>
<td>SU(3)</td>
</tr>
<tr>
<td>SO(3)</td>
<td>SO(3)</td>
</tr>
</tbody>
</table>

ASYMMETRIC ROTOR REVISITED

The hamiltonian of the asymmetric rotor assumes a very simple form when written in terms of projections of the angular momentum operator I onto the body-fixed symmetry axes of the system,

\[ H_{\text{ASR}} = \frac{1}{3} \sum_{\alpha=1}^{3} A_{\alpha} I_{\alpha}^2. \]  
(1)

In (1), the \( A_{\alpha} \) are inertia parameters. For rigid-body motion these are given by \( A_{\alpha} = \frac{1}{2} c_{\alpha} \) where the \( c_{\alpha} \) are eigenvalues of the inertia tensor,

\[ I_{\alpha\beta} = \int_{\mathbb{R}} (r^2 \delta_{\alpha\beta} - x_{\alpha} x_{\beta}) \, d^3 r. \]  
(2)

It will prove to be convenient to have everything expressed in terms of the eigenvalues, \( \lambda_{\alpha} \), of the traceless quadrupole operator,

\[ Q^c_{\alpha\beta} = \int_{\mathbb{R}} (3 x_{\alpha} x_{\beta} - r^2 \delta_{\alpha\beta}) \, d^3 r. \]  
(3)

The superscript "c" will be appended as necessary to denote collective model operators. The inertia and quadrupole tensors are related in a very simple way:

\[ I_{\alpha\beta} = \frac{1}{3} (\zeta \delta_{\alpha\beta} - Q^c_{\alpha\beta}), \text{ where } \zeta = 2 \int_{\mathbb{R}} r^2 \, d^3 r. \]  
(4)

It follows from this that \( c_{\alpha} = (\zeta - \lambda_{\alpha})/3 \), which is a result that can be used to define the moments of inertia of a system even if the motion is not rigid-rotor-like.

It should be clear that the asymmetric rotor hamiltonian is invariant under rotations about the principal axes; that is, \( H_{\text{ASR}} \) commutes with the transformation operators \( T_{\alpha} = \exp(i \lambda_{\alpha}). \) The set of operators \( (E, T_1, T_2, T_3) \), where \( E \) is the identity, forms a realization of the Vierergruppe \( (D_4) \). As a consequence, eigenstates of the rotor can be classified according to their transformation properties under this four-element symmetry group. The symmetrized eigenstates can be
expressed in terms of eigenstates of the symmetric rotor for which \( A_1 = A_2 = A_3 \) and, as a consequence, the magnitude of \( K \), the eigenvalue of \( I_3 \), is a good quantum number.\(^1\)

\[
\psi^{(\lambda\mu)|K|I}_{\text{SYM}} M = \frac{1}{\sqrt{2(\lambda+\mu+1)}} \left( D^I_{\text{KM}} + (-1)^{\lambda+\mu} D^I_{-\text{KM}} \right),
\]

\[
\psi^{(\lambda\mu)|I|}_{\text{AR-M}} M = \sum_{K>0} C^{(\lambda\mu)|I|}_{K} \psi^{(\lambda\mu)|K|I}_{\text{SYM}} M.
\]  

In (5), the \( D^I_{\text{KM}} \) are rotation matrices and the prime on the summation for the asymmetric wavefunction means that only even or odd \( K \) values are to be included. The quantity \( \lambda \) can be either an even (0) or odd (1) integer while \( \mu \) must be even for \( K \) even and odd for \( K \) odd. The transformation properties of these functions under \( D_2 \) are given in Table 2. The quantity \( \nu \) is a running index used to give a unique labelling to the eigenstates since in general there are more than one of symmetry type \( (\lambda\mu) \) and spin \( I \).

### Table 2. Properties of eigenstates of the asymmetric rotor. The symmetry types are classes of the Vierergruppe \( (D_2) \) with elements \( \mathbf{T} = \exp(i\pi\alpha) \) that generate rotations \( \alpha \) by \( \pi \) about the principal symmetry axes of the system.

<table>
<thead>
<tr>
<th>Symmetry Type</th>
<th>Transformation</th>
<th>Index ( \lambda \mu )</th>
<th>Dimension ( I ) (even)</th>
<th>Dimension ( I ) (odd)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>1 1 1 1</td>
<td>e e</td>
<td>(I+2)/2</td>
<td>(I-1)/2</td>
</tr>
<tr>
<td>( B_{1\frac{1}{2}} + B_3 )</td>
<td>1 -1 -1 1</td>
<td>o e</td>
<td>I/2</td>
<td>(I+1)/2</td>
</tr>
<tr>
<td>( B_{2\frac{1}{2}} + B_2 )</td>
<td>1 -1 1 -1</td>
<td>o o</td>
<td>I/2</td>
<td>(I+1)/2</td>
</tr>
<tr>
<td>( B_{3\frac{1}{2}} + B_1 )</td>
<td>1 1 -1 -1</td>
<td>e o</td>
<td>I/2</td>
<td>(I+1)/2</td>
</tr>
</tbody>
</table>

*The first of the two indices on the B-type symmetry label is the usual one found in character tables for the crystallographic point groups while the second one indicates directly the axes of the rotation. In what follows, the first index will be suppress in favor of the second to gain the added simplicity of a direct relationship between the \( \mathbf{B}_\alpha \) and \( \mathbf{T}_\alpha \).

*The symbols e and o refer to even and odd character of the integer indices \( \lambda \) and \( \mu \) that are used to specify the symmetry type.

*The number of eigenstates with total angular momentum \( I \) of a specific \( D_2 \) symmetry type. Note that the sum of each column is \( 2I+1 \). For \( I=0 \) there is only A-type symmetry and for \( I=1 \) there are only B-types, each one occurring once.
The invariance of the Hamiltonian under $D_2$ means that for angular momentum $I$ the $(2I+1)$-dimensional matrix equation \((K=I, -I+1, ..., I)\) breaks up into block diagonal form. The dimension of each of these submatrices, which can be labelled by its $D_2$ symmetry type or, equivalently, by the odd or even character of the integer indices $\lambda$ and $\mu$, is also given in Table 2. Since most even-even rotational nuclei have a positive parity $I=0, 2, 4, ...$ ground-state band rotational sequence, it seems that type A symmetry applies. However, there is no apriori reason to exclude the others. Indeed, as will be shown below, the shell model supports all four symmetry types.

Because of the invariance of $H_{ASR}$ under $D_2$, the eigenenergies of the rotor satisfy simple sum-rule relations. These form a powerful tool to use in checking whether or not a spectrum is rotational for they apply independent of the choice of the rotor's inertia parameters. Probably the best known example, which applies for A-type symmetry, is $E_{1^2}\ E_{2^2}\ E_{3^2}$; that is, the sum of the eigenenergies for \(I=2\) is equal to the sum of the eigenenergies for \(I=3\). Indeed, for A-type symmetry it can be shown that the trace of the submatrix of $H_{ASR}$ for spin $I$ is equal to the trace of the submatrix of $H_{ASR}$ for spin $(I+1)$.

\[
\text{Tr}(H_{ASR})_I = \text{Tr}(H_{ASR})_{I+1}.
\] (6)

A summary of analytic results for traces of $H_{ASR}$, from which results such as (6) can be deduced, are given in Table 3. Analytic results for traces of the square of $H_{ASR}$ can also be written down. By combining the two, analytic results for the centroids \[c = d^{-1}\text{Tr}(H)\] and variances \[\sigma^2 = d^{-1}\text{Tr}(H - c)^2\] of rotor spectra can be given:

\[
\begin{align*}
\epsilon_{ASR}/S_c &= \frac{1}{6}(I+1)/3 & \text{I(even)}
\end{align*}
\]

\[
\begin{align*}
\sigma_{ASR}/S_\sigma &= \frac{1}{45} \begin{cases} 
(4I-3)(I+4) & \text{I(even)} \\
(I-3)(4I+7) & \text{I(odd)}
\end{cases}
\end{align*}
\] (7)

In (7), $S_c = A_1A_2A_3$ and $S_\sigma = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$. Centroid and variance measures will be used below in a comparison of rotor and $SU(3)$ + $SO(3)$ shell-model results.

Before proceeding to a consideration of the $SU(3)$ + $SO(3)$ algebra, it is important to further prepare the way by giving a frame independent representation for the rotor Hamiltonian. To achieve this consider the following three rotational scalars,

\[
I^2 = \sum_\alpha I_\alpha^2 = I_1^2 + I_2^2 + I_3^2
\]

\[
\begin{align*}
X_3^2 &= \sum_{\alpha, \beta} q_{\alpha}^C q_{\beta}^C I_{\alpha, \beta} = \lambda_1 I_1^2 + \lambda_2 I_2^2 + \lambda_3 I_3^2
\end{align*}
\]

\[
\begin{align*}
X_4^2 &= \sum_{\alpha, \beta, \gamma} q_{\alpha}^C q_{\beta}^C q_{\gamma}^C I_{\alpha, \beta, \gamma} = \lambda_1^2 I_1^2 + \lambda_2^2 I_2^2 + \lambda_3^2 I_3^2.
\end{align*}
\] (8)
In (8), $Q_{\alpha \beta}^C$ is the quadrupole operator defined by (3). The last forms given for the $X_3^2$ and $X_4^2$ operators follow because they are rotational scalars and can therefore be evaluated without loss of generality in the principal axis system where $Q_{\alpha \beta}^C = \lambda_{\alpha \beta} 0_{\alpha \beta}$. Equations (8) can be inverted to yield expressions for the $I_\alpha^2$ in terms of $I_2^2$, $X_3^2$, $X_4^2$:

$$I_\alpha^2 = \frac{\lambda_{\beta \gamma} I_2^2 + \lambda \chi_{\alpha 3} X_3^2 + \lambda \chi_{\alpha 4} X_4^2}{(2\lambda_{\alpha \gamma}^2 + \lambda \chi_{\alpha \beta} \gamma)},$$

where $(\alpha, \beta, \gamma) \overset{\text{cyclic}}{\longrightarrow} (1, 2, 3)$. \hspace{1cm} (9)

It follows that

$$H_{\text{ASR}} = a I_2^2 + b X_3^2 + c X_4^2;$$

$$a = \sum_{\alpha, \alpha'} a_{\alpha \alpha'} \lambda_{\beta \gamma}/(2\lambda_{\alpha \gamma}^2 + \lambda \chi_{\alpha \beta} \gamma),$$

$$b = \sum_{\alpha, \alpha'} b_{\alpha \alpha'} \lambda_{\beta \gamma}/(2\lambda_{\alpha \gamma}^2 + \lambda \chi_{\alpha \beta} \gamma),$$

$$c = \sum_{\alpha, \alpha'} c_{\alpha \alpha'} = 1/(2\lambda_{\alpha \gamma}^2 + \lambda \chi_{\alpha \beta} \gamma).$$ \hspace{1cm} (10)

Since this new form for $H_{\text{ASR}}$ does not explicitly display the $D_2$ symmetry of the rotor, it might appear to represent a step backward rather than forward. While that may be so if the objective is only to solve the rotor problem, it is not so if the objective is to explore the underpinnings of rotational behavior in microscopic systems. We will now show that the $SU(3) \rightarrow SO(3)$ group structure supports a faithful realization of the rotor dynamics; that is, there is an $SU(3) \rightarrow SO(3)$ hamiltonian that is the microscopic image of (10) and its eigenstates display all the features of eigenstates of the rotor, including the $D_2$ invariance!

Table 3. Sum rules for eigenenergies of the rotor. Traces of $H_{\text{ASR}}$ are given in terms of $S=(A_1+A_2+A_3)/6$ and $S = A_\alpha/2$ where the $A_\alpha$ are the inertia parameters of the rotor.

<table>
<thead>
<tr>
<th>Symmetry Type</th>
<th>$I(\text{even})$</th>
<th>$\text{Tr}(H_{\text{ASR}})I$</th>
<th>$I(\text{odd})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$I(I+1)(I+2)S$</td>
<td>$I(I-1)(I+1)S$</td>
<td></td>
</tr>
<tr>
<td>$B_1$</td>
<td>$I(I-1)(I+1)S + I(I+1)S_1$</td>
<td>$I(I+1)(I+2)S - I(I+1)S_1$</td>
<td></td>
</tr>
<tr>
<td>$B_2$</td>
<td>$I(I-1)(I+1)S + I(I+1)S_2$</td>
<td>$I(I+1)(I+2)S - I(I+1)S_2$</td>
<td></td>
</tr>
<tr>
<td>$B_3$</td>
<td>$I(I-1)(I+1)S + I(I+1)S_3$</td>
<td>$I(I+1)(I+2)S - I(I+1)S_3$</td>
<td></td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>$2I(I+1)(2I+1)S$</td>
<td>$2I(I+1)(2I+1)S$</td>
<td></td>
</tr>
</tbody>
</table>
SU(3) + SO(3) Algebra

The use of SU(3) in physics can be traced back to the pioneering work of Racah who, in his quest to provide a simple interpretation of regularities in atomic spectra, led him to explore various group theoretical coupling schemes.\(^1\) The SU(3) \(\rightarrow\) SO(3) structure enters for p-shell applications. The objective was to find a set of operators with simple eigenvalues that could be used to form a complete orthonormal labelling scheme for basis states of multiplet configurations. The hope was that the eigenvalues of these operators would be associated with important physics.

The SU(3) algebra is spanned by eight generators, three angular momentum operators, \(L_\mu; \mu=0,1,2\), which are associated with its SO(3) subgroup, and five quadrupole operators, \(Q_\mu; \mu=0,1,2\). (It is important to note that whereas the angular momentum operator \(L\) is the microscopic equivalent of the collective model \(I\), the spherical tensor operators \(Q_\mu\) of the SU(3) algebra are not the microscopic equivalent of the collective model cartesian operators \(Q_{\alpha\beta}\). They are the combination of \(r^2 Y_{2\mu}(q,\phi,\rho)\) and \(p^2 Y_{2\mu}(q,\phi,\rho)\) that in a harmonic oscillator basis have no matrix elements coupling different major shells. However, within a major shell equivalence holds.) Since SU(3) is a rank 2 group it has two Casimir invariants, one of degree two in the generators, which we label \(C_2\), and one of degree three which we label \(C_3\). The eigenvalues of these operators are related to the irreducible representation labels \((\lambda_\mu)\) in a simple way,

\[
\begin{align*}
\langle C_2 \rangle(\lambda_\mu) &= \lambda^2 + \lambda_\mu + \mu^2 + 3\lambda + 3\mu \\
\langle C_3 \rangle(\lambda_\mu) &= 2\lambda^3 + 3\lambda^2 - 3\lambda_\mu^2 - 2\mu^2 + 9\lambda^2 - 9\mu^2 + 9\lambda - 9\mu
\end{align*}
\]  

(11)

Three labels are required to specify states within an irreducible representation. When SU(3) is reduced with respect to SO(3), the angular momentum and its projection provide two of these three labels. One additional state-labelling operator is necessary.

Racah recognized that there are two candidates for resolving the SU(3) \(\rightarrow\) SO(3) state-labelling problem, \(X_3^2\langle LQL\rangle^0\) and \(X_4^2\langle LQQL\rangle^0\). But neither he nor his students were able to find a linear combination of these operators that has simple eigenvalues and yields an orthonormal resolution of multiply occurring \(L\) values.\(^2\) In addition to this state-labelling problem, SU(3) is not simply reducible; that is, in the reduction of a product of two SU(3) representations a given representation may occur more than once. These two features make it the simplest example of the situation that is usually encountered in algebraic theories. For these technical reasons and other physical ones, there has been an abundance of theoretical work on the SU(3) \(\rightarrow\) SO(3) group structure.\(^3\) Only SO(3), which is simply reducible and multiplicity free, has been better studied.\(^4\)

Perhaps the most significant of the theoretical developments relates to the introduction and use of the integrity basis concept. It provides a simple answer to the question, What is the complete set of state-labelling operators? Or, stated differently and in a way that elevates the question beyond the state-labelling problem, What SO(3) scalars can be built out of the SU(3) generators? Each such scalar or combination of scalars is a candidate for a state-labelling
operator. In this form the question can be identified as a classic problem of group theory that is addressed, for example, in the work of Molien, Noether, and Weyl. The solution is simple: all the SO(3) scalar operators that can be built out of the generators of SU(3) can be expressed as polynomial functions of a finite subset of the SU(3) → SO(3) scalars. That set is called the SU(3) → SO(3) integrity basis.

The SU(3) → SO(3) integrity basis consist of six operators. These can be chosen to be the second and third order Casimir invariants of SU(3), C_2 and C_3, the Casimir invariant of SO(3), L^2, and three non-Casimir invariant scalar operators, one of degree three in the generators of SU(3), X_3 = (LxQxL) = (LQL)^0, one of degree four, X_4 = (LxQ)^k x (QxL)^k with X_4 = (LQL)^0, and an antihermitian one of degree six that can be represented by the commutator of X_3 and X_4, X_6 = [X_3, X_4]. The square of X_6, which is necessarily hermitian, has a polynomial representation in terms of the other integrity basis operators so in any polynomial expansion X_6 should only enter linearly.

In probing how rotations might emerge in an SU(3) → SO(3) algebraic theory, we had to ask and answer the question, What is the most general SO(3) scalar interaction the SU(3) algebra supports? Clearly this is the same question as the one raised in seeking a resolution of the state-labelling problem. The answer is a polynomial function of the integrity basis operators,

\[ H = \sum_{abcde} h_{abcde} (L^a)^b (X^c)_3 (X^d)_3 (X^e)_3. \]  

(12)

Note that the antihermitian operator X_6 is not included because we have assumed time reversal invariance applies. Now within a single representation of SU(3) the Casimir invariants C_2 and C_3 act as a simple multiple of the identity; they only contribute to a shift in the centroid energy of the representation. The effective interaction can therefore be written as a polynomial function of just three operators, (L^2, X_3, X_4). For an H of maximum degree k in the generators of SU(3) we have

\[ H(k) = \sum_{2a+3b+4c=k} h_{abc} (L^a)^b (X^c)_3. \]  

(13)

We have published results which show that there exists a fourth order SU(3) → SO(3) integrity basis interaction that reproduces the eigenvalue spectrum of an axially symmetric rotor hamiltonian,

\[ \frac{1}{2a} L^2 + \frac{1}{2b} - \frac{1}{2b} I_3^2 \equiv H_{\text{SYM}} \leftarrow H_{\text{SU3}} \equiv aL^2 + bX_3 + cX_4 + dL^4. \]  

(14)

Actually, the mapping can be established without including the L^4 term in H_{SU3}. It represents a centrifugal stretching or antistretching "correction" that is only present in a higher-order collective model theory.
Before proceeding further with the derivation of analytic results for mapping between collective (macroscopic) and algebraic (microscopic) rotor Hamiltonians, it seems appropriate to give explicit results for matrix elements of the integrity basis operators for then regardless of the form chosen for H, because of (12), its matrix elements can be evaluated, \[ \langle \lambda_k \lambda_l | \mathbf{L}^2 | \lambda_k \lambda_l \rangle = L(L+1) \delta_{k',k} \]

\[ \langle \lambda_k \lambda_l | \mathbf{L} | \lambda_k \lambda_l \rangle = L(L+1) \Delta(L) \delta_{k',k} \]

\[ \langle \lambda_k \lambda_l | \mathbf{X}_3 | \lambda_k \lambda_l \rangle = L(L+1) \Delta(L) \delta_{k',k} \]

\[ \langle \lambda_k \lambda_l | \mathbf{Q}_3 | \lambda_k \lambda_l \rangle = Q_3(L) \delta_{k',k} \]

In (15), W denotes an SO(3) Racah coefficient and the double-barred matrix elements of the quadrupole operator are given by

\[ \langle \lambda_k \lambda_l | \mathbf{Q}_3 | \lambda_k \lambda_l \rangle = (-1)^{\frac{1}{2} \sum \frac{1}{2} L} \langle \lambda_k \lambda_l | \mathbf{Q}_3 | \lambda_k \lambda_l \rangle \]

\[ \langle \lambda_k \lambda_l | \mathbf{Q}_3 | \lambda_k \lambda_l \rangle = Q_3(L) \delta_{k',k} \]

The phase factor \(\phi\) is 1 if \(\mu \neq 0\) and 0 if \(\mu = 0\). As in (11) above, the factor \(C_2\) represents the expectation value of the second order Casimir invariant of SU(3) and \(\langle \lambda_k | \mathbf{L} | \lambda_l \rangle_{\rho=1} \) is a reduced SU(3) Wigner coefficient. \(C_2\) is normalized so that \(Q^2 = 3C_2\). The symbol \(\kappa\) is used in place of the Elliott K label to denote the fact that an orthogonal labelling of the multiple occurrences of \(L\) values in the \(\lambda_k\) representation of SU(3) is being employed. The \(\kappa\)-scheme preserves as best possible the physical significance of the Elliott state-labelling prescription. In particular, all \(\kappa=0\) states are pure Elliott K=0 states, a \(\kappa=2\) state is an Elliott K=2 state with sufficient K=0 admixture to make it orthogonal to the \(\kappa=0\) state, etc.

There is an obvious choice for the SU(3) \(\rightarrow\) SO(3) image of an asymmetric rotor Hamiltonian,

\[ H_{SU3} = aL^2 + bX_3 + cX_4 \]

This follows from (10) under the substitution \(I \leftrightarrow L\) and \(Q^2 \leftrightarrow Q\). What about the values of the \(a, b, c\) parameters? In the collective model they are given in terms of the inertia parameters of the system. In the SU(3) \(\rightarrow\) SO(3) model we have, until now, taken them as parameters to be determined in any particular case by a least squares fit to the data. However, one can do better than that. Analytic expressions for the \(a, b, c\) parameters of the SU(3) \(\rightarrow\) SO(3) theory can be obtained by establishing a relationship between the \(\lambda\) and the representation labels \(\lambda\) and \(\mu\) of SU(3).

Under the Elliott prescription for projecting states of good angular momentum from the intrinsic highest weight SU(3) state,

\[ \langle Q_0 | = \langle 2|X_3 - N_1 - N_2 | \sim 2\lambda + \mu \]

\[ \langle 2A_0 | = \langle X_1 - N_2 | \sim \mu \]

(18)
In (18), $N_\alpha$ counts the number of oscillator quanta in the $\alpha$-th direction. Since $<x^2_\alpha>_\alpha = N_\alpha$ for the oscillator, it follows that $<N_\alpha>_\alpha = \lambda_\alpha$ and, accordingly,

$$
\begin{align*}
\lambda_1 &\sim (-\lambda + \mu)/3 + \sigma_1 \\
\lambda_2 &\sim (-\lambda - 2\mu)/3 + \sigma_2 \\
\lambda_3 &\sim (2\lambda + \mu)/3 + \sigma_3.
\end{align*}
$$

(19)

In (19) the $\sigma_\alpha$ are constants that satisfy $\sigma_1 + \sigma_2 + \sigma_3 = 0$. To fix the $\sigma_\alpha$ and hence uniquely specify the relationship between the $\lambda$ and $\lambda$ and $\mu$, we require that the collective model invariants $\text{Tr}[(Q^2)^2]$ and $\text{Tr}[(Q^2)^3]$ go over into invariants of the SU(3) algebra. For this $\sigma_1 = 0$, $\sigma_2 = 1$, $\sigma_3 = 1$, in which case

$$
\begin{align*}
\text{Tr}[(Q^2)^2] &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \frac{2}{3}(\lambda^2 + \lambda\mu + \mu^2 + 3\lambda + 3\mu + 3) = \frac{2}{3} <C_2> + 2 \\
\text{Tr}[(Q^2)^3] &= \lambda_1^2\lambda_2^2 + \lambda_3^2 - \frac{1}{9}(2\lambda^2 + 3\lambda\mu + 3\mu^2 + 2\lambda + 2\mu + 2\lambda\mu + 9\lambda^2 - 9\mu^2 - 9\lambda - 9\mu) = \frac{1}{9} <C_3>.
\end{align*}
$$

(20)

The microscopic equivalent of the rotor hamiltonian can now be given. Care must be exercised in associating the cartesian operators of the rotational model to the spherically coupled tensor operators of the integrity basis: $X_3 \leftrightarrow (6/\sqrt{10})X_3^C$ and $X_4 \leftrightarrow -(18/5)X_4^C$.

Taking these differences into account we finally have

$$
\begin{align*}
\text{AI}_1^2 + \text{BI}_2^2 + \text{CI}_3^2 &\equiv H_{\text{ASR}} \longleftrightarrow H_{\text{SU3}} \equiv aL^2 + bX_3 + cX_4
\end{align*}
$$

$$
\begin{align*}
A &= a - \frac{2}{\sqrt{10}} f_2 b + \frac{2}{15} f_1 (\frac{f_2^2}{f_3^2}) c \\
B &= a - \frac{2}{\sqrt{10}} f_2 b + \frac{2}{15} f_0 (\frac{f_2^2}{f_1^2}) c \\
C &= a - \frac{2}{\sqrt{10}} f_2 b + \frac{2}{15} f_0 (\frac{f_2^2}{f_2^2}) c
\end{align*}
$$

$$
\begin{align*}
a &= -\frac{1}{9} (\frac{f_2 f_3}{g_2 g_3} A + \frac{f_3 f_1}{g_3 g_2} B + \frac{f_1 f_2}{g_1 g_2} C) \\
b &= \frac{1}{18} (\frac{f_1}{g_2} A + \frac{f_2}{g_3} B + \frac{f_3}{g_1} C) \\
c &= \frac{5}{18} (\frac{1}{g_2 g_3} A + \frac{1}{g_3 g_2} B + \frac{1}{g_1 g_2} C)
\end{align*}
$$
\[
\begin{align*}
\text{where } & \quad f_1 = \lambda - \mu \\
& \quad g_1 = \lambda + \mu + 2 \\
& \quad f_2 = \lambda + 2\mu + 3 \\
& \quad g_2 = -1 \\
& \quad f_3 = -2(\lambda + \mu + 3) \\
& \quad g_3 = -1 \\
\end{align*}
\] (21)

**SOME EXAMPLES**

So what? By some formal manipulations we have arrived at what appears to be an SU(3) + SO(3) integrity basis interaction that is the image of the asymmetric rotor hamiltonian. But does the mapping reproduce rotor dynamics? In particular, do eigenstates of \(H_{SU3}\) yield rotor values for energies, electromagnetic transition rates, etc.? And what about the \(D_2\) invariance; can the eigenstates of \(H_{SU3}\) be classified according to symmetry classes of the Vierergruppe? This seems quite unlikely for even simpler things do not seem to match up. For example, the \(\lambda\mu\) representation of SU(3) is finite dimensional with the number of occurrences of a specific \(L\) value given by the rather awkward formula \(^{32}\)

\[
d(\lambda\mu, L) = \frac{[(\lambda+\mu+2-L)/2] - [(\lambda+1-L)/2] - [(\mu+1-L)/2]}{2}.
\] (22)

In contrast, for the rotor there are \(2I+1\) states of spin \(I\). The heavy brackets in (22), \([\ ]\), denote the greatest integer function. Furthermore, there is no angular momentum cutoff for the rotor whereas for SU(3) the maximum \(L\) value in the \(\lambda\mu\) representation is \(\lambda\mu\). Despite all this, we will now show that \(H_{SU3}\) is indeed a true image of \(H_{ASR}\), with eigenstates that reproduce rotor values for observables and even possess \(D_2\) invariance!

First consider the dimension question. It is a simple exercise to verify, using (22), that for both \(\lambda\) and \(\mu\) even and the angular momentum \(L\) less than or equal to the \(\text{min}(\lambda, \mu)+1\), the multiplicity of the \(L\) value in the \((\lambda\mu)\) representation of SU(3) is \((L+2)/2\) for even \(L\) and \((L-1)/2\) for odd \(L\). This is the same result as the one that applies for the A-type, \(D_2\) symmetry subspace of the rotor, see Table 1. Similarly, for either \(\lambda\) or \(\mu\) odd or both \(\lambda\) and \(\mu\) odd, the dimensions agree with those for the B-type, \(D_2\) symmetries. For \(L\) values greater than \(\text{min}(\lambda, \mu)+1\), the \((\lambda\mu)\) representation produces less states than exist for the rotor. This is a direct consequence of the microscopic underpinnings of the algebraic theory. For example, if the low-lying states of \(^{24}\text{Mg}\) are taken to be \((s)^4(p)^{12}(d)^8\) shell-model configurations, there is only one way to make an \(L=12, S=0, T=0\) state. It is found in the leading \((\lambda\mu)=(8,4)\) representation of SU(3). For the rotor, on the other hand, there are seven \(I=12\) states of A-type symmetry. Six of those seven are blocked by the finite-space/fermion-statistics constraint of the shell-model picture.

In Figure 1, six sets of spectra are given, three for \(H_{ASR}\) and three corresponding ones for \(H_{SU3}\). The three sets for the asymmetric rotor are labelled by the asymmetry parameter,

\[
\kappa = (2A_1-A_3-A_2)/(A_3-A_2).
\] (23)

By convention, \(A_2 < A_1 < A_3\). Accordingly, the asymmetry parameter has
Figure 1. A comparison of the spectra of $H_{ASR}$ and $H_{SU3}$ for prolate, most asymmetric, and oblate rotor geometries: $A_2=10$ and $A_3=200$ with $A_1=10, 105, 200$ corresponding to values $\kappa = -1, 0, +1$ for the asymmetry parameter which is defined by $\kappa = (2A_1-A_3-A_2)/(A_3-A_2)$. The algebraic results are for the $(\lambda_\mu) = (40,8)$ representation with the interaction parameters determined using the connection formulae. Only for the prolate case is $K$ a good quantum number; the separation of the spectra into bands for the other two cases is only to aid in making a comparison of the spectra.
a value -1 for a prolate shape \( A_2 = A_1 < A_3 \), 0 for what can be called the most asymmetric shape \( A_2 < A_1 = (A_2 + A_3)/2 < A_3 \), and +1 for an oblate shape \( A_2 < A_1 = A_3 \). In the Figures, \( A_2 = 10 \) and \( A_3 = 200 \) so \( A_1 = 10,105,200 \) for the \( \kappa = -1,0,1 \) results, respectively. Only for the prolate case \( (\kappa=1) \) is \( K \) a good quantum number; for the most asymmetric \( (\kappa=0) \) and oblate \( (\kappa=+1) \) cases \( K \) is not a good quantum number. So the spectra, as shown, are somewhat misleading because the groupings suggest a \( K=0 \) and \( K=2 \) band structure in all three cases whereas in reality that labelling only applies to the first.

The SU(3) results shown in Figure 1 are for \((\lambda,\mu)=(40,8)\) with the parameters \( a, b, c \) of \( H_{SU3} \) set by the connection formula, (21). For example, for the \( \kappa=0 \) case with \( A_1 = 105 \), \( B_2 = 10 \), \( C_3 = 200 \) one finds that \( a = 183.03, b = 2.929, c = 0.045769 \). Notice how closely the SU(3) eigenenergies track the rotor results. Whereas one might be tempted to dismiss the prolate results as elementary and fortuitous for there are only really two important features to the spectra, the \( L[I+1] \) spacing and the \( K \)-band splitting, in the most asymmetric and oblate cases one cannot for the rotor spectra are quite complex and show very little regularity. Nonetheless, the algebraic results reproduce the rotor results with comparable quality in all three cases. In particular, note how accurately the 7-6 and 5-4 "inversions" are reproduced in the \( \kappa=0 \) and \( \kappa=\pm 1 \) cases, respectively.

The dimensionality arguments given above suggest, at least for \( L \) values less than \( \min(\lambda,\mu)+1 \), that the \( \lambda \) and \( \mu \) labels might provide a key to gaining an understanding of the transformation properties of SU(3) + SO(3) basis states under the action of generators of the Vierergruppe. To see how this comes about, we draw on known symmetry relations for the transformation coefficients between SU(3) + SU(2) x U(1) and SU(3) + SO(3) basis states.\(^{32}\) For SO(3) projection from an intrinsic highest weight SU(3) state one has that

\[
\langle \lambda\mu | l(\lambda\mu) | KLM \rangle = (-1)^{\lambda+\mu+L} \langle \lambda\mu | l(\lambda\mu) | KLM \rangle \tag{24}
\]

Since the \( l(\lambda\mu) \) form a complete set and the phase factor in (24) is independent of \( \alpha \), it follows that

\[
| l(\lambda\mu) | KLM \rangle = (-1)^{\lambda+\mu+L} | l(\lambda\mu) | KLM \rangle \tag{25}
\]

The SU(3) + SO(3) states can therefore be reorganized to have the same transformation properties under the action of the generators of \( D_2 \) as eigenstates of the symmetric rotor,

\[
| l(\lambda\mu) | KLM \rangle \equiv \frac{1}{\sqrt{2(1+\kappa)\kappa}} | l(\lambda\mu) | KLM \rangle + (-1)^{\lambda+\mu+L} | l(\lambda\mu) | KLM \rangle \tag{26}
\]

And just as for the rotor, the odd-even character of \( \lambda \) and \( \mu \) then dictate the \( D_2 \) symmetry class to which the SU(3) + SO(3) eigenstates belong, see Table 2.

To provide further support for this association between the odd-even character of the \( \lambda \) and \( \mu \) values and \( D_2 \) rotor symmetry types, we calculated and compared sum-rule measures for eigenenergies of \( H_{SU3} \) and \( H_{ASR} \). The results are given in Figure 2(a,b). The solid curves are \( \text{Tr}(H_{ASR})/L[I[I+1]] \) which, according to the results given in Table
3, depend linearly on 1. The symbols label the corresponding $H_{SU3}$ measures for selected $(\lambda_\mu)$ values. (The $(\lambda_\mu)=(30,8)$ representation is the leading one in a pseudo SU(3) description of the rare earth nucleus $^{168}\text{Er}$. The other $(\lambda_\mu)$ values are its even-odd, odd-even, and odd-odd neighbors.) In each case the results shown are for the most asymmetric rotor geometry ($\kappa=0$) with $a,b,c$ determined by the mapping formulae, (21). The agreement is remarkable; for $L_{\text{min}}(\lambda_\mu)+1$ the deviations fall below the 1% level. This can also be seen in Figure 3(a,b) where results for centroids and variances of the rotor and algebraic, $(\lambda_\mu)=(30,8)$, theories are compared. For $L_{\text{min}}(\lambda_\mu)+1=9$, the agreement is again excellent, better for the centroids than for the variances. This is to be expected for the variance is a measure of the spread in the eigenvalues of a spectrum about the centroid which serves to fix its location. Note that $\omega_{SU3}^2 = 0$ for $L=37$ and $L=38$ as is required for each of these $L$ values occurs only once in the $(\lambda_\mu)=(30,8)$ representation.

Figure 2. A comparison of rotor and algebraic results for traces [divided by $L(L+1)$] of the interaction $H_{ASR}$ and $H_{SU3}$ in the $A,B,\alpha$-type symmetry subspaces of $D_2$. The same set of inertia parameters were used in all four cases with $(\lambda_\mu)$ values chosen as indicated. The parameters of $H_{SU3}$ were fixed from those of $H_{ASR}$ by using the connection formulae. Whereas algebraic results exist for $H_{ASR}$ (Table 3) the $H_{SU3}$ numbers were determined by first calculating the various $L$-submatrices and explicitly evaluating their traces.
Figure 3. A comparison of centroid and variance measures for the asymmetric rotor ($\kappa=0$) and its algebraic $[(\lambda, \mu) = (30,8)]$ image. Up to $L=\text{min}(\lambda, \mu)+1=9$, every state of the rotor has an algebraic image. The fall-off for $L>9$ is a direct consequence of "missing" states in the algebraic theory. Note that for $L=37$ and $L=38$ the variance, which is a measure of the spread in eigenvalues, goes to zero as it must for there is but one of each of those states in the $(30,8)$ representation.

In Figure 4 a typical spectrum for $H_{\text{ASR}}(\kappa=0)$ is shown. The corresponding results for $H_{\text{SU3}}$, with $(\lambda, \mu)=(8,4)$ and the $a, b, c$ parameters determined using the connection formulae (21), are also given. A comparison of the two gives a good indication of both the accuracy and limitations of the mapping procedure. All states of the $(8,4)$ representation were included in generating the SU(3) eigenvalue spectrum: $L=0^1,2^2,3^2,4^2,5^2,6^2,7^2,8^2,9^2,10^2,11^2,12^1$. These states are a shell-model image of those associated with the eigenvalues enclosed in the box on the rotor spectrum. Note that for the ground state band, which terminates at $L=12$ in the SU(3) theory, the agreement between eigenenergies is almost perfect. The agreement is also excellent for the lowest members of the second band ($L \leq 7$). Bigger differences are found between eigenenergies of excited states of the third band. Note also that it is only for $L$ value up to five
Figure 4. Asymmetric rotor ($\kappa=0$) excitation spectra and its algebraic image for the $(\lambda\mu)=(8,4)$ representation of SU(3). Only rotor levels within the broken-boxed area have an SU(3) image in the $(8,4)$ representation. Note that the agreement is excellent for ground-band states. It deteriorates slightly with increasing angular momentum in the excited bands with the onset of significant deviations occurring for lower angular momenta in the higher bands.
that the \((8,4)\) representation accounts for all the \(L\) values of the rotor; the rotor has four \(L=6\) states whereas in the \((8,4)\) representation there are only three, etc. These results are typical: the finite space constraint of an SU(3) theory results in deviations from rotor values for observables that increase with increasing \(L\) within each band with the onset occurring at lower \(L\) values in the higher bands. But even for a relatively small representation like the \((8,4)\), which is the leading one in a \((s^{4}p^{12}(d_{5/2})^{8}\) shell-model description of \(^{24}\text{Mg}\), the algebraic and rotor results agree amazingly well!

As the final example, we now give some results for \(^{24}\text{Mg}\). The experimental and calculated spectra are shown in Figure 5. Using a nonlinear-leastsquares procedure we first determined the parameters of the asymmetric rotor that yielded eigenenergies that best reproduce the experimental excitation energies:

\[ A_1 = 195.79, \]
\[ A_2 = 195.30, \]
\[ A_3 = 848.50 \] (keV). This implies a symmetric \((\kappa = 0.998)\) rotor configuration for \(^{24}\text{Mg}\). We then used the mapping formulae to determine the parameters of the SU(3) Hamiltonian under the assumption that the leading representation is \((\lambda_{\mu}) = (8,4): a = 239.61, b = 20.955, c = 1.4370\). The results shown are for this SU(3) + SO(3) integrity basis interaction. As already seen in Figure 3 for the most asymmetric case, the \(H_{\text{SU3}}\) and \(H_{\text{ASR}}\) results differ very little for states of the \((8,4)\) representation with \(L \leq 7\). It is for that reason the rotor results are not shown. The fit to the data, determined in this way, is quite reasonable. However, one can do better by applying the

![Energy Spectrum Diagram](Figure 5. A comparison of experimental (EXP) and theoretical (SU3) energy spectra for the ground band and gamma band states of \(^{24}\text{Mg}\). The theoretical spectra is the result of applying the connection formulae for the \((\lambda_{\mu}) = (8,4)\) representation to inertia parameters of the rotor where the latter were determined by a best fit to the data.)
least-square procedure directly to the SU(3) theory without going through the intermediate steps of fitting to the rotor and using the connection formulae. This suggests that the SU(3) picture, while it can be used to reproduce rotor results, actually goes beyond it!

CONCLUSION

We have shown that the SU(3) × SO(3) group structure provides for an algebraic realization of the rotor. Given the three inertia parameters of the rotor and the representation labels $\lambda$ and $\mu$ of SU(3), the connection formulae, (21), can be used to determine the parameters of an SU(3) × SO(3) integrity basis interaction that "reproduces" rotor results when acting in the $(\lambda,\mu)$ space. (Although we have not given results for electromagnetic transition rates, they compare in a way that mirrors the agreement between eigenenergies of the two theories. In particular, for low L values of the lowest bands the algebraic and rotor results for E2 transition rates, both intra and inter-band, are nearly identical.) The theory applies equally to boson (e.g., IBM) and fermion (e.g., MCM) realizations of the SU(3) × SO(3) algebra. The difference in the physics enters through the representations of SU(3) that occur. For example, in the simplest IBM theory only even $\lambda$ and $\mu$ values enter. Accordingly, such a theory supports only A-type, D-type rotor configurations. In contrast, in an MCM theory all symmetry types can be realized.

A particularly intriguing and important feature is that the constants of the motion for the rotor are carried over into Casimir invariants of the algebraic theory. This suggests that the concept of shape has significance in both pictures. In particular, if $\langle Q_{2}^{C} \rangle \sim k \beta \cos(\gamma)$, $\langle Q_{4}^{C} \rangle \sim 0$, and $\langle Q_{6}^{C} \rangle \sim 1/2 k \beta \sin(\gamma)$, where as usual $\beta$ and $\gamma$ are the shape parameters of the hydrodynamical model, then

$$
\begin{align*}
\lambda_1 &\sim -(1/2)k \beta [\cos(\gamma)-3\sin(\gamma)] \sim -(\lambda-\mu)/3 \\
\lambda_2 &\sim -(1/2)k \beta [\cos(\gamma)+3\sin(\gamma)] \sim -(2\lambda+\mu+3)/3 \\
\lambda_3 &\sim k \beta \cos(\gamma) \sim +(2\lambda+\mu+3)/3
\end{align*}
$$

and

$$
\begin{align*}
\text{Tr}[\langle Q_{2}^{C} \rangle^2] &\sim (3/2)k^2 \beta^2 \sim (2/3)C_2 \sim 2 \\
\text{Tr}[\langle Q_{4}^{C} \rangle^3] &\sim (3/4)k^3 \beta^3 \cos(3\gamma) \sim (1/9)C_3.
\end{align*}
$$

In this way $\beta$ and $\gamma$ are seen to be determined by $C_2$ and $C_3$ and are therefore the same for all members of a representation.

We have said very little about how this fits into a shell-model picture of nuclear structure. To see how this goes, consider the Elliott model. The full shell-model space is first partitioned into its space [irreducible representations of U(N(N+1)/2) for the Nth oscillator shell] and supermultiplet [irreducible representations of U(4)] parts. The space geometry is therefore given.
by the group chain \( U(N(N+1)/2) \rightarrow SU(3) \rightarrow SO(3) \) with the corresponding state labels \([\mathcal{f}]_{\lambda}(\lambda_{\mu})\). Likewise, the supermultiplet symmetry must be reduced with respect to its spin-isospin subgroup to gain the \( S \) and \( T \) state labels, \( U(4) \rightarrow SU_{S}(2) \times SU_{T}(2) \) which yields the state labels \([\tilde{\mathcal{f}}]_{\rho\tau}(\rho\tau)\). Antisymmetric states of good total angular momentum are formed by coupling \([\mathcal{f}] \) and \([\tilde{\mathcal{f}}] \) to \([I^{D}] \) and \( L \) and \( S \) to \( J \),

\[
|\varphi\rangle \rightarrow |\left[I^{D}\right];[\mathcal{f}]_{\lambda}(\lambda_{\mu})\rangle_{\rho\tau};|\tilde{\mathcal{f}}\rangle_{\rho\tau}(\rho\tau)\rangle_{\rho\tau};|J_{\rho\tau}\rangle.
\]

(29)

In this way the full space is partitioned into representations of \( SU(3) \). If the true hamiltonian for the many-body system has the form

\[
H = H_{SU3} + H'
\]

(30)

with the model space norm of \( H' \) small compared to that of \( H_{SU3} \), the shell-model dynamics will appear to be that of a collection of rotors interacting through the \( H' \) term.

An interesting feature of this scenario is that if \( H \) is truly a microscopic interaction, the \( H_{SU3} \) part of \( H \) will have the same form in all representations of \( SU(3) \). The connection formulae can then be used to determine the effective inertia parameters for the various \((\lambda_{\mu})\) that occur. Suppose, as an example, that in the \( ^{24}\text{Mg} \) case \( H \) gives rise to an \( H_{SU3} \) that corresponds to the most asymmetric shape for the leading \((\lambda_{\mu})=(8,4)\) representation: \( a=49.400 \), \( b=0.82264 \), \( c=-0.41887 \) for which \( A=50 \), \( B=100 \), and \( C=150 \) so \( \nu=0 \). That same interaction will produce different asymmetry measures in other \( SU(3) \) representations. Examples are given in Table 4. The end result is that if \( H \) can be written in the form suggested by (30), then the shell model can be viewed as a collection of interacting rotors of mixed symmetry types and different inertia parameters. Of course, if \( H' \) dominates, the rotor features will be lost. But there is good evidence suggesting that it does not, and in that case the next task is to discover how to extract \( H_{SU3} \) and \( H' \) from \( H \) and give appropriate measures for their relative importance. In this the methods of statistical spectroscopy can be brought to bear.35

<table>
<thead>
<tr>
<th>((\lambda_{\mu}))</th>
<th>(A_1)</th>
<th>(A_2)</th>
<th>(A_3)</th>
<th>(\kappa)</th>
<th>(D_2) Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lambda_{(8,4)})</td>
<td>100</td>
<td>50</td>
<td>150</td>
<td>0.00</td>
<td>A</td>
</tr>
<tr>
<td>(\lambda_{(7,3)})</td>
<td>84</td>
<td>50</td>
<td>127</td>
<td>-0.12</td>
<td>B_1</td>
</tr>
<tr>
<td>(\lambda_{(8,1)})</td>
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<td>54</td>
<td>127</td>
<td>-0.53</td>
<td>B_2</td>
</tr>
<tr>
<td>(\lambda_{(4,6)})</td>
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<td>51</td>
<td>107</td>
<td>+0.75</td>
<td>A</td>
</tr>
<tr>
<td>(\lambda_{(5,4)})</td>
<td>84</td>
<td>49</td>
<td>107</td>
<td>+0.21</td>
<td>B_3</td>
</tr>
</tbody>
</table>
Of course there are also other open questions. For example, the simplicity of our results suggest that it might be possible to find analytic expressions for matrix elements of the integrity basis operators.\footnote{36} Also, it should be clear that the \text{SU}(3) - \text{SO}(3) algebra offers more than just rotor dynamics; we have not explored features that may enter with the inclusion of higher-order terms in \(\mathfrak{h}_{\text{SU}(3)}\).

Obviously the \(L^2\) term that is usually associated with centrifugal distortions in the collective model is there, but what about terms like \(L^2\mathbf{X}_3\) or \(X_qX_4\), what do they do? And since the eigenfunctions of the asymmetric rotor are Lame functions, can more be said about analytic forms for basis states of \text{SU}(3)? In short, it seems that that \text{SU}(3) remains a fertile field for future farming!

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