ALGEBRAIC BRIDGE BETWEEN MACROSCOPIC AND MICROSCOPIC MODELS OF NUCLEAR STRUCTURE

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Abstract

Algebraic models developed over the past 2-3 decades have helped to bridge the gap between macroscopic (collective-model) and microscopic (shell-model) theories of nuclear structure. Consequences of a correspondence between the triaxial rotor model and a generic SU(3) theory are the subject of this review. While rotations are relatively easy to understand, vibrations are not and this raises important questions regarding the potential energy surface concept.

1. Introduction

A long-time objective of nuclear physics has been to bridge the gap between macroscopic (collective-model) and microscopic (shell-model) descriptions of observed phenomena. Progress has been slow because of the difficulty in carrying out microscopic calculations, and the success of the much simpler macroscopic pictures. Algebraic shell-model theories come closest to realizing this objective. There are two basic types of algebraic models: those based on a boson description of the dynamics, such as the Interacting Boson Model (IBM) [1], and others that can be grouped under an Algebraic Fermion Model (AFM) logo [2], which treat the nucleons as fermions. In this paper we will focus on the latter - bosons will only enter when they appear naturally as a consequence of physical circumstances such as low particle or high hole occupancy so Pauli Principle constraints are not violated.

The first and most familiar algebraic fermion model is the Elliott SU(3) scheme. It is known to work well for light (N ≤ 20) nuclei [3]. Another is the Sp(3,R) (denoted Sp(6,R) sometimes) or symplectic model which is a natural multi-flavor extension of the Elliott scheme [4]. For heavier systems (A ≥ 150) there are currently two algebraic models being employed: the so-called Fermion Dynamical Symmetry Model (FDMSym) which identifies s and d fermion pair operators that form an algebra which closes under commutation (the SO(8) group for the n=4 shell and Sp(6) for n=5 and n=6) and gives a possible microscopic interpretation of the IBM [5], and the pseudo-SU(3) model and its pseudo-symplectic extension which build on the concept of good pseudo-spin symmetry in heavy nuclei [6,7,8].

This talk will focus on the SU(3) model and its Sp(3,R) extension. First we will show how rotational motion can be realized within this framework and then we will go on to explore the shell-model underpinnings of β and γ vibrational motion as it is normally portrayed in the collective model. The resultant theory leads to a deeper understanding of the strengths and limitations of collective-model phenomenology and gives a shell-model interpretation of the potential energy surface concept. It also demonstrates the importance of multi-flavor shell-model couplings that enter naturally in the symplectic extension of the Elliott SU(3) theory.

2. Rotational motion

A shell-model interpretation of any motion can be given whenever the Hamiltonian which generates that motion can be expressed in a frame-independent form because that expression can then be directly rewritten in terms of the corresponding microscopic operators. The rotor is a particularly elegant example because expressing the Hamiltonian in a frame-independent form is relatively easy and leads immediately to its shell-model representation. Furthermore, the operators that enter into the expression have historical significance, dating back to Racah's pioneering work on the SU(3)×SO(3) symmetry group [9].

The triaxial rotor Hamiltonian is given by

\[
H_{\text{ROT}} = A_{1}I_{1}^{2} + A_{2}I_{2}^{2} + A_{3}I_{3}^{2}
\]

where \(I_{\alpha} (\alpha = 1, 2, 3)\) is the projection of the total angular momentum on the \(\alpha\)-th body-fixed symmetry axis and \(A_{\alpha}\) is the corresponding inertia parameter. This principal-axis form can be rewritten in a frame-independent representation using three special scalar operators:

\[
L_{\alpha}^{2} = \sum_{\alpha} I_{\alpha}^{2}, \quad X_{\alpha}^{2} = \sum_{\alpha} I_{\alpha}Q_{\alpha}^{2}, \quad X_{\alpha}^{4} = \sum_{\alpha} I_{\alpha}Q_{\alpha}^{4}
\]

The \(L_{\alpha}\) and \(Q_{\alpha}^{2}\) in this equation are Cartesian forms for the total angular momentum and collective quadrupole operators, respectively. (The superscript \(c\) appended to the \(Q\) denotes the fact that the \(Q_{\alpha}^{c}\) are collective quadrupole operators which have non-vanishing matrix elements between major shells (\(n' = n, n \pm 2\)), in contrast with the algebraic quadrupole operators, \(Q_{\alpha}^{a}\), which have non-vanishing matrix elements only within a major shell, \(n' = n\)). The last expression given for each scalar is the form these operators take in the body-fixed, principal-axis system where the eigenvalues of \(Q_{\alpha}^{2}\) are presumed to be sharp: \(Q_{\alpha}^{a} = \lambda_{\alpha}a_{\alpha}^{\prime}\). Solving these three equations for the \(I_{\alpha}^{2}\) in terms of \(L_{\alpha}^{2}, X_{\alpha}^{2}\), and \(X_{\alpha}^{4}\) yields the following:

\[
H_{\text{ROT}} = aL_{1}^{2} + bX_{1}^{2} + cX_{1}^{4}
\]

where

\[
a = \sum_{\alpha} a_{\alpha}A_{\alpha}, \quad b = \sum_{\alpha} b_{\alpha}A_{\alpha}, \quad c = \sum_{\alpha} c_{\alpha}A_{\alpha},
\]

\[
a_{\alpha} = \frac{\lambda_{\alpha}^{2}\lambda_{\alpha}^{4}}{2\lambda_{\alpha}^{2} + \lambda_{\alpha}^{4}}, \quad b_{\alpha} = \frac{\lambda_{\alpha}^{4}}{2\lambda_{\alpha}^{2} + \lambda_{\alpha}^{4}}, \quad c_{\alpha} = \frac{1}{2\lambda_{\alpha}^{2} + \lambda_{\alpha}^{4}}.
\]

A shell-model image of the rotor Hamiltonian can now be obtained by substituting single-particle forms for the collective \(L_{\alpha}\) and \(Q_{\alpha}^{a}\) operators: \(L_{\alpha} = \sum_{\alpha} I_{\alpha}(s)\) and \(Q_{\alpha}^{a} = \sum_{\alpha} Q_{\alpha}^{a}(s)\). It is important to remember that while the \(L_{\alpha}\) have non-vanishing matrix elements only within a major oscillator shell, the \(Q_{\alpha}^{a}\) couple shells differing by two quanta (\(n' = n, n \pm 2\)). Indeed, the off-diagonal (\(n' = n \pm 2\)) couplings are about equal in magnitude to the diagonal (\(n' = n\)) ones. It follows from this that operators like \(Q_{2}^{2}\) and \(Q_{2}^{1}\) and the \(X_{1}^{2}\) and \(X_{1}^{4}\) will destroy the shell structure. This catastrophe can be avoided by setting all off-diagonal couplings between major shells to zero, an action which corresponds to replacing the \(Q_{\alpha}^{a}\) operators by their algebraic counterparts \(Q_{\alpha}^{c}\). Elliott was the first one to recognize that the \(Q_{\alpha}^{a}\) operators, along with the
where $H_0$ is the harmonic oscillator Hamiltonian.

Shell-model values for the $\lambda_i$ follow by equating invariants of the two theories. Because SU(3) is a rank two group it has two invariants: $C_2$ with eigenvalues $\lambda + \mu$, and $C_3$ with eigenvalues $\lambda + 2 \mu$ and $3(\lambda + \mu)$. The trigonometric representation labels the spherical group. The representation of the rotor $[T_3 \times \text{SO}(3)]$ also has two invariants: the trace of the square $[\text{Trace}(Q^2)]$ and the trace of the cube $[\text{Trace}(Q^3)]$ of the collective quadrupole. The eigenvalues of these forms are $\lambda^2 + \lambda^2 + \lambda^2$ and $\lambda + 2 \mu$ and $3(\lambda + \mu)$, respectively, where $\beta$ are the shape variables of the collective model. Requiring a linear correspondence between these two sets of invariants yields:

$$\lambda_1 = -(\lambda - \mu)/3, \quad \lambda_2 = -(\lambda + 2 \mu)/3, \quad \lambda_3 = +(2 \lambda + \mu + 3)/3.$$  

(5)

This correspondence sets up a relationship between the $(\beta, \gamma)$ and $(\mu, \lambda)$:

$$\beta^2 = \frac{4\pi}{5\lambda_1^2} \left[ \frac{2 \lambda + \mu + 3}{3} \right]^2, \quad \gamma = \tan^{-1} \left( \frac{\sqrt{3}(\mu + 1)}{2 \lambda + \mu + 3} \right).$$  

(6)

Since $\lambda$ and $\mu$ are positive integers, this translates into a regular grid when superimposed on a traditional $(\beta, \gamma)$ plot, with $\beta$ the radius vector and $\gamma$ the azimuthal angle:

$$k \beta_0 = k \beta \sin(\gamma) \left( \frac{2 \lambda + \mu + 3}{3} \right), \quad k \beta_0 = k \beta \sin(\gamma) \left( \frac{2 \lambda + \mu + 3}{3} \right).$$  

(7)

where $k = \frac{\lambda_1^2}{\lambda_1^2}$. Each $(\mu, \lambda)$-irrep corresponds to a unique value for the $(\beta, \gamma)$-pair. A consequence of the $(\beta, \gamma)$-$(\lambda, \mu)$ mapping is that constraints are placed on the $(\beta, \gamma)$-values of the collective model. This follows because the set of $(\mu, \lambda)$ irrep for any particular nucleus is fixed by the shell structure and particle statistics. The $(\beta, \gamma)$ $(\lambda, \mu)$ mapping also teaches us that the collective model picture is overly restrictive because within the shell-model picture a given $(\lambda, \mu)$ can occur multiple times. So whereas the $(\beta, \gamma)$ $(\lambda, \mu)$ mapping implies constraints within the $(\beta, \gamma)$ plane, it also points to a rich configuration space extending above the plane which is missing in the collective-model picture.

A further condition that follows from the $(\beta, \gamma)$ $(\lambda, \mu)$ mapping is band termination. In the collective model description the rotational bands extend indefinitely but because SU(3) is a compact group with finite dimensional representations its bands all terminate. For a prolate geometry the former follows the $L(L + 1)$ energy rule, while the latter shows deviation from this simple behavior at $L$ values greater than about 1/3 the maximum, which for the $(\lambda, \mu)$ irrep is given by $L_{\text{max}} = \lambda + \mu$. Accompanying this deviation from the $L(L + 1)$ energy rule of the rotor is a fall-off in predicted B(E2) strengths. Since B(E2) values are determined by the square of an SU(3) coupling coefficient, for the $\mu = 0$ case an analytic results can be given:

$$B(E2 : L \rightarrow L - 2) = \frac{3(\lambda + L + 1)(L - L - 2)L(L + 1)}{2(\lambda + 3)(2L - 3)(2L - 1)}.$$  

(8)

The rotor result is recovered from this SU(3) expression in the $\lambda \rightarrow \infty$ limit of the theory.

It is also instructive to push the $(\beta, \gamma)$ $(\lambda, \mu)$ connection to a consideration of the two-rotor picture which is commonly used to describe heavy nuclei – one rotor representing the protons (p) and another the neutrons (n). For prolate geometries $(\gamma_p = 0$ and $\gamma_n = 0$) it is convenient to introduce an angle $\theta$ which measures the angle between the principal axes of the two distributions. (The scissors mode used to describe the so-called giant B(M1) strengths gets its name from this simple two-rotor picture. Also note that the Exclusion Principle is not violated because the two distributions are made up of different particle types). For $\theta = 0^\circ$ the two axially symmetric ellipsoids overlap maximally (aligned principal axes) whereas when $\theta = 90^\circ$ the principal axes are perpendicular to one another and the resulting overlap of the two distributions is a minimum. The $(\beta, \gamma)$ value of the joint distribution can be determined once $\beta$, $\gamma$, and $\theta$ are specified. This picture corresponds to the $(\lambda_p, \mu_p) = (0, 0) \rightarrow (\lambda, \mu)$ coupling in the SU(3) scheme which is known to be simply reducible, that is, each allowed $(\lambda, \mu)$ irrep in the product $(\lambda_p, \mu_p) \otimes (\lambda, \mu)$ is $(\lambda + \lambda_p, \mu + \mu_p)$, $(\lambda + \lambda_p - 2, \mu + \mu_p)$, $(\lambda + \lambda_p - 4, 2)$, etc., but only those $(\lambda, \mu)$ for which $\mu \leq \lambda$ remain. Once a discrete orientation angle given by $\theta$ is specified, the $(\lambda, \mu)$ irrep in the product $(\lambda_p, \mu_p) \otimes (\lambda, \mu)$ is $(\lambda + \lambda_p - 2n, \mu)$ where $n = 0 \rightarrow 0$. In general one must deal with trivial shapes ($\gamma = 0$, and $\gamma = \pi$) and a general product distribution: $$(\beta, \gamma) \times (\beta, \gamma) \rightarrow (\beta, \gamma).$$  

In this case the geometrical interpretation is more complicated because the variable $\theta$, measuring the angle between the major axes of the ellipsoids, together with the $(\beta, \gamma)$ and $(\beta, \gamma)$, is no longer sufficient to characterize the final $(\beta, \gamma)$ configuration. In general, the three Euler angles $(\psi, \theta, \phi)$ of the rotation relating the two sets of principal axis frames must be specified. For $(\psi, \theta, \phi) = (0, 0, 0)$ the major and minor axes of the sub-distributions coincide (maximum alignment) whereas when $(\psi, \theta, \phi) = (0, 0, 0)$ the semi-axes (y) remain aligned but the major (z) and minor (x) axes of the two systems are perpendicular to one another, etc. In the corresponding SU(3) case the product configurations are determined by the Littlewood Rules for the coupling of two three-rowed Young diagrams. There is a need for three $(\mu, \lambda) = (m, n)$ quantum label in this case as well: $(\lambda_p, \mu_p) \otimes (\lambda, \mu) = (\lambda + \lambda_p + m, \mu + n)$, where $m$ is a non-negative integer index $(m = 0, 1, 2, ..., n)$ labeling distinct occurrences of the same $(\lambda, \mu)$ that can occur in the $(\lambda_p, \mu_p)$ product. Working backwards, it should also be clear that the $(\beta, \gamma)$ $(\lambda, \mu)$ correspondence can be used to give a geometrical interpretation to the abstract group theoretical concept of the outer multiplicity – at least up until now seems to have escaped a simple physical interpretation.

It is also instructive to view the relationship between the rotor and SU(3) theories at a more fundamental level. This can be achieved by comparing the algebra of their symmetry groups. The symmetry group of the quantum rotor is the semi-direct product $[T_3 \times \text{SO}(3)]$ where $T_3$ is generated by the five independent components of the (spherical) collective quadrupole operator $Q^2$ and SO(3) is generated by the angular momentum operators $(L_i)$. The generators of SU(3), on the other hand, are the $Q_{ij}$ [see the discussion between eqs.(2) and (4)] and the $L_{ij}$ operators. If $Q_{ij}$ denotes a generic quadrupole operator, then the commutation relations of the $L_{ij}$ and the $Q_{ij}$ are given in general by:

$$[L_{ij}, Q_{kl}] = -\frac{\delta_{ik}Q_{jl} - \delta_{jk}Q_{il}}{\mu} + \frac{\delta_{il}Q_{jk} - \delta_{jl}Q_{ik}}{\mu},$$  

$$[L_{ij}, Q_{kl}] = -\frac{\delta_{ik}Q_{jl} - \delta_{jk}Q_{il}}{\mu} + \frac{\delta_{il}Q_{jk} - \delta_{jl}Q_{ik}}{\mu},$$  

$$[L_{ij}, Q_{kl}] = -\frac{\delta_{ik}Q_{jl} - \delta_{jk}Q_{il}}{\mu} + \frac{\delta_{il}Q_{jk} - \delta_{jl}Q_{ik}}{\mu},$$  

$$[Q_{ij}, Q_{kl}] = c < 2 \mu, \nu | 2 \mu, \nu \times 3 \mu, \nu >.$$  

(9)

where $c = 0$ for $T_3 \times \text{SO}(3)$, $(Q^2 = Q^2)$, $c = +\sqrt{10}$ for SU(3) $(Q^2 = Q^2)$, and $c = -\sqrt{10}$ for a heretofore not mentioned group $\text{SO}(3)$ $(Q^2 = Q^2)$, which is associated with shear degrees of freedom. In eq.(9) the $< - | -$ symbol denotes an $(\alpha, \beta)$ Clebsch-Gordan coefficient. All three of these groups, $T_3 \times \text{SO}(3)$, SU(3), and $\text{SO}(3)$, are subgroups of the symplectic group $\text{Sp}(3, R)$, which we will return to in Section 4. From these commutation relations it is easy to show that the SU(3) algebra reduces to that of $T_3 \times \text{SO}(3)$ if $Q^2$ is divided by the square root of the second order invariant of SU(3) $(Q^2 = Q^2/\sqrt{2})$ where by definition the $Q_{ij} = (Q^2 \cdot Q^2 + 3L^2)/4$ commutes with the $Q^2$ and $L_{ij}$ operators the first and second and
second commutators in eq. (9) remain unchanged, while the $L_{\alpha\beta}$ on the right-hand-side of the third goes over into $L_{\alpha\beta}/C_3$, and for low values of $L$ in large SU(3) irreps, $L_{\alpha\beta}/C_3 \to 0$. This renormalization of the $Q^2$ operator is called a contraction process and the arguments presented show that the SU(3) algebra reduces to the algebra of $T_W \otimes SO(3)$ in this limit, and consequently, that the SU(3) theory reduces to that of a quantum rotor. Differences between observables of the theories follow because SU(3) is a compact group with finite dimensional representations while $T_W \otimes SO(3)$ is non-compact with infinite dimensional representations. Band termination and a fall-off in B(E2) strengths are examples.

3. Vibrational modes

The collective model takes vibrational as well as rotational degrees of freedom into account plus coupling between these modes. The coupling enters through the inertia parameters which are functions of the deformation. The macroscopic Hamiltonian can be written as the sum

$$H_{MAC} = H_{VIB} + H_{ROT}$$

where $H_{ROT}$ is given by eq.(1) with $A_0 \to 1/4 B_0 \beta^2 \sin^2(\gamma + \frac{3}{2} \alpha)$ and $H_{VIB}$ has the form,

$$H_{VIB} = \frac{1}{2} B_{2\beta} \beta^2 + B_{2\gamma} \beta \beta \gamma + \frac{1}{2} B_{\gamma\gamma} \beta^2 \gamma^2 + V(\beta, \gamma)$$

where, in principle, $V(\beta, \gamma)$ can be any rotationally invariant function of $\beta$ and $\gamma$ [10]. An early formulation of the theory used a polynomial expansion in the scalars $\beta^2$ and $\beta \cos(3\gamma)$:

$$V(\beta, \gamma) = \sum_{p=1}^{2n} C_{p} P_{p} \beta^{2p-3q}(\cos(3\gamma))^q$$

For this potential it is clear that a second-order theory ($k = 2$) gives simple harmonic motion in the $\beta$-degree of freedom while a third-order ($k = 3$) term is required for harmonic vibrations in the $\gamma$ direction. The usual theory takes ($k = 6$), which for triaxial geometries ($\gamma \neq 0$) means four additional terms are possible: $\beta^4$, $\beta^2 \cos(3\gamma)$, $\beta^2 \gamma$ and $\beta \cos(3\gamma)$, which is then a 6th order polynomial in $\beta$ (with no linear term as this would shift the minimum) in the prolate ($\beta = 0$) limit. Recent enhancements to the theory extend this to include anharmonic and even numeric forms for the $V(\beta, \gamma)$ potential [11]. Since $\beta$ measures elongation along the symmetry axis, the $\beta$ excitations contribute no angular momentum to the system and in the harmonic limit gives excitation energies in multiples of $\hbar \omega_{\beta}$. The $\gamma$ variable, on the other hand, measures departures from axial symmetry and contributes 2 units of angular momentum per $\hbar \omega_{\gamma}$.

Our shell-model interpretation of rotational motion led to a simple relationship between $(\beta, \gamma)$ and $(\lambda, \mu)$, see eqs. (6) and (7). Under this mapping, changes in $\beta$ and $\gamma$ correspond to changes in $\lambda$ and $\mu$. The following linear relations result upon differentiation of eqs. (6):$

\beta = \kappa (\cos(\tilde{\gamma}) \lambda + \cos(\tilde{\gamma}) \mu) \quad \text{and} \quad \beta \gamma = \kappa (\sin(\tilde{\gamma}) \dot{\lambda} - \sin(\tilde{\gamma}) \dot{\mu})$

where $\tilde{\gamma} = \frac{\gamma}{2} - \frac{\pi}{4}$ and $\kappa = \frac{1}{2} k$. The $\gamma \leftrightarrow \tilde{\gamma}$ transformation represents a reflection across the $\gamma = \frac{\pi}{4}$ plane and corresponds to a $\lambda \leftrightarrow \mu$ interchange under the $(\beta, \gamma) \leftrightarrow (\lambda, \mu)$ mapping. These expressions for $\beta$ and $\beta \gamma$ in terms of $\lambda$ and $\mu$ can be substituted into the kinetic energy part of $H_{VIB}$ to determine its shell-model equivalent. Likewise, the potential energy can be given in terms of $\lambda$ and $\mu$ by expressing it in terms of the invariants of the theories. For the polynomial form given in eq.(12) this is particularly simple because $\beta^4 \leftrightarrow \kappa^2 C_2(\lambda, \mu) + 3$ and $\beta \gamma \leftrightarrow \kappa^2 C_3(\lambda, \mu)$. The apparent shell-model image of $H_{VIB}$ labeled $H_{SU(3)}$ in what follows, for reasons that will become clear - is then

$$H_{SU(3)} = \frac{1}{2} B_{1\lambda} \dot{\lambda}^2 + B_{1\mu} \dot{\mu}^2 + \frac{1}{2} B_{\mu \mu} \dot{\mu}^2 + V(\lambda, \mu)$$

where from eq.(12) the potential is given by

$$V(\lambda, \mu) = \sum_{p=1}^{2n} C_{p} P_{p} \left[ C_2(\lambda, \mu) + 3 \right]^p [C_3(\lambda, \mu)]^q$$

In eq.(14) the mass parameters are functions of $\gamma$ and can be written as a sum of symmetric ($s$) and antisymmetric ($a$) parts ($B = B^s + B^a$) under the $(\beta, \gamma) \leftrightarrow (\lambda, \mu)$ and $\beta^2 \leftrightarrow \kappa^2 C_2(\lambda, \mu) + 3$ and $\beta \gamma \leftrightarrow \kappa^2 C_3(\lambda, \mu)$:

$$B_{1\lambda}(\gamma) = \kappa^2 \left[ C_2(\gamma) B_{2\beta} + \sin^2(\gamma) B_{\gamma\gamma} \right] = B_{1\mu}(\gamma)$$

$$B_{1\mu}(\gamma) = -\kappa^2 \cos(\gamma) \sin(\gamma) B_{2\beta} = -B_{2\mu}(\gamma)$$

$$B_{1\mu}(\gamma) = \kappa^2 \cos(\gamma) \sin(\gamma) B_{2\beta} - \sin^2(\gamma) \sin(\gamma) B_{\gamma\gamma} = B_{2\mu}(\gamma)$$

It appears we have discovered the microscopic equivalent of the macroscopic Hamiltonian - with time derivatives replaced by conjugate momenta under second quantization:

$$H_{MAC} = H_{SU(3)} + H_{SU(3)}$$

However, as is shown below, it is in the interpretation of the potential, $V(\beta, \gamma) \leftrightarrow V(\lambda, \mu)$, where the theories differ from one another at a fundamental level.

For the special $k = 2$ case, the $V(\lambda, \mu)$ potential in eq.(15) reduces to $C_2(\lambda, \mu) + 3$ and this expression can be brought into a simple quadratic form by using eqs.(6) and (7):

$$C_2(\lambda, \mu) + 3 = \kappa^{-2} (\beta^2 + \beta^4)$$

It follows that if $C = C_{p} \geq 0$ (see discussion below), $H_{SU(3)}$ reduces to a coupled oscillator in localized regions of configuration space, and this explains our use of the SU(2) subscript:

$$H_{SU(3)} = \frac{1}{2} B_{2\beta} \beta^2 + B_{2\mu} \dot{\mu}^2 + \frac{1}{2} B_{\mu \mu} \dot{\mu}^2 + \frac{1}{2} C_{p} \beta^2 + \frac{1}{2} C_{q} \beta^4$$

Here the mass parameters are simple functions of those already given in eq.(16):

$$B_{2\beta} = \kappa^{-2} B_{1\beta}, B_{2\mu} = \frac{1}{\sqrt{3}} \kappa^{-2} (2B_{1\mu} - B_{1\lambda}), B_{\mu \mu} = \frac{1}{3} \kappa^{-2} (B_{1\lambda} + 4B_{1\mu} - 4B_{1\lambda})$$

(19)
Within this framework the \((n_2, n_4)\) state of the \(\Xi = 2\) band in the leading SU(3) irrep, proceeds both as they must be for a realization of \(A\)-type symmetry of the \(D_3\) symmetry group and that \(\lambda, \mu \geq 2\). To fix the position of the \((n_2, n_4) = (0,1)\) mode, interpreted in this way, at an excitation energy \(\hbar \omega_0\) above the \((n_2, n_4) = (0,0)\) ground state, requires the \(\omega_2\) operator, which has eigenvalue \(K^2\), to enter \(H_{\text{ROB}}\) with the strength \(\beta = \hbar \omega_0/4 - 3\alpha/2\) where \(\alpha\) is the parameter multiplying the \(J^2\) term in \(H_{\text{ROB}}\). However, this simple prescription fails to produce a \((n_2, n_4) = (0, 2)\) mode at the second harmonic \(2\hbar \omega_0\) level, giving instead a bandhead with \(L = 4\) at an excitation energy of \(4\hbar \omega_0 - \alpha\). There is a further difference because the \((n_2, n_4) = (0, 2)\) configuration of the collective model picture consists of two states, not just one, with angular momenta \(L = 0\) and \(L = 4\) where these two result from the symmetric coupling of a pair of \(L = 2\) phonons. (The \(L = 2\) configuration is ruled out by reflection symmetry that is implicit in the hydrodynamic assumption.) In summary, while this prescription yields a relatively simple interpretation of \(B\)-vibrational motion of the collective model it only accounts for the first of the \(\gamma\)-vibrational excitations - double \(\gamma\) modes are not part of the SU(3) picture.

Differences in the two prescriptions derive from the fact that the collective model adheres to the hydrodynamic assumption which attributes no structure to the nucleus other than its shape.

This is clearly an oversimplification - the existence of SU(3) irrep with odd \(\lambda\) and \(\mu\) values, which correspond to \(B_\lambda\) type \(D_3\) symmetries \((\alpha = 1, 2, 3)\), is a clear indication that nuclei do not have an internal structure not portrayed by a simple collective-model picture. This particular feature is especially important when attempting to interpret the structure of odd-A nuclei. There is also no fixed relationship between "shape" and "moments of inertia" within the shell-model framework - a given SU(3) irrep can display the full range of spectral characteristics: \(\lambda = 1\) (prolate), \(\lambda = 0\) (oblate), and \(\lambda = 2\) (oblate). Successes of the collective model and its odd-A extension which couples collective excitations to a Nilsson-type single-particle picture for the "unpaired" nucleon have had a mesmerizing (if not a paralyzing) effect on the nuclear physics community. We believe a deeper understanding of nuclear physics phenomena lies in probing these fundamental differences.

Other boson theories, as elegant and simple as they may be, can be shown to suffer similar types of deficiencies - too few or too many states, depending upon specifics of the theory and application. For example, it is easy to show that the leading SU(3) irrep \((2n, 0)\) in the SU(3) limit of the IBA is spurious when the boson number (equal to one-half the number of valence particles) exceeds \(\hbar \omega_0\), for the shell under consideration. Whenever \(n \geq n_{\text{max}}\), some of the \(2n\) quanta must be placed in other than the \(z\)-direction, and since \(\lambda = n_2 + n_4\) and \(n = n_2 - n_4\), this means \(\lambda < 2n\) and \(\mu < 0\). So while it can be argued with justification that the collective model, and the more sophisticated derivative theories based on boson algebra, can be used to catalog and help understand large bodies of nuclear structure information, a lack of respect for the Pauli Principle can lead to the wrong physics, especially for states of heavy systems where the structure is strongly influenced by competition between interactions among the nucleons and the particle antisymmetry requirements.

4. Energy surfaces

The \((\lambda, \mu) \rightarrow (\lambda, \mu)\) correspondence set down in Section 2 means that the important collective-model concept of a potential energy surface can also be given a shell-model interpretation [13]. Recall that within the framework of the collective model the potential energy of a particular nucleus is given by \(V(\lambda, \mu)\), see eq (12), where the parameters of the corresponding collective-model hamiltonian have been adjusted to give a best fit to the excitation energies and transition strengths of that system. For this picture to be given a shell-model interpretation it is necessary to generalize the potential energy surface concept to that of a total energy surface. This follows because the potential and kinetic energies of a system are not separately
measurable, only their sum (which is the total energy) can be determined experimentally. One possible procedure is to calculate the binding energy of a system by diagonalizing its Hamiltonian which is constrained to act in a region of configuration space where the expectation value of the square and the cube of the quadrupole matrix are $\sim \beta^2$ and $\sim \beta^2 \cos(3\gamma)$, respectively. By varying $\beta$ and $\gamma$, this procedure can be used to determine the functional dependence of the total energy on the deformation parameters. Alternatively, one could determine coherent states for the system by requiring sharp expectation values for the square and cube of the quadrupole matrix and then calculate the expectation value of the Hamiltonian with respect to these states. Regardless of the methodology selected, the concept is only meaningful if the observables ($E \sim <H>, \beta^2 \sim <Q \cdot Q>$, and $\beta^2 \cos(3\gamma) \sim <Q \cdot (Q \times Q) \cdot Q>$ of the system are, or are approximately, simultaneously diagonalizable.

The simplest and most straightforward procedure, since the $(\beta, \gamma) \rightarrow (\lambda, \mu)$ mapping has already been established, is to calculate the expectation of the Hamiltonian in SU(3) shell-model irreps. The expectation value should be evaluated in the $L = 0$ state, which because it is unique is also a Hamiltonian eigenstate for that irrep: $<H> = \beta_0 \epsilon_0 [\lambda, \mu] L = 0]$. The $L \neq 0$ states in each irrep usually occur more than once and always lie at some positive excitation energy relative to the $L = 0$ configuration. This procedure produces the left-hand portion ($\beta < \beta_0$, where $\beta_0$ is the value of $\beta$ for which $<H>$ is an absolute minimum) of the schematic curve shown in Figure 1. The near inverse quadratic dependence of $<H>$ on $\beta$ for values of the deformation close to zero follows from the fact that the quadrupole-quadrupole term, with expectation value proportional to $\beta^2$ in $L = 0$ states, enters the residual interaction with a negative sign. Within the horizontal $0\hbar \omega$ extension of the shell-model the leading $(3\hbar \omega)$ irrep is the most deformed and therefore lies lowest – the other Pauli allowed configurations correspond to $\beta < \beta_0$ values lying at some positive excitation energy relative to the leading $(\lambda_0, \mu_0)$ irrep, where $\beta_0$ and $\gamma_0$ are related to $\lambda_0$ and $\mu_0$ in the usual way through eq. (7).

The $\beta > \beta_0$ portion of the curves shown in Figure 1 cannot be understood in the context of a $0\hbar \omega$ shell-model theory. Larger deformations correspond to radial extensions of the nucleus and these can only be realized by going to a vertical extension of the shell-model, that is, by including $2n\hbar \omega$ configurations ($n = 1, 2, \ldots$) in the set of basis states. From an algebraic perspective this extension (restricting to one-particle excitations, which excludes multi-particle-hole states) takes one from the compact SU(3) group to Sp(3,R), a non-compact symplectic group with infinite dimensional irreps. The symplectic model is an extension of the SU(3) scheme which contains as its $0\hbar \omega$ limit. The fact that irreps of the symplectic group are infinite follows because $Q^n$, which couples the $n$ and $n'$ osculators in the usual way, $n' = n$ and $n \pm 2$, is a generator of Sp(3,R). Nonetheless, the underlying oscillator structure and the fact that Sp(3,R) contains SU(3) as a subgroup means the relevant symplectic irreps have a simple structure. To see this, consider the special case where $H$ is the harmonic oscillator Hamiltonian, $H = H_0$. All $2n\hbar \omega$ configurations are then degenerate at energy $2n\hbar \omega$ relative to the $n = 0$ shell. The SU(3) content of these symmetric states is given by the direct product of a starting $0\hbar \omega$ SU(3) irrep, denoted by $(\lambda, \mu)$, and the SU(3) irreps contained in the symmetric product of the elementary $(2, 0)$ representation taken $n$ times. When an interaction is added to the oscillator Hamiltonian, these representations split at each level in the same way as the $0\hbar \omega$ SU(3) irreps do at the $n = 0$ level. This is indicated in a schematic form in Figure 1(a) for the maximally deformed $(3, 2, \mu)$ irrep ($\lambda = \mu$), which is a one-dimensional space which results from the coupling of $(2, 0)$ to the $0\hbar \omega$ ($n = 0$) shell-model irrep, $(\lambda, \mu)$. This structure is repeated for each $(\lambda, \mu)$ at each of the $n = 2, 3, \ldots$ levels. The most deformed configuration at each level carries the $(3, 2, \mu)$ irrep label. Calculated results for $^{24}$Mg are also shown.

With no interaction the landscape is evenly terraced (upper levels overhanging that lower ones) with step sizes $2\hbar \omega$ as one moves from the ground level at $0\hbar \omega$ to the $n$-th level $2n\hbar \omega$ up. This is modified by the interaction, with the shell structure surviving so long as the gain in binding energy ($\Delta H_{BB} \rightarrow$ negative) due to an increase in the deformation is less than the excitation energy added to the system ($\Delta H_{BB} \rightarrow$ positive) to gain that shell-gap configuration energy. ($\Delta H_{BB} < \Delta H_{BB}$) or $\Delta B E [n(\lambda, \mu)] < 2n\hbar \omega$, so the net effect as one moves to larger and larger deformation is an increase in $\Delta H$. The most strongly deformed states lie near the base of cliffs where this condition is just satisfied. So rises in the expectation value of $H$ for $\beta > \beta_0$ results from competition between the increase in the binding due to the availability of configurations at larger deformation, and the energy that must be added to the system to reach that level. The potential energy surface concept of the collective model appears to be an over-simplification of the interplay between the shell structure which results from particle statistics and a residual effective nucleon-nucleon interaction that distorts the landscape to find an overall minimum energy configuration.

5. Conclusion

We have sketched in general terms a possible shell-model interpretation for rotational and vibrational modes of excitation in nuclei. The equivalence of the shell-model SU(3) and the collective-model $T_2 \wedge SO(3)$ symmetries for values of the angular momentum that are small compared to the $\lambda + \mu$ maximum that can occur in a $(\lambda, \mu)$ irrep, explains the appearance of rotational-like behavior in nuclei. The $(\beta, \gamma) \rightarrow (\lambda, \mu)$ mapping discovered in establishing the shell-model image of rotational motion was shown to lead to a shell-model interpretation of the collective-model vibrational kinetic energy. However, there are irreconcilable differences in the collective and shell-model interpretations of the potential. This follows because in the collective model all regions of the $(\beta, \gamma)$ plane are equally accessible – only the potential gives preference to specific $(\beta, \gamma)$ shapes. On the other hand, the fact that nucleons are fermions imposes a structure that even in the absence of residual interactions blocks out some regions of configuration space. At the same time the shell model picture adds a third dimension to the problem because there can be multiple independent solutions for any set of $(\lambda, \mu)$ and therefore
\((\beta, \gamma)\) values. The macroscopic-microscopic comparison is indicated schematically as follows:

\[
H_{MAC} = TVIB + V_{VIB} + H_{ROT} \\
\downarrow \quad \downarrow \\
H_{MIC} = TV_{SU2} + V_{SU3} + H_{SU3}
\]

Two of these terms in the macroscopic (collective-model) Hamiltonian have unambiguous microscopic (shell-model) counterparts. The macroscopic and microscopic potentials are quite different, however. Paradoxically, the macroscopic form is both more general and restrictive than the microscopic one: more general because solutions are continuous functions of \((\beta, \gamma)\) and not constrained by other than the form of the potential, and more restrictive because it gives only one solution per \((\beta, \gamma)\) value. In the microscopic case the allowed \((\lambda, \mu)\) irreps are determined by the \([\ell] \rightarrow (\lambda, \mu)\) group plethysm. These irreps are coupled by \(SU(3)\) symmetry breaking interactions which soften the \((\lambda, \mu)\) grid – moving the shell-model picture closer in line with the collective-model picture of the continuous \((\beta, \gamma)\) variables. The collective model will remain a purely phenomenological theory unless a means is found for incorporating into the potential restrictions imposed by antisymmetrization requirements, and likewise a shell-model theory that does not include multi-\(\pi\) configurations should not be expected to give a proper account of the structure of strongly deformed systems.

References