Chapter 2

Comments on Infinite Series and Complex numbers

Comments on Chapters 1 and 2 of Boas:

February 27, 2002

[cm02v01.tex   Physics 2221   Classical Mechanics   R. Haymaker   Spring 2002]

DRAFT

These notes are intended to be a study aid for the course but not a polished set of notes. I ask the class to please bring typos, errors, gaps in pedagogy etc. to my attention. I will update these notes occasionally. You may wish to use these as a starting point to create your own Latex notes.

2.1 Comments on Chapter 1 of Boas - Infinite Series, Power series

The principal use of infinite series in this course is the Taylor expansion of a function, i.e. a power series. Choose the expansion point at the origin for convenience and then

$$\begin{aligned} f(x) &= f(0) + \frac{x^1}{1!}\left.\frac{df}{dx}\right|_{x=0} + \frac{x^2}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x=0} + \frac{x^3}{3!}\left.\frac{d^3 f}{dx^3}\right|_{x=0} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!}\left.\frac{d^n f}{dx^n}\right|_{x=0} . \end{aligned}$$

(Check that the zeroth order term is correct in the last line.)

We will avoid problematic series with tricky convergence questions and always have the understanding that for small enough x, the series will converge. This is important since we will often approximate functions using this expression and keep only the first few terms. This procedure may be meaningless if the series does not converge. If the series does converge, then taking x smaller and smaller generally gives a better and better approximation for typical applications. The taylor series is a calculus theorem but some taylor series are clearly also understandable from purely algebraic considerations, such as the geometric series (see below)

2.1.1 Examples

It is worth memorizing the general terms for a few series so that you can recall quickly the first few terms to use in an approximation.

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots \\ (1+x)^{\alpha} &= 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots \\ e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ \sin x &= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \end{aligned}$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

- The first one is the "geometric series". Note from the pattern of terms we can deduce

$$\begin{aligned} S &= 1 + x + x^2 + x^3 + \cdots \\ xS &= x + x^2 + x^3 + \cdots \\ S &= 1 + xS \\ S &= \frac{1}{1-x} \end{aligned}$$

  It converges for $|x| < 1$.

- The second one is the binomial expansion. It terminates for positive integer values of α giving a polynomial. However it is a convergent infinite series for any α for sufficiently small x.

- The next three converge for any value of x. It is clear that no matter how large the value of x, the factorial in the denominator will always eventually dominate and drive the terms quickly to zero.

  Notice that $\sin x$, being odd, contains only odd powers and $\cos x$, being even, contains only even powers.

2.2 Comments on Chapter 2 of Boas - Complex numbers

Complex analysis is not just an extension of real analysis, it is a wholly new subject with results that are surprising and unifying. We will call upon properties of complex numbers as needed. If necessary I will refer back to this chapter in Boas. When the opportunity comes for you to pick a math elective, consider Complex Variable and Linear Vector Spaces as equally high priorities.

Lets mention a few basics, including the standard notation of z, x, and y. Think of a complex number as a point in the complex plane.

2.2.1 Arithmetic Operations

$$
\begin{aligned}
z &= x + iy \\
i^2 &= -1 \\
z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2) \\
z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i\,(x_1y_2 + y_1x_2)
\end{aligned}
$$

where this is the cartesian representation of the number. The polar representation is

$$
\begin{aligned}
z &= \rho e^{i\theta} \\
\rho e^{i\theta} &= \rho\,(\cos\theta + i\sin\theta) \qquad (2.1) \\
z_1 z_2 &= \rho_1 \rho_2 e^{i(\theta_1+\theta_2)} \\
\sqrt{z} &= \pm\sqrt{\rho}e^{i\theta/2}
\end{aligned}
$$

ρ is called the 'modulus' (physicspeak 'magnitude'), θ the 'argument'(physicspeak 'phase').

2.2.2 Complex Conjugate

The complex conjugate of a complex number is

$$
\begin{aligned}
z &= x + iy \\
\bar{z} \equiv z^* &= x - iy
\end{aligned}
$$

Mathematicians use $\bar{z}$, physicists use both $\bar{z}$ and z^*.

2.2.3 The n nth roots of unity

$$
1^{1/n} = e^{i0},\ e^{i2\pi/n},\ \cdots e^{i2\pi(n-1)/n}.
$$

2.2.4 complex exponentials and complex logarithms

$$\begin{aligned} z &= \rho e^{i\theta} \\ z &= e^{w} \\ w &= \ln(z) \\ &= \ln(\rho) + i(\theta + 2\pi n); \qquad n = \text{integer} \end{aligned}$$

Note that there are an infinite number of values of the logarithm. But that is no surprise, one can add $2\pi n$ to theta in an exponential to give the same value.

2.2.5 exponential of a complex number

Consider

$$\begin{aligned} e^{z} &= e^{x+iy} \\ &= e^{x}(\cos y + i \sin y) \end{aligned}$$

2.2.6 complex powers of complex numbers

Consider the complex numbers z and ζ:

$$\begin{aligned} z &= \rho e^{i\theta} \\ z^{\zeta} &= e^{\zeta \ln(z)} \\ &= e^{\zeta\left(\ln(\rho) + i(\theta + 2\pi n)\right)} \end{aligned}$$

The expression in the exponential is the product of two complex numbers. But that can be expressed as a complex number. Note that there are an infinite number of values to this expression.

2.2.7 trigonometric and hyperbolic functions

$$\begin{aligned}
\sin\theta &= \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) \\
\cos\theta &= \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \\
\sinh\theta &= \frac{1}{2}\left(e^{\theta} - e^{-\theta}\right) \\
\cosh\theta &= \frac{1}{2}\left(e^{\theta} + e^{-\theta}\right)
\end{aligned}$$

2.2.8 circle of convergence of power series

One of the beautiful results of complex analysis is an extemely simple understanding of the confergence of power series. The details of this require a proper course in complex variable but we can give a simple rule for the convergence of a power series.

Consider

$$f(z) = \sum_{n=1}^{\infty} \frac{f^n(0)}{n!}(z - z_0)^n$$

Plot the expansion point z_0 in the complex plane. Draw a cirle around it. Expand the circle until it encounters a singularity. The power series will converge for every value of z inside that circle. I will explain with examples in class.

2.3 Problem Set

Due in class, Wednesday, January 30.

1. Find the power series for

$$\frac{1}{(1-x)^2}$$

three ways: (i) square the geometric series, (ii) take a derivative of the geometric series (iii) using the binomial series (second example). Work out these examples from zeroth through fourth order.

2. Verify Eqn.(2.1) using the power series in Sec. 2.1 generalized to complex numbers for terms up to θ^5.

3. Evaluate i^i, (note there are an infinite number of values)

4. consider the function

$$f(z) = \frac{1}{z^2 + 4}$$

Expand this function in a power series about the point $z = 0$, determining the coeficients. Find the radius of the circle of convergence. Take a value of z with a modulus twice the radius of the circle of convergence. Evaluate numerically the first 5 terms in the series and appraise the possibility that the series might converge or not.