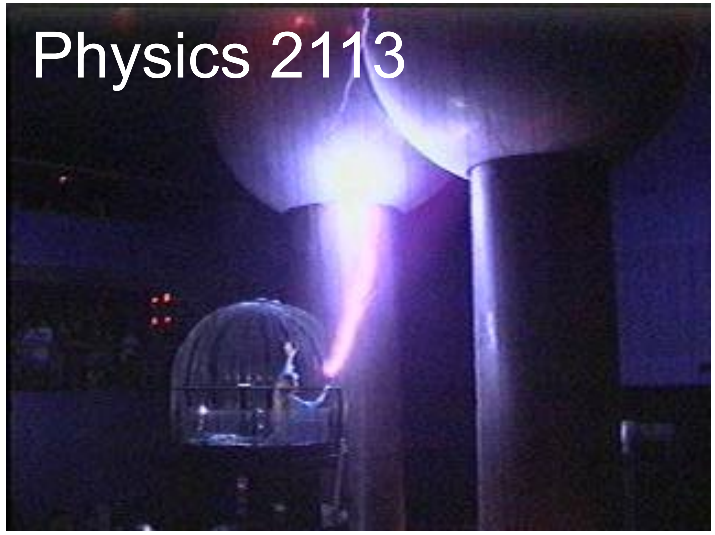
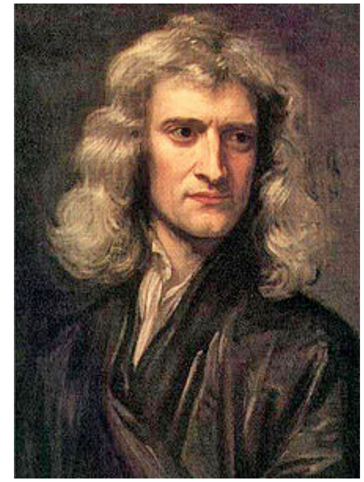
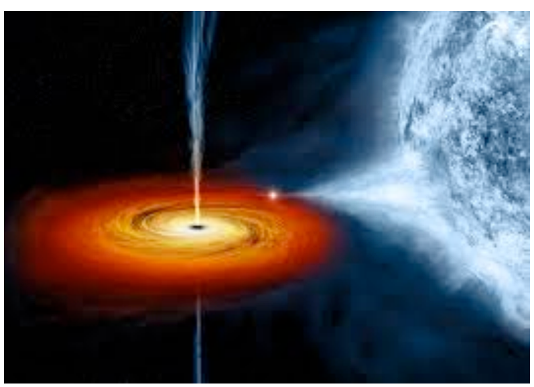




Isaac Newton
(1642–1727)

Physics 2113
Lecture 09: MON 15 SEP

CH23: Gauss' Law



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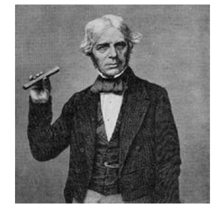
23-6 A Charged Isolated Conductor 612

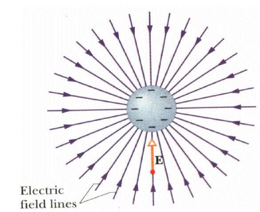
Michael Faraday
(1791–1867)

Carl Friedrich Gauss 1777-1855

Developed a mathematical theorem that was put into its simplest form, in terms of pictures, by Michael Faraday.

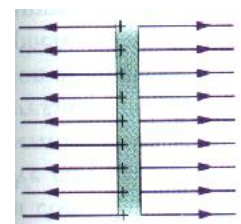


The basic idea can be inferred from observations we have actually already made:



Field of a point charge decays as $1/r^2$

Field of an infinite line of charge decays as $1/r$

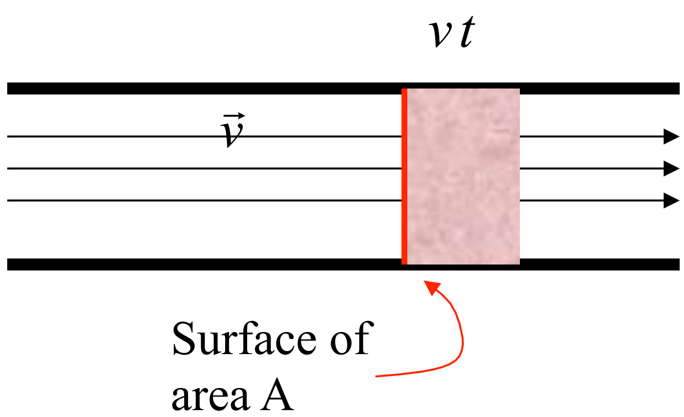


Field of a charged infinite plane is constant.

Intuitively: field lines behave like a fluid flow, that is only disrupted by the presence of charges. Otherwise, the "amount of flow" has to be conserved.

Electric flux:

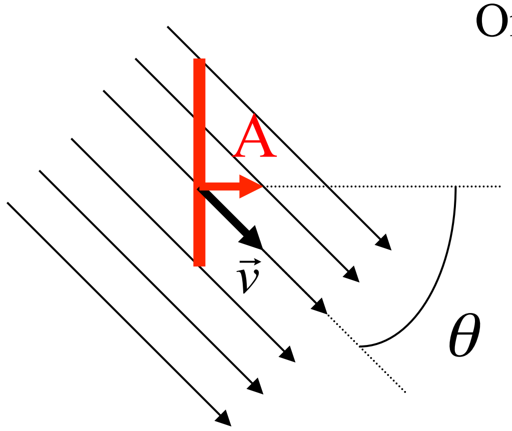
Consider again water flowing through a pipe



Volume of water flowing through the surface in time t: $v\,t\,A$

Volume of water flowing per unit time: $v\,A$

If velocity is not perpendicular to surface,



Only the perpendicular component contributes

$$\text{flux} = vA\cos\theta = \vec{v}\cdot\vec{A}$$

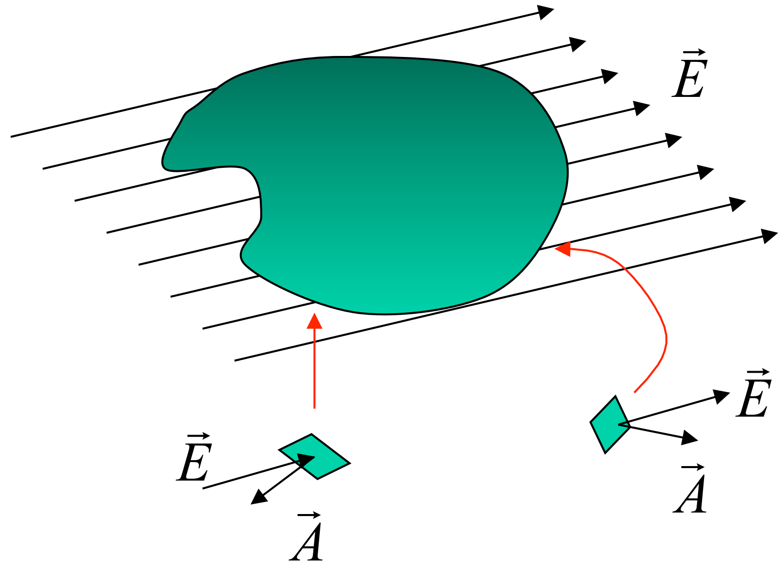
Where we have introduced a vector $\vec{A}$ of magnitude equal to the area of the surface and direction perpendicular to it

This mathematical construction can be applied to any vector. The resulting quantity is called the

flux of the vector.

Electric flux: $\Phi = \vec{E} \cdot \vec{A}$ Units : N m/C^2

What if the surface is not a plane?



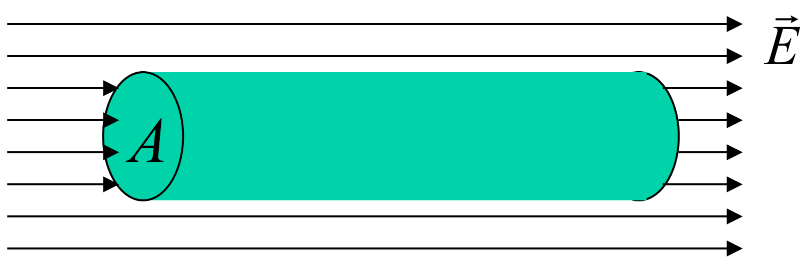
Break it up into planar little pieces, sum.

If you make the pieces infinitesimally small, you end up with the flux integral:

$$\Phi = \int_S \vec{E} \cdot d\vec{A}$$

This is usually a complicated surface integral.

Example: cylinder

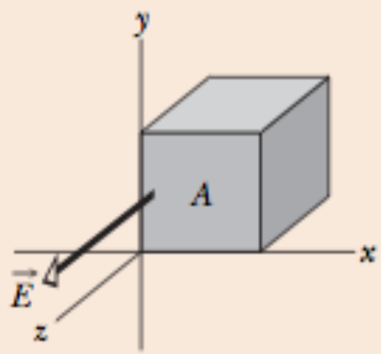


$$\Phi = \int_{\text{cover 1}} + \int_{\text{side}} + \int_{\text{cover 2}}$$

$$\Phi = -EA \quad +0 \quad +EA = 0$$

CHECKPOINT 1

The figure here shows a Gaussian cube of face area A immersed in a uniform electric field $\vec{E}$ that has the positive direction of the z axis. In terms of E and A, what is the flux through (a) the front face (which is in the xy plane), (b) the rear face, (c) the top face, and (d) the whole cube?



Sample Problem

Flux through a closed cube, nonuniform field

A *nonuniform* electric field given by $\vec{E} = 3.0x\hat{i} + 4.0\hat{j}$ pierces the Gaussian cube shown in Fig. 23-5*a*. (E is in newtons per coulomb and x is in meters.) What is the electric flux through the right face, the left face, and the top face? (We consider the other faces in another sample problem.)

KEY IDEA

We can find the flux Φ through the surface by integrating the scalar product $\vec{E} \cdot d\vec{A}$ over each face.

Right face: An area vector $\vec{A}$ is always perpendicular to its surface and always points away from the interior of a Gaussian surface. Thus, the vector $d\vec{A}$ for any area element (small section) on the right face of the cube must point in the positive direction of the x axis. An example of such an element is shown in Figs. 23-5*b* and *c*, but we would have an identical vector for any other choice of an area element on that face. The most convenient way to express the vector is in unit-vector notation,

$$d\vec{A} = dA\hat{i}.$$

From Eq. 23-4, the flux Φ_r through the right face is then

$$\begin{aligned}\Phi_r &= \int \vec{E} \cdot d\vec{A} = \int (3.0x\hat{i} + 4.0\hat{j}) \cdot (dA\hat{i}) \\ &= \int [(3.0x)(dA)\hat{i} \cdot \hat{i} + (4.0)(dA)\hat{j} \cdot \hat{i}] \\ &= \int (3.0x\, dA + 0) = 3.0 \int x\, dA.\end{aligned}$$

We are about to integrate over the right face, but we note that x has the same value everywhere on that face—namely, $x = 3.0$ m. This means we can substitute that constant value for x. This can be a confusing argument. Although x is certainly a variable as we move left to right across the figure, because the right face is perpendicular to the x axis, every point on the face has the same x coordinate. (The y and z coordinates do not matter in our integral.) Thus, we have

$$\Phi_r = 3.0 \int (3.0)\, dA = 9.0 \int dA.$$

The integral $\int dA$ merely gives us the area $A = 4.0\ \text{m}^2$ of the right face; so

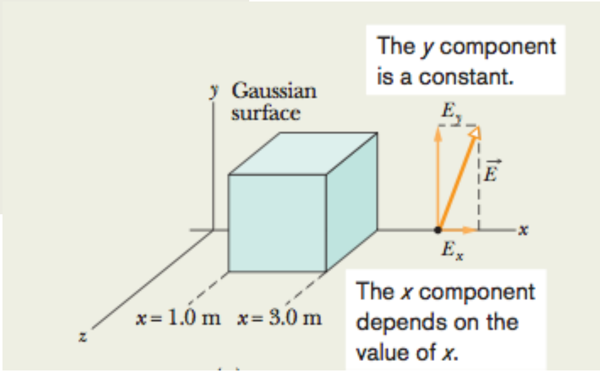
$$\Phi_r = (9.0\ \text{N/C})(4.0\ \text{m}^2) = 36\ \text{N} \cdot \text{m}^2/\text{C}. \qquad \text{(Answer)}$$

***Left face*:** The procedure for finding the flux through the left face is the same as that for the right face. However, two factors change. (1) The differential area vector $d\vec{A}$ points in the negative direction of the x axis, and thus $d\vec{A} = -dA\hat{i}$ (Fig. 23-5*d*). (2) The term x again appears in our integration, and it is again constant over the face being considered. However, on the left face, $x = 1.0$ m. With these two changes, we find that the flux Φ_l through the left face is

$$\Phi_l = -12\ \text{N} \cdot \text{m}^2/\text{C}. \qquad \text{(Answer)}$$

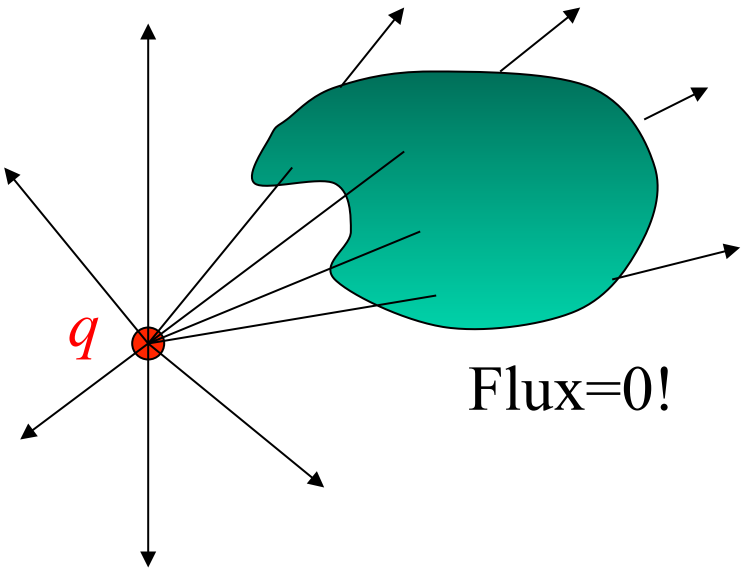
Top face: The differential area vector $d\vec{A}$ points in the positive direction of the y axis, and thus $d\vec{A} = dA\hat{j}$ (Fig. 23-5*e*). The flux Φ_t through the top face is then

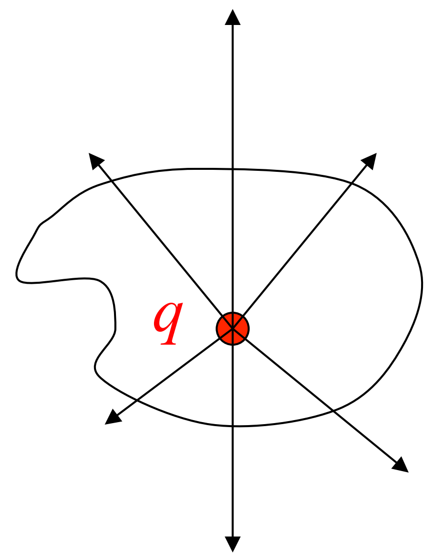
$$\begin{aligned}\Phi_t &= \int (3.0x\hat{i} + 4.0\hat{j}) \cdot (dA\hat{j}) \\ &= \int [(3.0x)(dA)\hat{i} \cdot \hat{j} + (4.0)(dA)\hat{j} \cdot \hat{j}] \\ &= \int (0 + 4.0\, dA) = 4.0 \int dA \\ &= 16\ \text{N} \cdot \text{m}^2/\text{C}.\end{aligned}$$



Gauss' law:

Given an arbitrary closed surface, the electric flux through it is proportional to the charge enclosed by the surface.





$$\boxed{\Phi \equiv \oint_{\text{Surface}} \vec{E} \cdot d\vec{A} = \frac{q}{\varepsilon_0}}$$

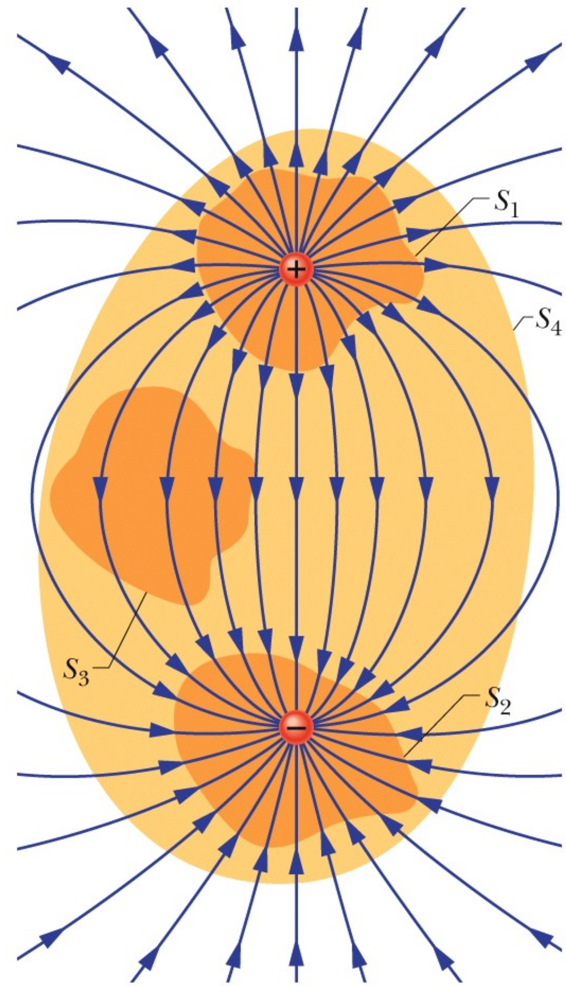
Two charges, equal in magnitude but opposite in sign, and the field lines that represent their net electric field. Four Gaussian surfaces are shown in cross section.

Surface S1.The electric field is outward for all points on this surface. Thus, the flux of the electric field through this surface is positive, and so is the net charge within the surface, as Gauss' law requires

Surface S2.The electric field is inward for all points on this surface. Thus, the flux of the electric field through this surface is negative and so is the enclosed charge, as Gauss' law requires.

Surface S3.This surface encloses no charge, and thus $q_{enc} = 0$. Gauss' law requires that the net flux of the electric field through this surface be zero. That is reasonable because all the field lines pass entirely through the surface, entering it at the top and leaving at the bottom.

Surface S4.This surface encloses no net charge, because the enclosed positive and negative charges have equal magnitudes. Gauss' law requires that the net flux of the electric field through this surface be zero. That is reasonable because there are as many field lines leaving surface S4 as entering it.



Who cares about the flux?

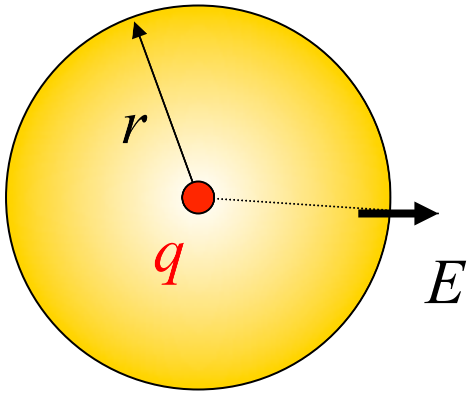
In general, it is true that knowing the flux is not of much use.

However, in situations of high symmetry, it is easy to compute the electric field knowing the flux.

In situations of high symmetry the integral involved in defining the flux in terms of the electric field can be immediate to compute.

In symmetric cases knowing the flux is tantamount to knowing the field.

Example: spherical symmetry



One can pick any Gaussian surface one likes, but in this case it would be foolish to pick anything but a sphere.

The electric field is everywhere perpendicular to the surface (it is radial). Moreover, on the sphere |E| is constant. Therefore the flux is simply given by |E| times the area of the sphere:

$$\Phi = |E|\,A = |E|\,4\pi r^2$$

But according to Gauss' law: $\Phi = \dfrac{q}{\varepsilon_0}$

$$\boxed{|E| = \frac{q}{4\pi\varepsilon_0 r^2}}$$

Coulomb's law!

By the way, this proves that the field outside a uniformly charged sphere is independent of the radius of a sphere. In particular, we can replace it with a point charge.

Example:

6E. In Fig. 24-29, the charge on a neutral isolated conductor is separated by a nearby positively charged rod. What is the net flux through each of the five Gaussian surfaces shown in cross section? Assume that the charges enclosed by S_1, S_2, and S_3 are equal in magnitude.

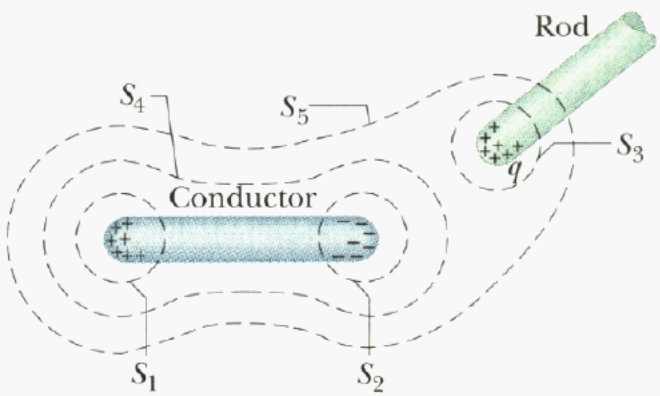


FIGURE 24-29 Exercise 6.

Charge in S_1 equals q.

Flux in S_1 (using Gauss' law)

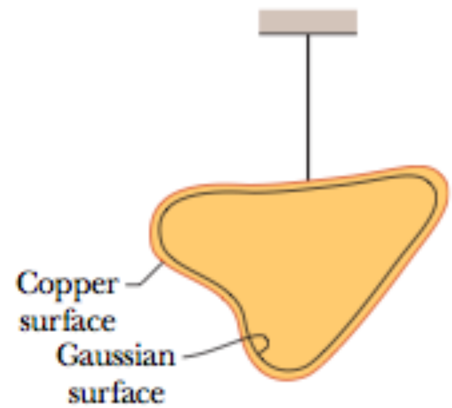
$$\Phi_1 = q/\varepsilon_0 = -\Phi_2 = \Phi_3$$

The conductor was, and still is, uncharged. Therefore $\Phi_4 = 0$

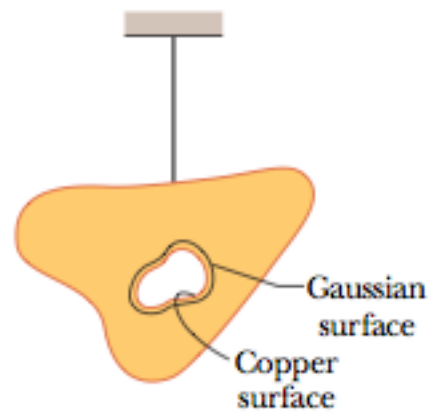
And as a consequence,

$$\Phi_5 = q/\varepsilon_0$$

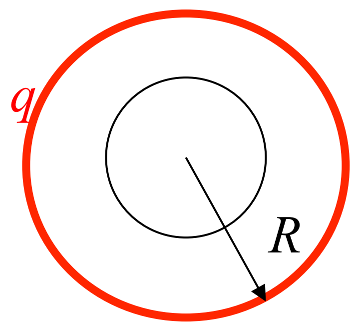
A charged conductor:



A conductor with a cavity:



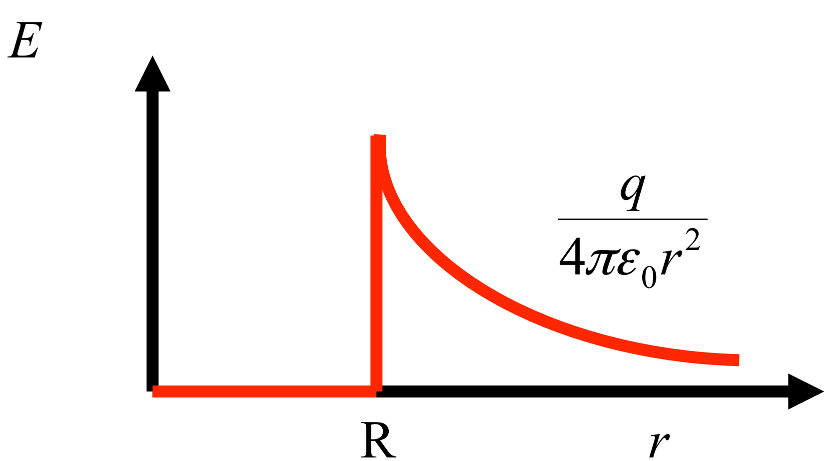
Example: Charged spherical shell, compute field as a function of r.



By symmetry all fields are radial, and it is wise to take spherical Gaussian surfaces. All fluxes aretherefore given by field times area (important!).

Pick Gaussian surface inside the shell. It encloses no charge. Flux is zero. Field is zero.

Pick spherical Gaussian surface outside the shell. The enclosed charge is q. Field is radial. Therefore the field is given by that of a point charge q at the center.



Summary:

- Gauss' law provides a very direct way to compute the electric flux.
- In situations with symmetry, knowing the flux allows to compute the fields reasonably easily.