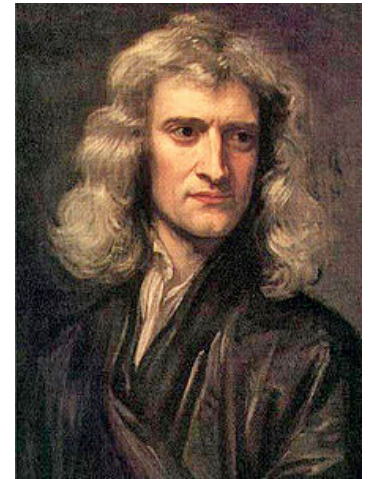
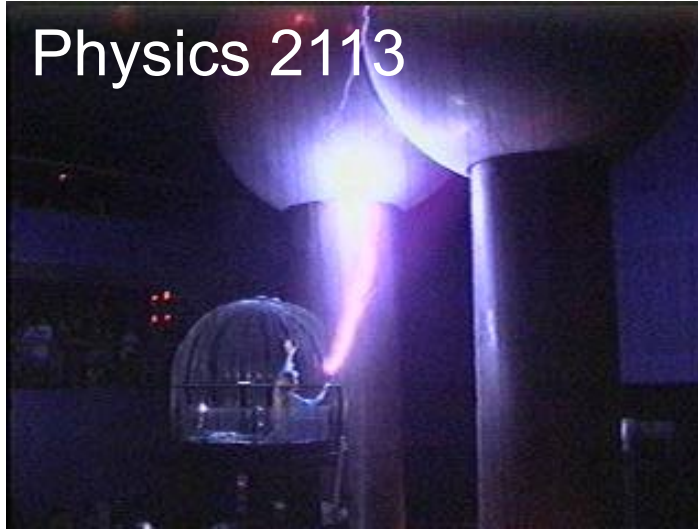


Physics 2113



Isaac Newton
(1642–1727)

Physics 2113

Lecture 02: WED 27 AUG

CH13: Gravitation II



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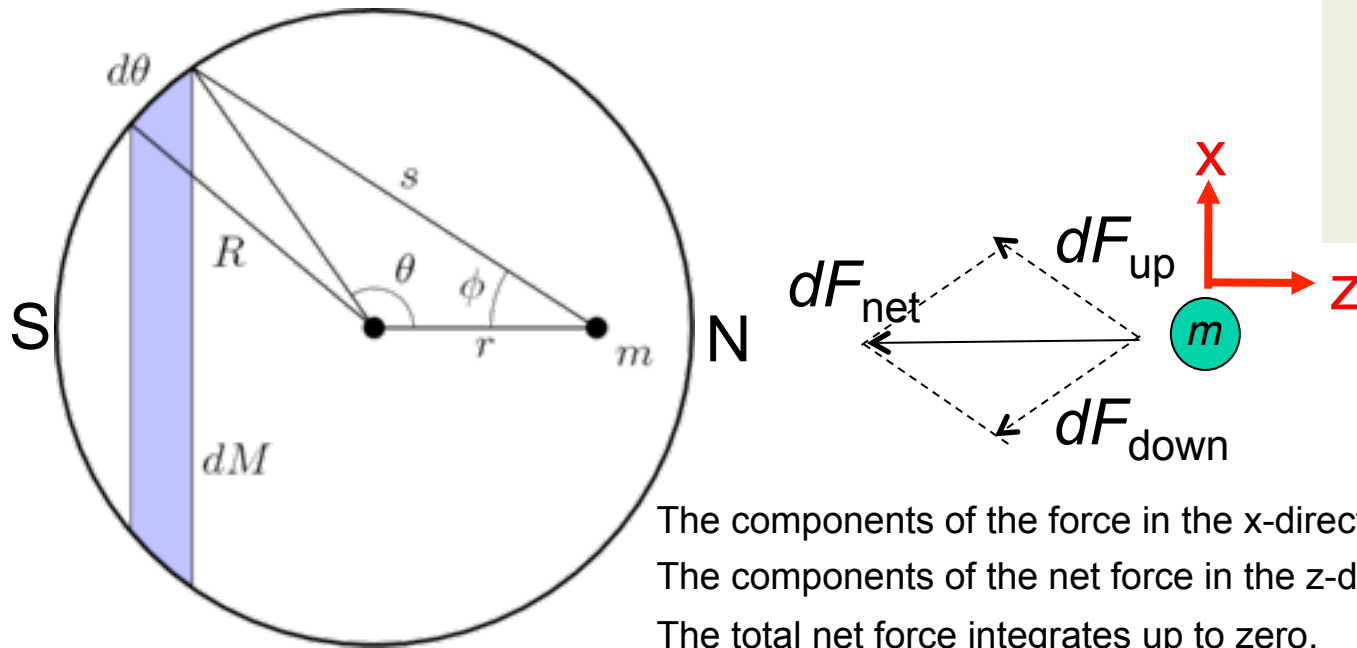


Michael Faraday
(1791–1867)

13.4: Gravitation Inside Earth: Shell Game II

http://en.wikipedia.org/wiki/Shell_theorem#Inside_a_shell

A uniform shell of matter exerts no net gravitational force on a particle located inside it.

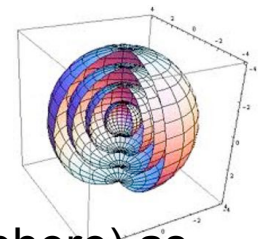
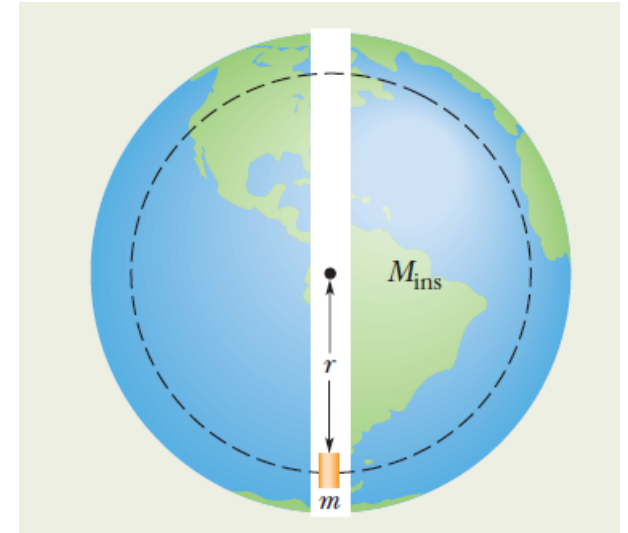


Force From GREATER Shell
Mass FARTHER From m
Exactly Cancels the Force from
LESSER Shell Mass CLOSER
to m due to inverse-square law!

Proof same as for $r > R$ but with different limits of integration!

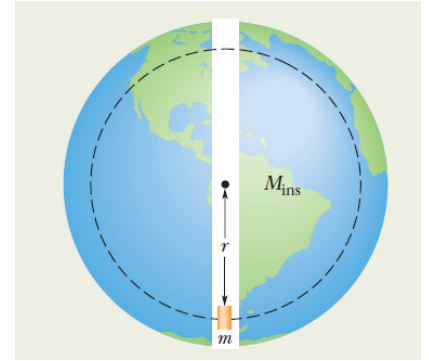
$$F_{\text{net}} = \frac{GMm}{4r^2R} \int_{R-r}^{R+r} \left(1 + \frac{r^2 - R^2}{s^2} \right) ds = 0$$

Applying the shell law to concentric shells proves can treat Earth (uniform sphere) as if all mass are in shells $\leq r$ are at center and NO shells with radius $> r$ contribute any force at all!

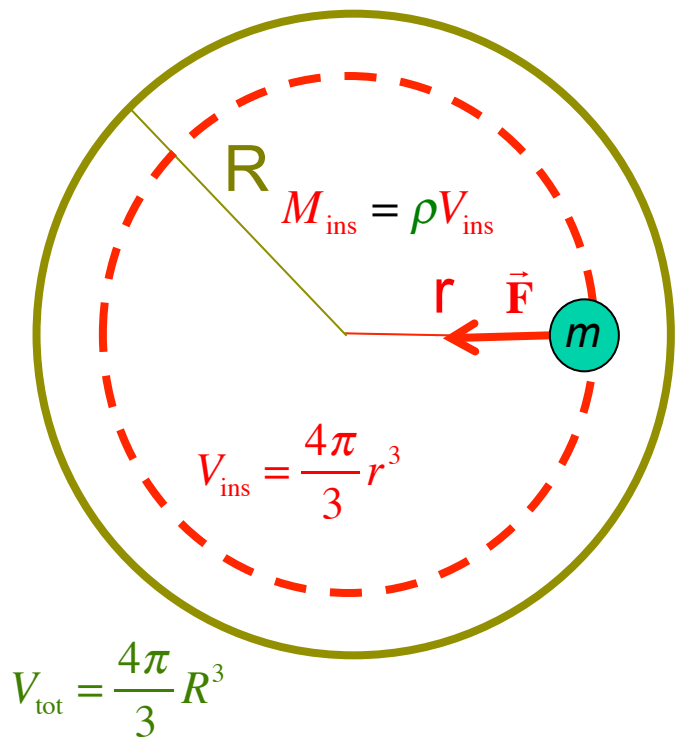


13.4: Gravitation Inside Earth: Shell Game II

1. A uniform shell of matter exerts no net gravitational force on a particle located inside it.
2. A uniform shell of matter exerts a force on a particle located outside it as if all the mass was at the center.



ICPP:
What is Net Force On m
at Center of Earth?



$$\text{density} = \rho = \frac{M_{\text{tot}}}{V_{\text{tot}}}$$

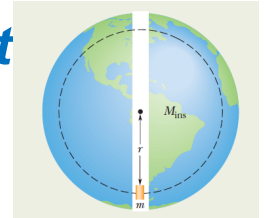
$$M_{\text{ins}} = \rho V_{\text{ins}} = \frac{M}{V_{\text{tot}}} V_{\text{ins}} = M \frac{r^3}{R^3}$$

$$\text{force} = F = \frac{GmM_{\text{ins}}}{r^2} = \frac{Gm}{r^2} \left(M \frac{r^3}{R^3} \right) = \frac{GmM}{R^3} r$$

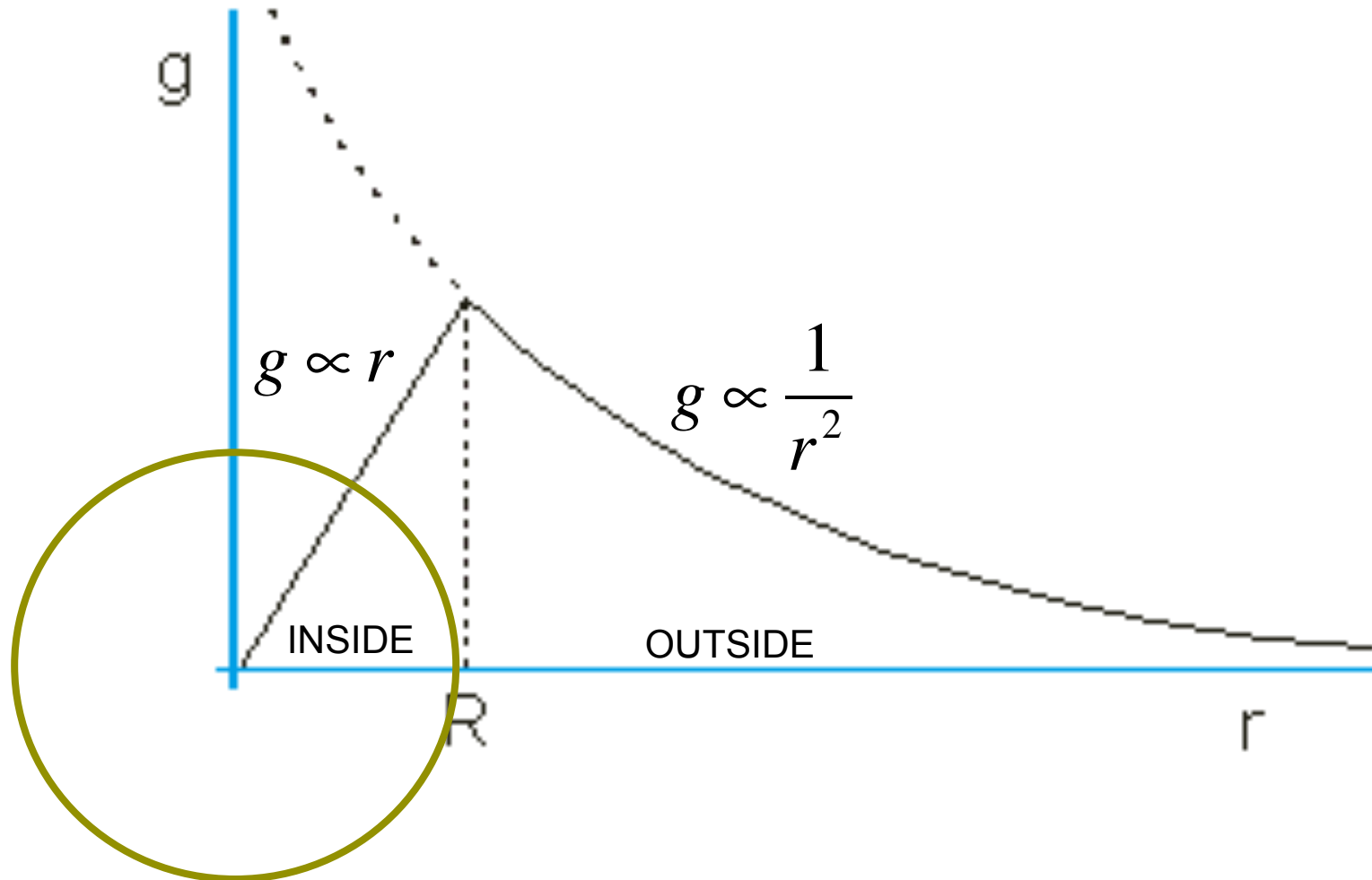
$$\text{field} = g = \frac{GM_{\text{ins}}}{r^2} = \frac{G}{r^2} \left(M \frac{r^3}{R^3} \right) = \frac{GM}{R^3} r$$

Inside the Earth the Force and Field Scale LINEARLY with r .
This is like Hooke's Law for a Mass on a Spring.

13.4: Gravitation Inside Earth: Summary Moving From Center Out

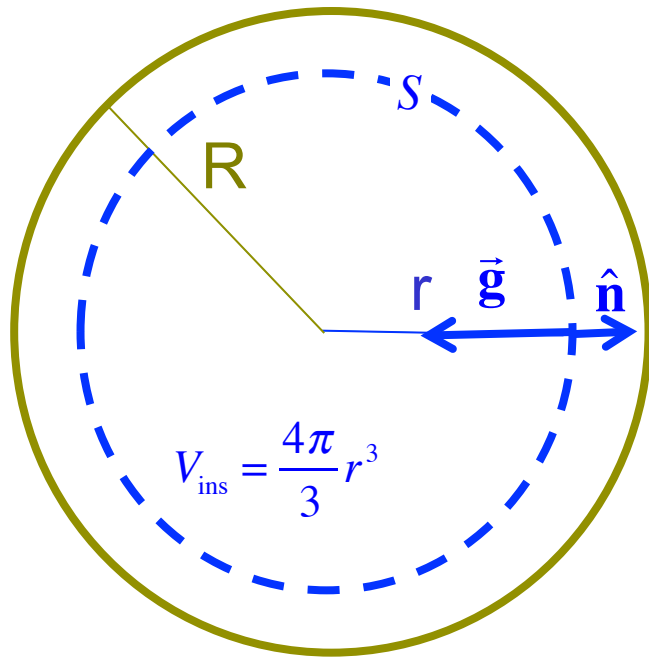
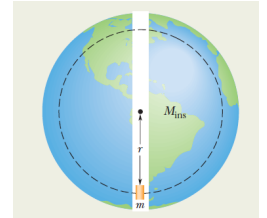


1. **INSIDE** a uniform sphere field/force **INCREASES** like r
2. **OUTSIDE** a uniform sphere field/force **DECREASES** like $1/r^2$



13.4: Gravitation Inside Earth: Gauss's Law for Gravity

http://en.wikipedia.org/wiki/Shell_theorem#Derivation_using_Gauss.27s_law



Case I: $r > R$

$$M_{\text{ins}} = M$$

$$g(r)4\pi r^2 = -4\pi GM$$

$$g(r) = -\frac{GM}{r^2} \quad \checkmark$$

$$\int_S \vec{g} \cdot d\vec{S} = -4\pi GM_{\text{ins}} \quad \text{Gauss's Law}$$

$$\vec{g} = g(r)\hat{n}$$

Surface Integral

$$\int_S \vec{g} \cdot d\vec{S} = \int_S \vec{g} \cdot \hat{n} dS = g(r) \int_S dS = g(r)4\pi r^2$$

Case II: $r < R$

$$M_{\text{ins}} = M \frac{r^3}{R^3}$$

$$g(r)4\pi r^2 = -4\pi GM \frac{r^3}{R^3}$$

$$g(r) = -\frac{GM}{R^3} r \quad \checkmark$$

13.4: Gravitation Inside Earth

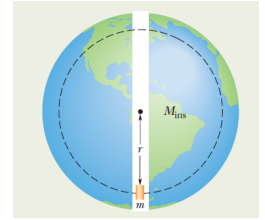
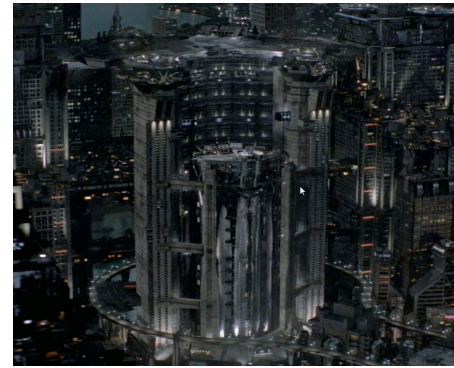
A uniform shell of matter exerts no net gravitational force on a particle located inside it.

Sample Problem

In the 2012 remake of the film *Total Recall*, Colin Farrell rides a train that falls through the center of the Earth.

In the film Farrell experiences normal gravity until he hits the core, then experiences a moment of weightlessness at the core, and then resumes normal gravity (in the opposite direction) as the train continues to the other side of the Earth.

Decide if this is what really would happen (or if it is complete Hollywood BS) by finding the gravitational force on the capsule of mass m when it reaches a distance r from Earth's center. Assume that Earth is a sphere of uniform density ρ (mass per unit volume).



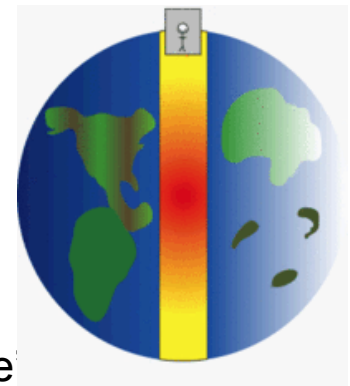
BS or not BS?

Calculations:

$$F = \frac{GmM_{\text{ins}}}{r^2}.$$

$$M_{\text{ins}} = \rho V_{\text{ins}} = \rho \frac{4\pi r^3}{3}$$

$$F = \frac{4\pi Gm\rho}{3} r.$$

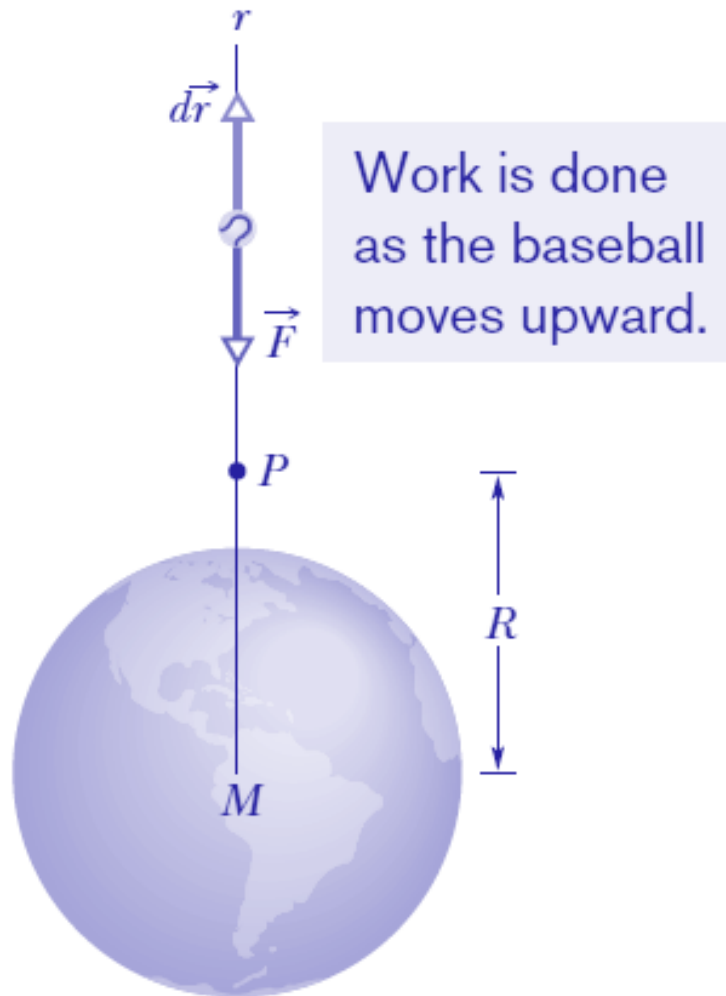


Hooke
Like mass on spring!

The force magnitude depends linearly on the capsule's distance r from Earth's center. Thus, as r decreases, F also decreases, until it is zero at Earth's center.

However the train and occupants are both in free fall would be weightless the entire time! Complete Hollywood BS!

13.6: Gravitational Potential Energy



Let us shoot a baseball directly away from Earth along the path in the figure. We want to find the gravitational potential energy U of the ball at point P along its path, at radial distance R from Earth's center.

The work W done on the ball by the gravitational force as the ball travels from point P to a great (infinite) distance from Earth is:

$$W = \int_R^{\infty} \vec{F}(r) \cdot d\vec{r}.$$

$$\vec{F}(r) \cdot d\vec{r} = F(r) dr \cos \phi = -\frac{GMm}{r^2} dr,$$

$$W = -GMm \int_R^{\infty} \frac{1}{r^2} dr = \left[\frac{GMm}{r} \right]_R^{\infty}$$

$$= 0 - \frac{GMm}{R} = -\frac{GMm}{R},$$

where W is the work required to move the ball from point P (at distance R) to infinity. Work can also be expressed in terms of potential energies as

$$U_{\infty} - U = -W. \quad \Rightarrow \quad U = W = -\frac{GMm}{R}.$$

13.6: Gravitational Potential Energy

The gravitational potential energy of the two-particle *system* is:

$$U = -\frac{GMm}{r}$$

$U(r)$ approaches zero as r approaches infinity and that for any finite value of r , the value of $U(r)$ is negative.

If the system contains more than two particles, consider each pair of particles in turn, calculate the gravitational potential energy of that pair with the above relation, as if the other particles were not there, and then algebraically sum the results. That is,

$$U = -\left(\frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}}\right).$$

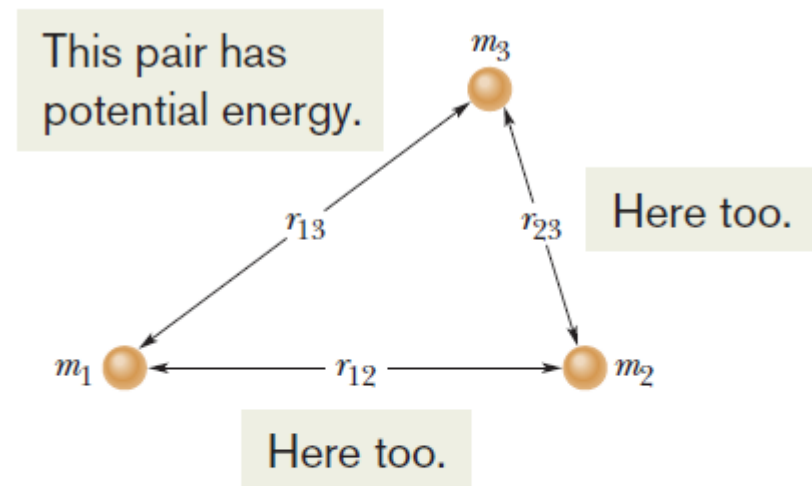


Fig. 13-8 A system consisting of three particles. The gravitational potential energy *of the system* is the sum of the gravitational potential energies of all three pairs of particles.

Gravitational **Potential Energy** U vs. Gravitational **Potential** V

**Gravitational
Potential Energy:**
(Units: Joules = J)

$$U = -\frac{GmM}{r}$$

**Gravitational
Potential:**
(Units: J/kg)

$$V = -\frac{GM}{r}$$

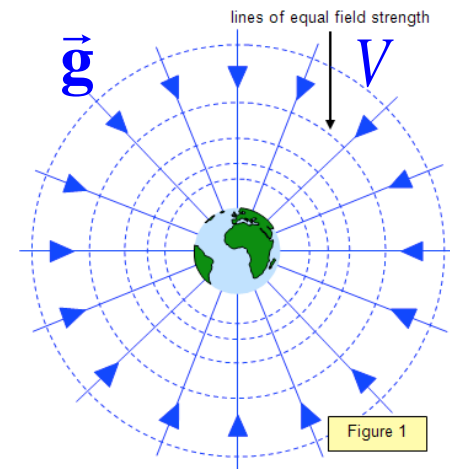
**Given the **Potential
Energy**, Find the
Potential:**

$$U = mV$$

**Given the
Potential,
Find the **Field****

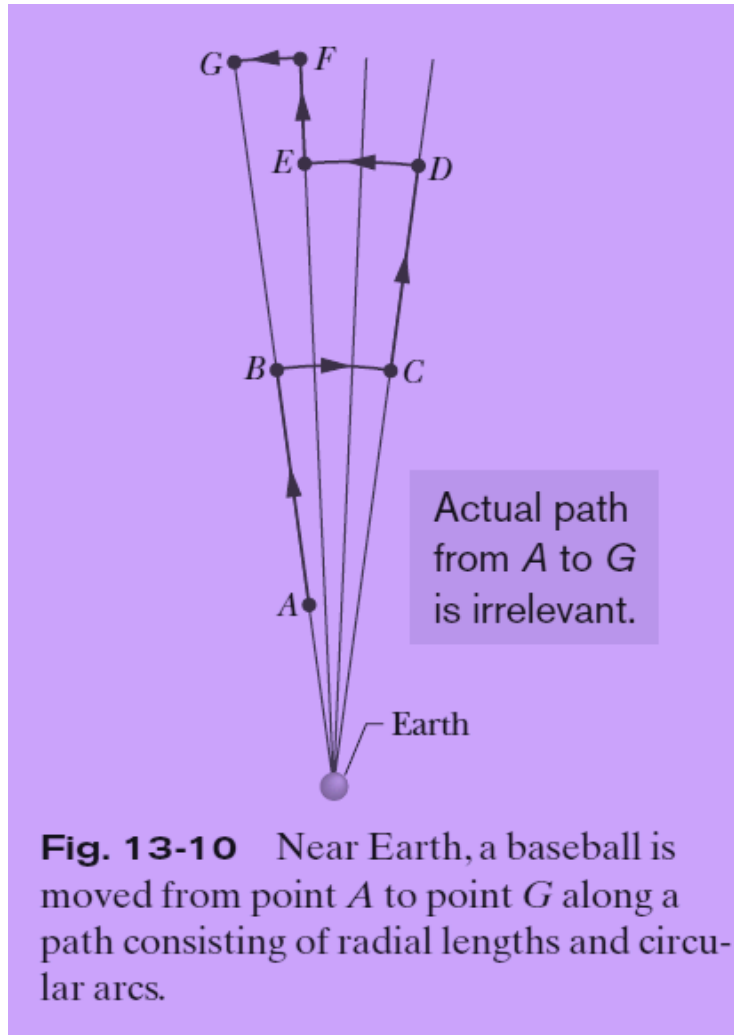
$$g = -\frac{dV}{dr}$$

Equal Potential Lines Perpendicular to Field Lines



Note: Potential
Exists in Empty
Space Whether Test
Mass m is There or
Not!

13.6: Gravitational Potential Energy Path Independence



The work done along each circular arc is zero, because the direction of \mathbf{F} is perpendicular to the arc at every point. Thus, W is the sum of only the works done by \mathbf{F} along the three radial lengths.

The gravitational force is a conservative force. Thus, the work done by the gravitational force on a particle moving from an initial point i to a final point f is independent of the path taken between the points. The change ΔU in the gravitational potential energy from point i to point f is given by

$$\Delta U = U_f - U_i = -W.$$

Since the work W done by a conservative force is independent of the actual path taken, the change ΔU in gravitational potential energy is **also independent** of the path taken.

13.6: Gravitational Potential Energy: Potential Energy and Force

$$(F(x) = -dU(x)/dx)$$



$$F = -\frac{dU}{dr} = -\frac{d}{dr} \left(-\frac{GMm}{r} \right)$$



$$= -\frac{GMm}{r^2}.$$

The minus sign indicates that the force on mass m points radially inward, toward mass M .

13.6: Gravitational Potential Energy: Potential and Field

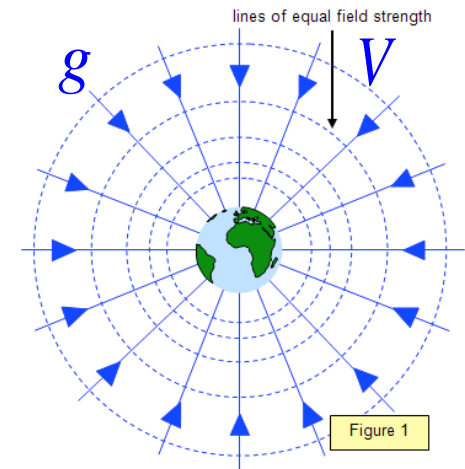
$$g = -\frac{dV}{dr}$$



$$V = -\frac{GM}{r}$$



$$g = -\frac{dV}{dr} = -\frac{d}{dr}\left(-\frac{GM}{r}\right) = -\frac{GM}{r^2}$$



The minus sign indicates that the field points radially inward, toward mass M.

$$\vec{g} = -\vec{\nabla} V \quad (\text{vector form})$$

13.6: Gravitational Potential Energy: Escape Speed

If you fire a projectile upward, there is a certain minimum initial speed that will cause it to move upward forever, theoretically coming to rest only at infinity.

This minimum initial speed is called the (Earth) **escape speed**.

Consider a projectile of mass m , leaving the surface of a planet (mass M , radius R) with escape speed v . The projectile has a kinetic energy K given by $\frac{1}{2}mv^2$, and a potential energy U given by:

$$U = -\frac{GMm}{R}$$

When the projectile reaches infinity, it stops and thus has no kinetic energy. It also has no potential energy because an infinite separation between two bodies is our zero-potential-energy configuration. Its total energy at infinity is therefore zero. From the principle of conservation of energy, its total energy at the planet's surface must also have been zero, and so

$$K + U = \frac{1}{2}mv^2 + \left(-\frac{GMm}{R}\right) = 0.$$

This gives the escape speed

$$v = \sqrt{\frac{2GM}{R}}.$$

13.6: Gravitational Potential Energy: Escape Speed

$$v = \sqrt{\frac{2GM}{R}}$$

Table 13-2

Some Escape Speeds

Body	Mass (kg)	Radius (m)	Escape Speed (km/s)
Ceres ^a	1.17×10^{21}	3.8×10^5	0.64
Earth's moon ^a	7.36×10^{22}	1.74×10^6	2.38
Earth	5.98×10^{24}	6.37×10^6	11.2
Jupiter	1.90×10^{27}	7.15×10^7	59.5
Sun	1.99×10^{30}	6.96×10^8	618
Sirius B ^b	2×10^{30}	1×10^7	5200
Neutron star ^c	2×10^{30}	1×10^4	2×10^5

^aThe most massive of the asteroids.

^bA *white dwarf* (a star in a final stage of evolution) that is a companion of the bright star Sirius.

^cThe collapsed core of a star that remains after that star has exploded in a *supernova* event.

ICPP: How Would You Find the Ratio M/R for a Black Hole Where $v \geq c$, the speed of light?

$$v = \sqrt{\frac{2GM}{R}} \geq c \quad (\text{Black Hole})$$

$$\frac{M}{R} \geq \frac{c^2}{2G} = 6.70 \times 10^{26} \frac{\text{kg}}{\text{m}}$$

$$M \cong 8.2 \times 10^{36} \text{ kg} = 4.1 \text{ Million Solar Masses} \quad (\text{Mass of Object at Center of Galaxy})$$

$$R \cong 1.2 \times 10^{10} \text{ m} \quad (\text{Radius of Object at Center of Galaxy})$$

$$\frac{M}{R} = 6.83 \times 10^{26} \quad (\text{Center of Galaxy is Super-Massive Black Hole!})$$

Example:

An asteroid, headed directly toward Earth, has a speed of 12 km/s relative to the planet when the asteroid is 10 Earth radii from Earth's center. Neglecting the effects of Earth's atmosphere on the asteroid, find the asteroid's speed v_f when it reaches Earth's surface.

KEY IDEAS

Because we are to neglect the effects of the atmosphere on the asteroid, the mechanical energy of the asteroid–Earth system is conserved during the fall. Thus, the final mechanical energy (when the asteroid reaches Earth's surface) is equal to the initial mechanical energy. With kinetic energy K and gravitational potential energy U , we can write this as

$$K_f + U_f = K_i + U_i. \quad (13-29)$$

Also, if we assume the system is isolated, the system's linear momentum must be conserved during the fall. Therefore, the momentum change of the asteroid and that of Earth must be equal in magnitude and opposite in sign. However, because Earth's mass is so much greater than the asteroid's mass, the change in Earth's speed is negligible relative to the change in the asteroid's speed. So, the change in Earth's kinetic energy is also negligible. Thus, we can assume that the kinetic energies in Eq. 13-29 are those of the asteroid alone.

Calculations: Let m represent the asteroid's mass and M represent Earth's mass (5.98×10^{24} kg). The asteroid is initially at distance $10R_E$ and finally at distance R_E , where R_E is

Earth's radius (6.37×10^6 m). Substituting Eq. 13-21 for U and $\frac{1}{2}mv^2$ for K , we rewrite Eq. 13-29 as

$$\frac{1}{2}mv_f^2 - \frac{GMm}{R_E} = \frac{1}{2}mv_i^2 - \frac{GMm}{10R_E}.$$

Rearranging and substituting known values, we find

$$\begin{aligned} v_f^2 &= v_i^2 + \frac{2GM}{R_E} \left(1 - \frac{1}{10} \right) \\ &= (12 \times 10^3 \text{ m/s})^2 \\ &\quad + \frac{2(6.67 \times 10^{-11} \text{ m}^3/\text{kg} \cdot \text{s}^2)(5.98 \times 10^{24} \text{ kg})}{6.37 \times 10^6 \text{ m}} 0.9 \\ &= 2.567 \times 10^8 \text{ m}^2/\text{s}^2, \end{aligned}$$

and

$$v_f = 1.60 \times 10^4 \text{ m/s} = 16 \text{ km/s}. \quad (\text{Answer})$$

At this speed, the asteroid would not have to be particularly large to do considerable damage at impact. If it were only 5 m across, the impact could release about as much energy as the nuclear explosion at Hiroshima. Alarmingly, about 500 million asteroids of this size are near Earth's orbit, and in 1994 one of them apparently penetrated Earth's atmosphere and exploded 20 km above the South Pacific (setting off nuclear-explosion warnings on six military satellites). The impact of an asteroid 500 m across (there may be a million of them near Earth's orbit) could end modern civilization and almost eliminate humans worldwide.

Summary

- A uniform shell exerts no gravitational force inside it.
- For a uniform ball, gravitational field grows inside of it and decreases outside of it.
- Gauss' law: easy way to see all this.