The Structure, Stability, and Dynamics of Self-Gravitating Systems

Origin of Virial Equations

The 2nd-order tensor virial equation (as opposed to the 1st-order virial equation[1]) is derived by taking the "first moment" of the Euler equation. Here we begin the derivation by referring specifically to the

Standard Lagrangian Representation
of Euler's Equation,

$$\mathbf{D}\mathbf{v} = -(1/\rho)\nabla P - \nabla\Phi .$$

[Equation I.A.1]

Multiplying this equation [I.A.1] through by the mass density ρ produces the relation (shown earlier in the context of our discussion of the numerous different forms of Euler's equation),

$$\rho\mathbf{D}\mathbf{v} = -\nabla P - \rho\nabla\Phi .$$

[Equation I.A.3]

In terms of its individual Cartesian vector components, this equation [I.A.3] takes the form,

$$\rho\mathbf{D}v_j = -\nabla_j P - \rho\nabla_j\Phi ,$$

[Equation I.Z.1]
=
EFE, Chapter 2, §11, Eq. (38)[2]

where ∇_j refers to the j^{th} component of the gradient operator. The first moment of Euler's equation is then obtained by multiplying equation [I.Z.1] through by each of the components x_i of the Cartesian position vector $\mathbf{x}$, then integrating the resulting equations over the entire volume of the physical system under consideration. Performing these steps initially produces a set of nine integral equations (one each for i = 1,2,3 and j = 1,2,3), each containing the following three terms:

$$\underbrace{\int_V [\, x_i\rho\mathbf{D}v_j \,]\, d^3x}_{\text{I}} = \underbrace{-\int_V [\, x_i \nabla_j P \,]\, d^3x}_{\text{II}} \underbrace{-\int_V [\, x_i\rho\nabla_j\Phi \,]\, d^3x}_{\text{III}} .$$

[Equation I.Z.2]

Before we attempt to manipulate each one of the identified terms in this equation into a more recognizable form, we must remind the reader of **Gauss's Theorem** which, in its most familiar form is also known as the

Divergence Theorem:

The integral of the divergence of any vector ***F*** *over a volume V is equal to the surface integral of the normal component of the vector over the surface S that is bounding V.* That is,

$$\int_V \nabla\cdot \mathbf{F}\, d^3x = \int_S \mathbf{F}\cdot\mathbf{n}\, da .$$

[Equation I.Z.3a]
=
Arfken, §1.11, Eq. (1.94a)

We shall also need to utilize a similar relation involving the volume integral over the gradient of any scalar function G(**x**), which we will refer to as,

Form B
of
Gauss's Theorem

$$\int_V \nabla G\, d^3x = \int_S G\, \mathbf{n}\, da .$$

[Equation I.Z.3b]
=
Arfken, §1.11, Eq. (1.102)

Furthermore, it will be useful to notice that, for any fluid variable "q",

$$\begin{aligned}\rho\mathbf{D}q &= \mathbf{D}(\rho q) - q\,\mathbf{D}\rho \\ &= \partial_t(\rho q) + \mathbf{v}\cdot\nabla(\rho q) + (\rho q)\,\nabla\cdot\mathbf{v} \\ &= \partial_t(\rho q) + \nabla\cdot[\,(\rho q)\mathbf{v}\,]\,,\end{aligned}$$

where in order to derive the second line we have utilized both expression [VI.M.14], which relates the Lagrangian time-derivative to the Eulerian time-derivative, and the standard Lagrangian representation of the continuity equation [I.B.1]. Hence, via the Divergence Theorem,

$$\begin{aligned}\int_V (\rho\mathbf{D}q)\, d^3x &= \int_V \partial_t(\rho q)\, d^3x + \int_S (\rho q\,\mathbf{v})\cdot\mathbf{n}\, da \\ &= \partial_t \int_V (\rho q)\, d^3x + \int_S (\rho q\,\mathbf{v})\cdot\mathbf{n}\, da .\end{aligned}$$

[Equation I.Z.4]

This last step -- in which we have moved the partial time-derivative outside of the volume integration -- is justified as long as the

$$\int_{V_L} Q\, d^3x\ ,$$

is the total mass enclosed by the Lagrangian volume V_L and its time-derivative,[3]

$$D \int_{V_L} Q\, d^3x\ ,$$

should be zero. Well, via a very careful proof (which we will not reproduce here) of the

Reynolds Theorem
which states:

$$D \int_{V_L} Q\, d^3x = \partial_t \int_V Q\, d^3x + \int_S (Q\, \mathbf{v})\cdot\mathbf{n}\, da\ ,$$

[Equation I.Z.5]
=
Tassoul '78, §3.2, Eq. (17)

Tassoul (1978; see specifically §3.2, pp. 46-47), has demonstrated that the right-hand-side of equation [I.Z.4] is precisely the Lagrangian time-derivative[3] of the global variable defined by the volume integral,

$$\int_{V_L} (\rho q)\, d^3x\ .$$

So, by combining equations [I.Z.4] and [I.Z.5] we recognize the following relationship, which has been presented by Chandrasekhar (1987) in the form of a

Lemma:

If $q(\boldsymbol{x}\ ,t)$ is any attribute of a fluid element, then

$$D \int_{V_L} (\rho q)\, d^3x = \int_V (\rho \mathbf{D} q)\, d^3x\ .$$

[Equation I.Z.6]
=
EFE, Chapter 2, §11, Eq. (40)
Tassoul '78, §3.3, Eq. (23)[3]

We will find this Lemma and Gauss's Theorem to be very useful as we examine, in turn, each of the identified terms in equation [I.Z.2].

Term I:

The expression under the integral sign on the left-hand-side of equation [I.Z.2] may be manipulated into a variety of different f
Notice, first, that by bringing the component of the position vector inside the time-derivative and realizing that $v_i = \mathbf{D}x_i$, we ma
write,

$$x_i \rho \mathbf{D} v_j = \rho \, [\mathbf{D}(x_i v_j) - v_i v_j] \, .$$

[Equation I.Z.7]

Hence, Term I in equation [I.Z.2] may be rewritten as,

$$\int_V (x_i \rho \mathbf{D} v_j) \, d^3x \; = \int_V \rho \, \mathbf{D}(x_i v_j) \, d^3x - \int_V (\rho v_i v_j) \, d^3x$$

$$= \mathbf{D} \int_{V_L} (\rho x_i v_j) \, d^3x - 2 \, T_{ij} \, ,$$

[Equation I.Z.8]
=
Tassoul '78, §3.7, Eq. (136)

where, in deriving the second line of this relation, we have employed Lemma [I.Z.6] and have adopted the following standard

Definition of the
Kinetic Energy Tensor

$$T_{ij} \; \equiv \; (1/2) \int_V \rho v_i v_j \, d^3x \, .$$

[Equation I.Z.9]
=
EFE, Chapter 2, §9, Eq. (9)
BT87, Chapter 4, Eq. (4-74b)
Tassoul '78, §3.7, Eq. (137)

Now we note that both terms on the right-hand-side of equation [I.Z.2] are symmetric in the indices *i* and *j*. Hence, the
sub-expressions that contribute to Term I on the left-hand-side of the equation must be symmetric in these indices as well. We r
therefore rewrite the integrand of the first term on the right-hand-side of equation [I.Z.8] as,

$$\rho x_i v_j \; = \; (1/2) \, \rho \, (x_i v_j + x_j v_i)$$

$$= \; (1/2) \, \rho \, (x_i \, \mathbf{D} x_j + x_j \, \mathbf{D} x_i)$$

$$= \; (1/2) \, \rho \, \mathbf{D}(x_i x_j) \, .$$

[Equation I.Z.10]

Performing a volume integral over this last expression and, once again employing Lemma [I.Z.6], we conclude that,

$$\int_V (\rho \, x_i v_j) \, d^3x \; = \; (1/2) \int_V \rho \, \mathbf{D}(x_i x_j) \, d^3x$$

$$= \; (1/2) \, \mathbf{D} \int_{V_L} (\rho x_i x_j) \, d^3x$$

$$= \ (1/2)\ D\ I_{ij}\ .$$

[Equation I.Z.11]

where, in deriving this last line, we have adopted the following standard

Definition of the
Moment of Inertia Tensor

$$I_{ij} \ \equiv \ \int_V \rho x_i x_j \, d^3x\ .$$

[Equation I.Z.12]
=
EFE, Chapter 2, §9, Eq. (4)
BT87, Chapter 4, Eq. (4-76)
Tassoul '78, §3.7, Eq. (146)

Finally, then, by combining relation [I.Z.8] with relation [I.Z.11] we see that Term I may be written simply as,

$$\int_V (x_i \rho \mathbf{D} v_j)\, d^3x \ = \ (1/2)\ D^2 I_{ij} - 2\ T_{ij}\ ,$$

[Equation I.Z.13]
=
Tassoul '78, §3.7, Eq. (136)

Term II:

Integrating the first term on the right-hand-side of equation [I.Z.2] by parts (and assuming that at all locations in space the gas pressure is isotropic[4]), we obtain,

$$-\int_V [\ x_i\ \nabla_j P\ ]\, d^3x \ = \ -\int_V [\ \nabla_j(\ P x_i\ )\ ]\, d^3x + \delta_{ij}\int_V P\, d^3x\ .$$

[Equation I.Z.14]

Adopting the following

Definition of the
Total Internal Energy

$$U \ \equiv \ \int_V (\ \varepsilon\rho\ )\, d^3x\ ,$$

[Equation I.Z.15]

using Form B [II.A.4] of the ideal gas equation of state to relate the product $\varepsilon\rho$ to the gas pressure P, and applying Form B of Gauss's Theorem to the first term on the right-hand-side of this equation [I.Z.14], we further conclude that,

$$-\int_V [\, x_i \nabla_j P \,]\, d^3x \;=\; -\int_S [\, P x_i \,]\, n_j\, da + \delta_{ij} (\gamma - 1)\, U ,$$

[Equation I.Z.16]

where n_j is the j^{th} component of unit vector that is normal to the surface. Finally, then, if we demand that the volume over which the virial is evaluated always be constructed such that the gas pressure goes to zero everywhere on its surface S, we can write Term II simply as,

$$-\int_V [\, x_i \nabla_j P \,]\, d^3x \;=\; \delta_{ij} (\gamma - 1)\, U .$$

[Equation I.Z.17]

Term III:

If we adopt the following

Definition of the
Chandrasekhar Potential Energy Tensor

$$W_{ij} \;\equiv\; \int \rho(\mathbf{x})\, x_i\, a_j\, d^3\mathbf{x},$$

[Equation I.Z.17]
=
BT87, Chapter 2, Eq. (2-123)
EFE, Chapter 2, §10, Eq. (18)
Tassoul '78, Chapter 3, Eq. (140)

where a_j is the j^{th} component of the (vector) gravitational acceleration defined earlier[3] in terms of the gravitational potential as,

$$\mathbf{a} = -\nabla\Phi ,$$

[Equation I.H.4]

then Term III in equation [I.Z.2] becomes,

$$-\int_V [\, x_i \rho \nabla_j \Phi \,]\, d^3x \;=\; +\, W_{ij} .$$

[Equation I.Z.18]

As has been demonstrated explicitly in both BT87 and EFE, the components of the tensor W_{ij} may equivalently be expressed in terms of the following double volume integral,

$$W_{ij} = -\,(1/2)\, G \int\!\!\int \rho(\mathbf{x})\, \rho(\mathbf{x}')\, [\, (x_i' - x_i)(x_j' - x_j)\,]\, |\mathbf{x}' - \mathbf{x}|^{-3}\, d^3\mathbf{x}\, d^3\mathbf{x}',$$

[Equation III.R.6]

=

BT87, Chapter 2, Eq. (2-126)
EFE, Chapter 2, §10, Eqs. (14) & (15)
Tassoul '78, Chapter 3, Eq. (139)

demonstrating the "manifestly symmetric" nature of the tensor.

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which defines in a fundamental way the relationship between various global properties of isolated astrophysical systems. (Click on any one of these tensor variables to obtain its formal definition and click on the "origin" button to see a derivation of the equation.) A somewhat more familiar relation, usually referred to as the

Scalar Virial Theorem

$$(1/2)\ \mathbf{D}^2 I = 2\ E_{rot} + 3\ (\ \gamma - 1\ )\ U + E_{grav}\ ,$$

[Equation I.E.2]

results immediately from the trace of the 2nd-order tensor virial equation. This expression relates the second time-derivative of a system's moment of inertia to its gravitational potential energy E_{grav}, its total internal energy U, and its global kinetic energy (such as the energy tied up in rotation).

As we explain elsewhere in this H_Book, the 2nd-order tensor virial equation is not independent of the principal governing equations. Indeed, it is derived by taking the "first moment" of the Euler equation and integrating over the entire volume of the physical system under consideration. In his now classical analysis of ellipsoidal figures of equilibrium, Chandrasekhar has demonstrated how even higher order moments of Euler's equation can be used to examine the global stability of self-gravitating systems. Generally speaking, however, we shall not find it necessary to draw upon relationships that extend beyond the 2nd order tensor virial equation, although we may occasionally allude to the results that have been derived from them.

Footnotes

[1]As Chandrasekhar (1987) points out (see, specifically, Chapter 2, §11(a), p. 21 of EFE), the related 1st-order equation is obtained by simply (multiplying through by "1" then) integrating the Euler equation over the volume occupied by the fluid. This produces "an equation which simply expresses the uniform motion of the center of mass; it provides no essentially new information."

[2]Note that throughout his book, Chandrasekhar (1987) adopts a sign convention for the scalar gravitational potential that is opposite to the sign convention used herein.

[3]By using the color yellow for the operator D, we are following Tassoul's (1978) lead and drawing a distinction between a standard Lagrangian time-derivative and the time differentiation that is being carried out in this expression. Here, the time derivative is being taken of a (global) variable that exhibits no spatial variation and, hence, depends only on time (see footnote 3 on p. 45 of Tassoul '78). Neither Chandrasekhar (1987) nor Binney and Tremaine (1987) have adopted a notation that clearly draws a distinction between the operators **D** and D, but for added clarity we will do so throughout this H_Book.

[4]In discussing primarily stellar (rather than gas) dynamical systems, Binney and Tremaine (1987) include in the virial equations a more general pressure tensor Π_{ij} that is defined in terms of a nonisotropic velocity dispersion σ_{ij} as follows:

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