The Structure, Stability, and Dynamics of Self-Gravitating Systems

Integrals of Motion in Time-Invariant Potentials

The time-evolution of dynamical systems is usually much easier to understand, or at least the future behavior of such systems is much easier to predict, once we realize that certain constants or "integrals" of the motion are preserved in time. The most familiar integrals of motion are energy and angular momentum. It is important to realize, however, that when particles (or fluid elements) move under the influence of an external potential, these familiar integrals may not be constant in time, but other integrals of the motion may exist. We present here a general discussion of integrals of motion for systems of noninteracting particles (*i.e.* , effects of the gas pressure will be ignored) that move in time-invarient, external potentials.

Referring back to the principal governing equations, if the effects of pressure are ignored Euler's equation takes the form

$$\mathbf{Dv} = - \nabla\Phi .$$

[Equation VI.I.1]

Our objective is to find one or more "integrals" of the motion I_i for which the mathematical statement,

$$\mathbf{D}(I_i) = 0,$$

[Equation VI.I.2]

is both true and derivable from the equation of motion.

We'll focus, first, on the right-hand-side of the equation of motion. From the general (curvilinear form) of the gradient operator, we can write

$$- \nabla\Phi = - \mathbf{e}_1 (1/h_1) \partial_{\xi_1}\Phi - \mathbf{e}_2 (1/h_2) \partial_{\xi_2}\Phi - \mathbf{e}_3 (1/h_3) \partial_{\xi_3}\Phi .$$

[Equation VI.I.3]

Note, also, that from the definition of the Lagrangian time derivative of any scalar function, in general we can write,

$$\mathbf{D}\Phi = \partial_t\Phi + \mathbf{v}\cdot\nabla\Phi .$$

[Equation VI.I.4]

Here, physically, the Lagrangian (total) time-derivative $\mathbf{D}\Phi$ identifies the time-variation of the gravitational potential as seen by a moving particle (fluid element) while the Eulerian (partial) time-derivative identifies the actual time-variation of the potential as viewed by a stationary observer. Our discussions will be confined to time-invarient potentials, as desired, by simply demanding that

$$\partial_t\Phi = 0.$$

[Equation VI.I.5]

Hence, for our present discussions we may adopt the following relationship between the time- and spatial-variation of the gravitational potential, as seen by a moving particle:

$$\mathbf{D}\Phi = \mathbf{v}\cdot\nabla\Phi .$$

[Equation VI.I.6]

Now we turn our attention to the left-hand-side of the equation of motion. Realizing that the general (curvilinear coordinate) form of the velocity is

$$\mathbf{v} = \mathbf{e}_1\,(h_1 \mathbf{D}\xi_1) + \mathbf{e}_2\,(h_2 \mathbf{D}\xi_2) + \mathbf{e}_3\,(h_3 \mathbf{D}\xi_3),$$

[Equation VI.M.18]

we can, quite generally, write the time-derivative of **v** as,

$$\begin{aligned} \mathbf{Dv} \;=\; & \mathbf{e}_1 \{ d(h_1 \mathbf{D}\xi_1)/dt - (h_2 \mathbf{D}\xi_2)\, A - (h_3 \mathbf{D}\xi_3)\, B \} + \\ & \mathbf{e}_2 \{ d(h_2 \mathbf{D}\xi_2)/dt + (h_1 \mathbf{D}\xi_1)\, A - (h_3 \mathbf{D}\xi_3)\, C \} + \\ & \mathbf{e}_3 \{ d(h_3 \mathbf{D}\xi_3)/dt + (h_1 \mathbf{D}\xi_1)\, B + (h_2 \mathbf{D}\xi_2)\, C \}. \end{aligned}$$

[Equation VI.I.7]

where, as before [VI.M.17]:		
A	$\equiv f_1 \mathbf{D}\xi_2 - f_3 \mathbf{D}\xi_1$	$= [\mathbf{D}\xi_2]\,(1/h_1)\,\partial_{\xi_1} h_2 - [\mathbf{D}\xi_1]\,(1/h_2)\,\partial_{\xi_2} h_1$
B	$\equiv f_2 \mathbf{D}\xi_3 - f_5 \mathbf{D}\xi_1$	$= [\mathbf{D}\xi_3]\,(1/h_1)\,\partial_{\xi_1} h_3 - [\mathbf{D}\xi_1]\,(1/h_3)\,\partial_{\xi_3} h_1$
C	$\equiv f_4 \mathbf{D}\xi_3 - f_6 \mathbf{D}\xi_2$	$= [\mathbf{D}\xi_3]\,(1/h_2)\,\partial_{\xi_2} h_3 - [\mathbf{D}\xi_2]\,(1/h_3)\,\partial_{\xi_3} h_2$

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Utilizing the general expression for the time-derivative of any unit vector, derive the above expression [VI.I.7] for the acceleration (**Dv**) from the definition of **v**.

A

As a shorthand, it frequently is useful to adopt the variables,

$$\begin{aligned} v_1 &\equiv h_1 \mathbf{D}\xi_1, \\ v_2 &\equiv h_2 \mathbf{D}\xi_2, \\ v_3 &\equiv h_3 \mathbf{D}\xi_3, \end{aligned}$$

[Equation VI.I.8]

to denote the three components of the particle velocity. In terms of these variables, the above expressions for the velocity vector [VI.M.18] and acceleration vector [VI.I.7] take the following, simpler forms:

$$\mathbf{v} = \mathbf{e}_1\, v_1 + \mathbf{e}_2\, v_2 + \mathbf{e}_3\, v_3,$$

[Equation VI.I.9]

$$\mathbf{a} \equiv \mathbf{Dv} = \mathbf{e}_1 \{ dv_1 /dt - v_2 A - v_3 B \} + \mathbf{e}_2 \{ dv_2 /dt + v_1 A - v_3 C \} + \mathbf{e}_3 \{ dv_3 /dt + v_1 B + v_2 C \}.$$

[Equation VI.I.10]

Combining the above expressions for the left-hand-side [VI.I.10] and the right-hand-side [VI.I.3] of the equation of motion, we can write in the following very general form the

Three Components of the Equation of Motion [Equation VI.I.11]	
$\mathbf{e}_1$	$dv_1 /dt - v_2 A - v_3 B = - (1/h_1) \partial_{\xi_1} \Phi$
$\mathbf{e}_2$	$dv_2 /dt + v_1 A - v_3 C = - (1/h_2) \partial_{\xi_2} \Phi$
$\mathbf{e}_3$	$dv_3 /dt + v_1 B + v_2 C = - (1/h_3) \partial_{\xi_3} \Phi$

Building upon this presentation of the equation of motion written in curvilinear coordinates, the accompanying pages present derivations and discussions of:

- Energy as a valid integral of the motion in any time-invariant potential;
- Components of the angular momentum vector as integrals of the motion in spherically symmetric potentials;
- Components of the Laplace-Runge-Lenz vector as an integral of the motion in 1/r potentials;
- Integrals of motion in Kuzmin-like potentials;

Other Integrals

Situations in which $\Phi = \Phi(\xi_1)$

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