A quantum state of ultra-low phase noise

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A novel photon phase state is proposed that is similar to a recently advanced state by J.H. Shapiro, S.R. Shepard and N. Wong, Phys. Rev. Lett. 62 (1989) 2377. The new state given here has a phase distribution \( W_\phi \) that has a sharply-peaked central maximum with a superposed broad background. The phase noise associated with the width of the best-fitting gaussian to the central peak decreases exponentially with increasing mean photon number. The state is mathematically more tractable and perhaps physically more realizable than previous constructions.

Part of the motivation for the interest in squeezed states of light arises from their utility in constructing phase sensitive interferometers [1]. In this context it is not so much the percent of squeezing that is important but rather the power law that relates the phase uncertainty to the mean number of photons \( \langle m \rangle \). For a coherent state the phase noise decreases asymptotically as \( \langle m \rangle^{-1/2} \), whereas in a squeezed state it goes like \( \langle m \rangle^{-1} \). Hence it is possible to achieve the same phase sensitivity using a coherent state as with a squeezed state — but at the cost of needing a much larger number of photons to do it. This additional power offsets the advantage of the reduced phase noise in an interferometer. A phase state \( \varphi \) of precise phase has infinite mean photon number and is not physically realizable. For these reasons there has been an effort to find a state that minimizes phase uncertainty subject to the constraint of a fixed mean number of photons [2,3]. How one goes about this depends crucially upon the way one chooses to measure the phase uncertainty. Lack of an obvious phase operator [4] has led to a number of different proposals of phase uncertainty.

Shapiro and co-workers [2] have recently advanced a measure of quantum phase noise corresponding to the inverse peak height \( \delta \phi \equiv 1/W_0 \) of a centrally-peaked phase distribution \( W_\phi \). Pictorially, this amounts to replacing the distribution \( W_\phi \) with a rectangle of the peak height \( W_0 \) and area normalized to one, the area of \( W_\phi \). The measure \( \delta \phi \) is then just the width of this rectangle. For coherent and squeezed states \( W_\phi \) is gaussian and \( \delta \phi \) is a good measure of the gaussian width. For other states whose total area is not concentrated underneath the peak — \( \delta \phi \) does not seem to be as good a measure [5]. Nevertheless, one can still pose the variational problem: What state \( |\psi\rangle \),

\[
|\psi\rangle = \sum_{m=0}^{\infty} c_m |m\rangle, \quad c_m \geq 0,
\]

minimizes \( \delta \phi \)? Here \( |m\rangle \) is the \( m \)th photon number state. As Shapiro et al. [2] found, this problem has the solution

\[
|\psi\rangle \equiv \mathcal{N}(m_0) \sum_{m=0}^{m_0} (1+m)^{-1/2} |m\rangle,
\]

where \( m_0 \) is a cutoff parameter and \( \mathcal{N} \) is a normalization constant dependent on \( m_0 \). The cutoff is necessary to keep the mean number of photons \( \langle m \rangle \) finite. This state has the property that the Shapiro measure is \( \delta \phi \sim \langle m \rangle^{-2} \), hence an improvement over even squeezed states [2]. In addition the width of the central peak of this distribution, \( \Delta \phi \), as measured by the width of the best-fitting Gaussian, shows an exponential decay [5], \( \Delta \phi \sim \exp(-\langle m \rangle) \).

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The state $|\psi_\epsilon\rangle$ given in eq. (2) has some difficulties, however. Sharp cutoffs such as that at $m_0$ can lead to mathematical artifacts or “ringing” in the distribution that are nonphysical. In fact this does occur in the distribution $W_0[|\psi_\epsilon\rangle]$ and leads to difficulties in evaluating numerically the Fisher information needed by Caves et al. [6] in the context of a particular measurement scheme being used with the Shapiro state (2). The fact that $|\psi_\epsilon\rangle$ is defined only on a subset of the entire Hilbert space seems physically untenable. In analogy to coherent states, one would like to have a superposition of all Fock states $|m\rangle$ but with the higher $|m\rangle$ weighted with negligible probability. In addition we would like a smooth cutoff as $m\to\infty$ while still maintaining the interesting properties of $|\psi_\epsilon\rangle$. A state that accomplishes this is

$$|\epsilon\rangle = \mathcal{N}(\epsilon) \sum_{m=0}^{\infty} \frac{e^{-\epsilon m/2}}{1+m} |m\rangle \quad (\epsilon \geq 0).$$

(3)

This state may be treated analytically in closed form using the Lerch transcendent phi function [7]

$$f_\epsilon(w) = \Phi(e^{-w}n, 1) = \sum_{m=0}^{\infty} \frac{e^{-wm}}{(1+m)^n},$$

(4)

for which there exist various asymptotic expansions and integral representations. Two of the latter reduce to

$$f_1(w) = -e^w \ln(1-e^{-w}),$$

(5a)

$$f_2(w) = e^w \int_0^\infty \frac{t}{e^{t+w} - 1} dt.$$  

(5b)

The normalization condition $\langle \epsilon | \epsilon \rangle = 1$ determines

$$\mathcal{N} = \left[ f_2(\epsilon) \right]^{-1/2} \epsilon^{-1/2} \sqrt{6/\pi}.$$  

(6)

Using interference in phase space techniques [8] or other methods, one obtains the matrix element $\langle \phi | m \rangle = \exp(i\phi)/\sqrt{2\pi}$. This can be used to obtain the phase distribution

$$W_\phi[\epsilon] = |\langle \phi | \epsilon \rangle|^2 = \frac{|f_1(\epsilon/2 + i\phi)|^2}{2\pi f_2(\epsilon)}.$$  

(7)

This is much simpler than the equivalent expression for $W_\phi[\psi_\epsilon]$, given in ref. [5], due to the absence here of the sharp cutoff at $m_0$.

As $\epsilon \to 0^+$ the distribution approaches a background function

$$\lim_{\epsilon \to 0^+} W_\phi[\epsilon] = W_\phi^{(\text{back})}[\epsilon] = 3|f_1(i\epsilon)|^2/\pi^3 = 3[\ln^2(2\sin(|\phi|/2) + (|\phi|/2 - \pi/2)^2)]/\pi^3,$$

(8)

$\phi \neq 0$, that is the same as the Shapiro background $W_\phi^{(\text{back})}[\psi_\epsilon]$ obtained by letting $m_0 \to \infty$. However, at nonzero $\epsilon$ our $W_\phi[\epsilon]$ is free of the artificial ringing found to plague the Shapiro distribution at finite $m_0$. In fig. 1, for $\epsilon = 0.1$ we plot $W_\phi[\epsilon]$, the background distribution $W_\phi^{(\text{back})}[\epsilon]$, the Shapiro rectangle of width $\delta\phi$, and the best-fitting gaussian to $W_\phi[\epsilon]$. The distribution $W_\phi[\epsilon]$ is differentiable at the central peak and the peak height and second derivative there are, respectively,

$$W_{\phi = 0} = \frac{f_1^2(\epsilon/2)}{2\pi f_2(\epsilon)},$$

(9a)

$$\frac{\partial^2}{\partial \phi^2} W_\phi |_{\phi = 0} = \frac{2e^\epsilon[1-f_1(\epsilon/2)]}{2\pi f_2(\epsilon)[e^{\epsilon/2} - 1]^2}.$$  

(9b)

Using asymptotic expansions of the Lerch functions we may write the Shapiro measure $\delta \phi = W_0^{-1}$ as a series in $\epsilon$ and $\ln \epsilon$. The expression for mean photon number $\langle m \rangle$ can also be expanded in this fashion as

$$\langle m \rangle = f_1(\epsilon)/f_2(\epsilon) \sim -1 - (6/\pi^2) \ln \epsilon.$$  

(10)

Using this result to re-express the $\delta \phi$ expansion in terms of $\langle m \rangle$ rather than $\epsilon$ we have

$$\delta \phi[\epsilon] \sim 12\pi/\langle m \rangle^2.$$  

(11)

This should be compared to the result of using the original Shapiro state $|\psi_\epsilon\rangle$, eq. (2),

$$\delta \phi[\psi_\epsilon] \sim 12/\pi \langle m \rangle^2.$$  

(12)

We see that our new state maintains the same power law. For a given $\langle m \rangle$ the Shapiro measure $\delta \phi$ applied to the Shapiro state is slightly smaller by a factor of $\pi$ — the state was in fact chosen to minimize this measure. For large $\langle m \rangle$ this constant factor is of little consequence regarding the utility of the new state $|\epsilon\rangle$.

It has been argued [5] that although the Shapiro measure works well for coherent and squeezed states,
which have gaussian distributions, it does not properly measure the sharp peak width in states such as $|\psi_s\rangle$ or $|\epsilon\rangle$; see fig. 1. (This is because most of the area of $W_{\psi}[\epsilon]$ is not under the peak but in the broad background.) For a state such as $|\psi_s\rangle$ or $|\epsilon\rangle$ the width $\Delta\phi$ of the best-fitting gaussian to the peak, $W_{\psi}^{(peak)} \equiv W_0 \exp \left( -\phi/\Delta\phi \right)^2$, has been advanced [5]. This measure can be written as

$$\Delta\phi = \sqrt{2W_0/|W_0'|}$$

$$= f_1(\epsilon/2)[1-e^{-\epsilon^2/2}]/\sqrt{f_1(\epsilon/2)-1}$$

$$\sim (\pi/2\sqrt{6}) \langle m \rangle^{1/2} \exp(-\pi^2\langle m \rangle/6),$$

(13)

which shows a remarkable exponential decrease in phase noise with increasing $\langle m \rangle$ as was also found previously with the Shapiro state $|\psi_s\rangle$. The area under the peak -- approximated by the area of the fitted gaussian -- also vanishes exponentially; a fact that might make the exponentially small phase noise difficult to utilize in a measurement scheme. In fig. 2 we plot $W_{\psi}[\epsilon]$ for several values of $\epsilon$, along with the best-fitting gaussian of width $\Delta\phi$.

To illustrate clearly the different contributions from the background and the peak to the phase un-

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Fig. 1. The photon phase distribution $W_{\psi}[\epsilon]$, eq. (7), for the state $|\epsilon\rangle$, eq. (3), is plotted here as the solid line. The origin is arbitrary and has been chosen to correspond with the central peak of $W_{\psi}$. The cutoff parameter here is $\epsilon=0.1$, corresponding via eq. (10) to a mean photon number $\langle m \rangle \approx 0.79$ given by eq. (10). Notice the absence of the “ringing” in the wings that occurs in the Shapiro state distribution [5,6]. The dashed curve corresponds to the background distribution $W_{\psi}^{(back)}$ obtained by letting $\epsilon \rightarrow 0^+$ in eq. (8). The superposed rectangle has area one and peak height $W_0$, and hence its width corresponds to the Shapiro inverse peak height phase uncertainty $\delta\phi = W_0^{-1} \approx 0.90$. We see that this measure does not reflect the width of the narrow peak -- it is contaminated by the noise in the wings. A better measure of the peak noise is the width $\Delta\phi \approx 0.11$ of the best-fitting gaussian (dotted curve).

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Fig. 2. The phase distribution $W_{\psi}[\epsilon]$ is plotted by a solid curve for $\epsilon=0.5, 0.1, and 0.05$ in (a), (b), and (c), respectively. These values correspond to respective mean photon numbers $\langle m \rangle = 0.26, 0.79, and 1.09$. The dotted curve in each case represents the best-fitting gaussian to the peak. The width of this gaussian $\Delta\phi$, eq. (13), is taken as a sensible measure of the phase uncertainty associated only with the narrow peak -- excluding the noise of the broad background. We see that as $\epsilon \rightarrow 0^+$, $W_{\psi}$ is approaching the background distribution (the dashed curve of fig. 1) corresponding to $\langle m \rangle \rightarrow \infty$. At finite $\langle m \rangle$, however, there might exist a measurement scheme that could utilize the exponentially low phase noise in the peak.
certainty, we compute the variance of the distribution $W_\phi[\epsilon]$ asymptotically in large $\langle m \rangle$ as

$$\sigma_\phi^2 \equiv \int_{-\pi}^{\pi} \varphi^2 W_\phi[\epsilon]$$

$$\sim A \{ 1 - \langle m \rangle \exp[-(\pi^2/6)\langle m \rangle - 1] \}, \quad (14)$$

where

$$A \equiv \frac{\pi^2}{20} + \frac{6}{\pi^2} \int_{0}^{\pi} d\varphi \varphi^2 \ln^2(2 \sin \varphi/2) \approx 3.65331.$$

If we evaluate $\sigma_\phi^2$ using only the background distribution $W_\phi^{(\text{back})}[\epsilon]$ of eq. (8), we get

$$\sigma_\phi^{(\text{back})} \equiv \int_{-\pi}^{\pi} \varphi^2 W_\phi^{(\text{back})}[\epsilon] = A \cdot \quad (16)$$

Hence the constant term $A$ in the total variance $\sigma_\phi^2$, eq. (14), is associated with noise coming from the background. The exponential term in (14) is uncertainty associated with the width of the peak, $\Delta \phi$, eq. (13). One sees that the variance $\sigma_\phi^2$ decays exponentially with increasing $\langle m \rangle$ and approaches the constant value $A$.

In conclusion, a novel photon phase state $|\epsilon\rangle$, eq. (3), has been advanced that has Shapiro phase uncertainty measure $\delta \phi \sim \langle m \rangle^{-2}$ and gaussian peak measure $\Delta \phi \sim e^{-\langle m \rangle}$, as in the original Shapiro state [2], $|\psi_s\rangle$. Unlike the Shapiro state, this new state is a superposition of all Fock states $|m\rangle$, and is free of sharp cutoffs and the ringing associated with these cutoffs. This new state could have favorable consequences concerning a measurement scheme of Caves et al. [6] in which the ringing and the lack of a closed analytical form for the original Shapiro distribution $W_\phi[\psi_s]$ causes problems in the numerical analysis. The new state presented here can circumvent these problems since it is free of ringing and has a compact analytical form; an expression for the Fisher information of $W_\phi$ utilized by Caves and co-workers, may be given easily as an expansion in $\epsilon$. In addition, the form of the new state is similar to that of thermofield states [9] and may be realizable as the eigenstate of a sensible hamiltonian – a first step in the prescription for physical generation of this state. What is needed now is a measurement scheme that can take advantage of the exponential narrowness of the central peak of the phase distribution.

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