Throughout this homework set, the veritable obscenity $\epsilon_0$ will be taken as $\epsilon_0 = 1$. Theorists will rejoice. Experimenalists and engineers will kvetch. The mathematicians will ask me to disprove this by counterexample. You know I like all of you equally. :-)

**Problem 1 (Barton #4.1 p. 115)**

Given potentials $\psi_\alpha(r) = \alpha e^{-\lambda r}$ and $\psi_\beta(r) = \beta e^{-\lambda r}$, calculate charge density $\rho = -\nabla^2 \psi$ and total charge $Q = \int dV \rho$.

Hint: In 3D, $\rho = -\nabla^2 \psi = -\frac{1}{r^2} \frac{\partial}{\partial \theta} \left( r^2 \frac{\partial \psi}{\partial \theta} \right)$ and $\nabla^2 \left( \frac{1}{r} \right) = -4 \pi \delta(r)$ and in 2D $\rho = -\nabla^2 \psi = -\frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \frac{\partial \psi}{\partial \theta} \right)$.

(i) **3D**: For $\psi_\alpha(r) = \alpha e^{-\lambda r}$, we have $\rho_\alpha = \lambda \alpha (\frac{1}{r} - \lambda) e^{-\lambda r}$ so that $Q_\alpha = 4 \pi \lambda \alpha \int_0^\infty d \bar{r} \bar{e}^{-\lambda \bar{r}} (2 \bar{r} - \lambda \bar{r}^2) = 0 .\)

For $\psi_\beta(r) = \frac{\beta}{r} e^{-\lambda r}$, we note that $\rho_\beta = -\beta e^{-\lambda r} \nabla^2 \left( \frac{1}{r} \right) = -\frac{\beta}{r} \nabla^2 e^{-\lambda r} - 2 \nabla \left( e^{-\lambda r} \right) \cdot \nabla \left( \frac{1}{r} \right)$ or, using $\nabla^2 \left( \frac{1}{r} \right) = -4 \pi \delta(r)$ for the first term and $\frac{\rho_\alpha}{r}$ with $\alpha \to \beta$ for the second term, we have $\rho_\beta = 4 \pi \beta \delta(r) - \frac{\beta \lambda}{r} e^{-\lambda r} - 2 e^{-\lambda r} \nabla \left( e^{-\lambda r} \right) \cdot \nabla \left( \frac{1}{r} \right)$. Finally, since $\nabla \left( e^{-\lambda r} \right) \cdot \nabla \left( \frac{1}{r} \right) = \lambda \frac{\lambda}{r^2} - \frac{1}{r}$, we have $\rho_\beta = 4 \pi \beta \delta(r) - \frac{\beta \lambda}{r} e^{-\lambda r}$ and $Q_\beta = 4 \pi \beta - 4 \pi \lambda^2 \beta \int_0^\infty d \bar{r} \bar{e}^{-\lambda \bar{r}} \bar{r} = 4 \pi \beta - 4 \pi \beta = 0 .\)

(ii) **2D**: For $\psi_\alpha(r) = \alpha e^{-\lambda r}$, we have $\rho_\alpha = \lambda \alpha \left( \frac{1}{r} - \lambda \right) e^{-\lambda r}$ so that $Q_\alpha = 2 \pi \lambda \alpha \int_0^\infty d \bar{r} \bar{e}^{-\lambda \bar{r}} (1 - \lambda \bar{r}) = 0 .\)

For $\psi_\beta(r) = \frac{\beta}{r} e^{-\lambda r}$, we have $\rho_\beta = -\beta \left( \frac{1}{r^2} + \frac{\lambda}{r^2} \right) e^{-\lambda r}$ and, integrating, we have $Q_\beta = 2 \pi \beta \int_0^\infty d \bar{r} \bar{e}^{-\lambda \bar{r}} (1 - \lambda \bar{r}) = 0 .\)

The integral for $Q_\beta$ diverges as $r^{-1}$ as $r \to 0$, a nonphysical result, but I do believe that there is a way to formally remove this divergence. This has to do with a theorem called the Fredholm Alternative. Look it up on Wikipedia. Happy reading.

(iii) Using Gauss’ Law: $Q_{r < R} = \int_{r < R} \mathbf{E} \cdot d \mathbf{a}$ where $\mathbf{E} = -\nabla \psi$ and $d \mathbf{a} = R^2 \sin \theta \, d \phi \, r \, d \theta$ in 3D and $d \mathbf{a} = R \, d \phi \, \bar{r} \, d \theta$ in 2D. Since our potentials depend only on $r$, we can immediately write $Q = -4 \pi \lim_{R \to \infty} (\nabla \psi)_{r=R}$ in 3D and $Q = -2 \pi \lim_{R \to \infty} (\nabla \psi)_{r=R}$ in 2D. Furthermore, since the gradient operator in both 2D and 3D is simply $\nabla = \frac{\partial}{\partial r}$, all we really need is the the magnitude of the electric fields as $r \to \infty$. So, $\mathbf{E}_\alpha \cdot \bar{r} = -\nabla \psi_\alpha = -\alpha \lambda e^{-\lambda r}$ and $\mathbf{E}_\beta \cdot \bar{r} = -\nabla \psi_\beta = \beta \left( \lambda + \frac{1}{r} \right) e^{-\lambda r}$. Clearly, $\mathbf{E}_\alpha \cdot \bar{r} = \mathbf{E}_\beta \cdot \bar{r} = 0$ as $r \to \infty$, so in both 2D and 3D we find that $Q_\alpha = Q_\beta = 0 .\)

**Problem 2 (Barton #4.2 p. 115)**

Given: $\nabla^2 \psi = 0$ for $z \neq 0$ with the boundary conditions $\psi(z = 0) = \sin(k x)$ and $\psi(|z| \to \infty) = 0$.

(i) Since the function has no $y$ dependence, we can separate the solution as $\psi = X(x) Z(z)$ so that $\frac{\nabla^2 \psi}{\psi} = \frac{X''}{X} + \frac{Z''}{Z} = 0$ for $z \neq 0$.

Knowing that $Z(z)$ must dissappear as $|z| \to \infty$, we set $\frac{Z''}{Z} = -\alpha^2$ (with $\alpha > 0$) so that $Z(z) = A e^{-\alpha z} + B e^{\alpha z}$. The boundary condition at $|z| \to \infty$ implies that $A = 0$ for $z \leq 0$ and $B = 0$ for $z > 0$. Thus, we rewrite $Z(z) = C e^{-\alpha |z|}$. Since $\frac{X''}{X} = -\alpha^2$, and given that $\psi(z = 0) = \sin(k x)$, we can write $X(x) = \sin(\alpha x)$ where clearly $\alpha = k$. Thus, $\psi(x, z) = X(x) Z(z) = \sin(k x) e^{-k |z|}$ where the boundary condition at $x = 0$ implies that $C = 1$.

(ii) The charge distribution is determined as $\rho = -\nabla^2 \psi(x, z) = \left( \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} \right) \sin(k x) e^{-k |z|}$. To compute this, we note that $\frac{\partial^2}{\partial z^2} \sin(k x) e^{-k |z|} = \sin(k x) \left( -2 \delta(z) k e^{-k |z|} \right) + \frac{\partial^2}{\partial x^2} \sin(k x) e^{-k |z|} = -k^2 \sin(k x) e^{-k |z|}$ so that $\rho(x, z) = 2k \delta(z) \sin(k x) e^{-k |z|}$.
Problem 3 (Barton #4.3 p. 115)

Given the point dipole \( \mathbf{p} = \hat{z} p \) at the origin, calculate the potential \( \psi \), the electric field \( \mathbf{E} \), and show that \( \nabla^2 \mathbf{E} = 0 \) even at the \( \mathbf{r} \neq 0 \).

(i) We use the standard trick of placing point charges \( q \) and \(-q\) on the z-axis, centered at the origin, a distance \( d=p/q \) apart.

Summing the Green's functions \( G_N(r) = \frac{q}{4\pi |\mathbf{r}-\mathbf{r}_s|} \) over the discrete two-charge distribution, we get the potential \( \psi(\mathbf{r}) = \frac{1}{4\pi} \left( \frac{q}{r_s} - \frac{q}{r} \right) \)

where \( r_s = |\mathbf{r} + \frac{d}{2} \hat{z}|. \) Expanding this expression in \( d/r \ll 1 \), we obtain \( \psi(\mathbf{r}) = \frac{1}{4\pi} \frac{q d \cos(\theta)}{r^2} = \frac{1}{4\pi} \frac{p \cos(\theta)}{r^2} \) with \( \theta \) the polar angle.

The problem says to calculate \( \psi \), but this is a very standard result. Full credit to anyone who simply writes the correct potential.

(ii) The electric field is \( \mathbf{E} = \frac{p}{4\pi} \sqrt{\cos(\theta)/r^2} = \frac{p}{4\pi r^2} \left( 2 \cos(\theta), \sin(\theta), 0 \right). \) Note: Barton has a \( G_0 \) in the denominator; this is really \( \kappa_0 \).

(iii) The Laplacian \( \nabla^2 \mathbf{E} = \frac{p}{4\pi} \left( 8 \cos(\theta)/r^3, \cos(\theta) \cos(5 \sin(\theta))/r^3, 0 \right) = \frac{p}{4\pi r^3} \left( 8 \cos({\theta}), \csc(\theta) + 4 \sin(\theta), 0 \right) \) disappears only as \( r \to \infty \).

Problem 4 (Barton #4.4 p. 115)

Given that \( \nabla^2 \psi = 0 \) in the annular region \( a \leq r \leq 3a \), with \( \psi(a, \phi) = \cos(\phi) \) and \( \psi(3a, \phi) = \cos(3\phi) \).

(i) The “slice” \( \psi(2a, \phi) = A \cos(2\phi) \), with \( A \) constant, violates the condition that \( \nabla^2 \psi = 0 \). This slice will have local extrema in the \( \phi \) direction which, given the constancy of \( \psi \) in the radial direction, would render the function non-harmonic.

(ii) We have \( \psi(r, \phi) = e^{i\phi} \left( \frac{a_1}{a} + b_1 r \right) + e^{-i\phi} \left( \frac{a_1}{a} + b_3 r \right) + e^{3i\phi} \left( \frac{a_3}{a} + b_3 r \right) + e^{-3i\phi} \left( \frac{a_3}{a} + b_3 r \right) \), given the symmetry of the boundary conditions. Thus, \( \cos(\phi) = e^{i\phi} \left( \frac{a_1}{a} + b_1 a \right) + e^{-i\phi} \left( \frac{a_1}{a} + b_3 a \right) + e^{3i\phi} \left( \frac{a_3}{a} + b_3 a \right) + e^{-3i\phi} \left( \frac{a_3}{a} + b_3 a \right) \), which implies \( 0 = \frac{a_1}{a} + b_3 a^3 = \frac{a_3}{a} + b_3 a^3 \) and \( \frac{1}{2} = \frac{a_1}{a} + b_3 a = \frac{a_3}{a} + b_3 a \). It follows then that

\[
\psi(r, \phi) = e^{i\phi} \left( \frac{a_1}{a} + \frac{a_3 - 2a_1}{2a^2} r \right) + e^{-i\phi} \left( \frac{a_1}{a} + \frac{a_3 - 2a_1}{2a^2} r \right) + \left( \frac{1}{a^3} - \left( \frac{1}{3a} \right)^3 \right) \left( e^{3i\phi} a_3 + e^{-3i\phi} a_3 \right). \]

Thus, we see that

\[
\psi(r, \phi) = \left( \frac{9a}{16r} - \frac{r}{16a} \right) \cos(\phi) + \frac{27a^3}{1456} \left( \frac{r^2}{a^2} - \frac{1}{a^2} \right) \left( e^{3i\phi} + e^{-3i\phi} \right) \text{ or, rewriting in terms of the cosines we expect, we have}
\]

\[
\psi(r, \phi) = \left( \frac{9a}{16r} - \frac{r}{16a} \right) \cos(\phi) + \frac{27}{718} \left( \frac{r^2}{a^2} - \frac{1}{a^2} \right) \cos(3\phi).
\]

\[
\psi(r, \phi) \text{ for } a=1
\]
Problem 5 (Barton #4.5 p. 115)

In 3D, \( \rho_1 = \lambda e^{-r/\alpha} \) and \( \rho_2 = \lambda \frac{r}{\alpha} e^{-r/\alpha} = \lambda \frac{r}{\alpha} \cos(\theta) e^{-r/\alpha} = \lambda \cos(\theta) e^{-r/\alpha} \).

(i) First, \( Q_1 = \int dV \rho_1(r) = 4 \pi \int_0^\infty dr r^2 \lambda e^{-r/\alpha} = 8 \pi \alpha^3 \lambda \) so \( Q_1 = 8 \pi \alpha^3 \lambda \). Next, we see that, because of the angular integral in \( Q_2 = \int dV' \rho_2 = 2 \pi \alpha \int_0^\infty d\theta \sin(\theta) \int_0^\infty dr \frac{r}{\alpha} \cos(\theta) e^{-r/\alpha} \), the total charge is \( Q_2 = 0 \). Thus we construct the dipole moment \( \mathbf{p}_2 = \int dV' \rho_2(r) \mathbf{r} = 2 \pi \alpha \int_0^\infty d\phi \int_0^\infty d\theta \sin(\theta) \int_0^\infty dr r^2 \frac{r}{\alpha} \cos(\theta) e^{-r/\alpha} \left[ r \sin(\theta) \cos(\phi) \hat{x} + \sin(\theta) \sin(\phi) \hat{y} + r \cos(\theta) \hat{z} \right] \). Performing the integral over \( \phi \), \( \mathbf{p}_2 = 2 \pi \alpha \int_0^\infty d\theta \sin(\theta) \int_0^\infty dr dr^3 \frac{r}{\alpha} \cos(\theta) e^{-r/\alpha} = 8 \pi \alpha^4 \hat{z} \) so that \( \mathbf{p}_2 = 8 \pi \alpha^4 \hat{z} \).

(ii) Given that \( G(\mathbf{r} | \mathbf{r}') = \sum_{n=0}^\infty \sum_{m=1}^n \frac{\mathbf{r}_n^m}{2 \sin^{2} \frac{\theta}{2}} Y_{lm}(\Omega') Y_{lm}(\Omega) \), we calculate the potentials via \( \psi_i = \int dV' \rho_i(r) G(\mathbf{r} | \mathbf{r}') \). First, \( \rho_1 \) has no angular dependence, so \( \psi_1 = 4 \pi \int_0^\infty d\theta \frac{r}{\alpha} \cos(\theta) e^{-r/\alpha} = \frac{4}{\alpha} \int_0^\infty dr r^2 e^{-r/\alpha} \). Due to the presence of the \( r < a \), the integral becomes \( \psi_1 = \frac{4}{\alpha} \int_0^a dr \frac{r}{\alpha} \cos(\theta) e^{-r/\alpha} = \frac{4}{\alpha} \int_0^a dr r^2 e^{-r/\alpha} \) so that \( \psi_1(r, \Omega) = -\alpha^2 e^{-r/\alpha} + \frac{2 \lambda a^2}{r} \left( 1 - e^{-r/\alpha} \right) \). Next, we note that \( \rho_2 = \frac{2 \pi}{3} \int_0^\infty d\theta \sin(\theta) \cos(\theta) e^{-r/\alpha} \), so that \( \psi_2 = \frac{2 \pi}{3} \int_0^\infty d\theta \sin(\theta) \cos(\theta) e^{-r/\alpha} \) from which it follows \( \psi_2 = \frac{3}{4} \cos(\theta) \left( \frac{1}{2} \int_0^a dr' r'^2 e^{-r'/\alpha} + \int_0^\infty dr' e^{-r'/\alpha} \right) \) or \( \psi_2(r, \Omega) = \lambda \cos(\theta) \left( \frac{a^2}{2} e^{-r/\alpha} + \frac{2 \lambda a}{r} \left( \frac{r}{a} \right)^2 + 2 \frac{r}{a} + 2 \right) \).

Problem 6 (Barton #4.6 p. 116)

In 2D, we have \( \rho_1 = \lambda \frac{x}{r} e^{-r/\alpha} = \lambda \cos(\theta) e^{-r/\alpha} \) and \( \rho_2 = \lambda \left( 2 \frac{x^2}{r^2} - 1 \right) e^{-r/\alpha} = \lambda \left( 2 \cos(\phi)^2 - 1 \right) e^{-r/\alpha} = \lambda \cos(2\phi) e^{-r/\alpha} \) so that \( \psi_1 = \int d\lambda' \rho_1(r) G(\mathbf{r} | \mathbf{r}') \) where the Green's function is \( G(\mathbf{r} | \mathbf{r}') = \frac{1}{2\pi} \ln \frac{\mathbf{r}_1}{\mathbf{r}_2} + \sum_{m=1}^\infty \frac{\mathbf{r}_m^m}{(2\pi m)^m} \cos[m(\phi-\phi')] \). Note the factor of 2\pi that is missing in Barton's text. Simply checking that your answer yields \( \rho = -\nabla^2 \psi \) reveals this mistake!!!

The density \( \rho_1 \) depends only on \( \cos(\theta) \), so \( \psi_1 = \lambda \frac{1}{\alpha} \int_0^{2\pi} d\phi' \cos(\phi') \cos(\phi-\phi') \int_0^\infty dr' \frac{r}{r_0} e^{-r'/\alpha} = \frac{1}{2} \cos(\phi) \int_0^\infty dr' r_0 e^{-r'/\alpha} \). Thus, \( \psi_1 = \frac{1}{\alpha} \cos(\phi) \left( \frac{a^2}{2} e^{-r/\alpha} + \frac{a}{r} \left( \frac{r}{a} - 1 \right) \right) \).

The density \( \rho_2 \) depends only on \( \cos(2\phi) \), so \( \psi_2 = \lambda \int_0^{2\pi} d\phi' \frac{2 \pi}{2\pi} \cos(2\phi') \cos(2(\phi-\phi')) \int_0^\infty dr' \left( \frac{r_0}{r} \right)^2 e^{-r'/\alpha} \). It follows that \( \psi_2 = \frac{1}{4} \cos(2\phi) \int_0^\infty dr' \left( \frac{r_0}{r} \right)^2 e^{-r'/\alpha} \) or \( \psi_2 = \frac{1}{4} \cos(2\phi) \left( \frac{6a^2}{r^2} - \frac{a^2}{r^2} - \frac{6a^2}{r^2} + \frac{3a^2}{r^2} \right) e^{-r/\alpha} \).

The complete gamma function \( \Gamma(s, x) = \int_0^\infty e^{-x t} \) is given by \( \Gamma(0, \frac{r}{a}) = \int_0^\infty e^{-r t} \) dt... i.e., \( \Gamma(0, \frac{r}{a}) = \int_0^\infty e^{-r t} \) dt.
Problem 7 (Barton #4.7 p. 116)

The hemisphere \( r = a \) for \( z \geq 0 \) carries a charge density \( \sigma \), thus \( \rho(r) = \sigma \delta(r - a) \Theta(z) = \sigma \delta(r - a) \Theta(\pi/2 - \theta) \).

(i) So, we have

\[
\psi(0, 0, z) = 2 \pi \int_0^\infty \frac{r^2 \sin^2 \theta'}{4 \pi \sqrt{r^2 - a^2}} \, dr' \int_0^{\pi} \frac{\sin^2 \theta}{4 \pi \sqrt{r^2 - a^2}} \, d\theta' = \frac{\sigma a^2}{2} \left( \frac{1}{z - a} \right)
\]

or, rewriting a bit,

\[
\psi(0, 0, z) = \frac{\sigma a^2}{2} \sqrt{z^2 + a^2 - |z - a|}
\]

equivalently

\[
\psi(0, 0, z) = \frac{\sigma a^2}{\sqrt{z^2 + a^2 - |z - a|}}
\]

This is the dotted line in the plot below.

\[\psi(z)\] for \( a = \sigma = 1 \)

(ii) For \( z \to \pm \infty \), we expand to find

\[
\psi(0, 0, z) = \frac{\sigma a^2}{2z} \left( 1 + \frac{1}{z} \right)
\]

shown as the dark solid line in the plot above.

(iii) Center of charge is \( r_0 = \frac{\int_{V'} r \rho(r) \, dV}{\int_{V'} \rho(r) \, dV} = \frac{p}{Q} \).

By symmetry, the x and y components of the dipole moment are zero, so we need only compute

\[
p_z = 2 \pi \sigma \int_0^\infty \int_0^{\pi} \sin^2 \theta \, dr \, d\theta \int_0^{\pi} |z - a| \, \sin \theta \, \frac{\sin^2 \theta'}{4 \pi \sqrt{r^2 - a^2}} \, d\theta' = \frac{2 \pi \sigma a^3}{2} \int_0^{\pi} \sin^2 \theta \, d\theta = \pi \sigma a^3
\]

so \( p = \pi \sigma a^3 \hat{z} \). Also,

\[
Q = 2 \pi \sigma \int_0^\infty \int_0^{\pi} \sin^2 \theta \, d\theta \int_0^{\pi} \sin \theta \, \cos \theta \, \frac{\sin^2 \theta'}{4 \pi \sqrt{r^2 - a^2}} \, d\theta' = 2 \pi \sigma a^2 \int_0^{\pi} \sin \theta \, d\theta = 2 \pi \sigma a^2
\]

Thus, \( r_0 = \frac{p}{Q} = \frac{a \hat{z}}{2} \).

(iv) We have

\[
\frac{1}{|r - r'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'}}
\]

and, since \( r' \approx a \) is finite, we can expand

\[
\frac{1}{|r - r'|} = \frac{1}{r'} \left( 1 + \frac{r' - r}{r'} \right) = \frac{1}{r'} \left( 1 + \frac{r'}{r} \cos \theta' \right) = \frac{1}{r} \left( 1 + \frac{r'}{r} \cos \theta' \right) \sin \theta \sin \theta' \cos \phi \cos \phi' + \sin \theta \sin \theta' \sin \phi \sin \phi' + \cos \theta \cos \theta' \right]
\]

Therefore, the potential becomes

\[
\psi(r, \theta, \phi) = \frac{2 \pi \sigma a^2}{4 \pi r} \int_0^{\pi} \frac{1}{\sqrt{r^2 - a^2}} \left( 1 + \frac{a}{r} \right) \frac{\sin \theta \, d\theta'}{\sqrt{r^2 - a^2}} = \frac{\sigma a^2}{2} \left( 1 + \frac{a}{2r} \cos \theta \right)
\]

or

\[
\psi(r, \theta, \phi) = \frac{\sigma a^2}{2} \left( 1 + \frac{a}{2r} \cos \theta \right)
\]

Integrating over \( \phi' \), we obtain

\[
\psi(0, 0, \pm z) = \psi(0, 0, \phi) = \frac{\sigma a^2}{2z} \left( 1 + \frac{a}{2z} \right)
\]

and \( \psi(0, 0, \pm z) = \psi(0, \pi, \phi) = \frac{\sigma a^2}{z} \left( 1 - \frac{a}{z} \right) = \frac{\sigma a^2}{z} \left( 1 + \frac{a}{2z} \right) \). This agrees with our answers in part (ii).
Problem 8 (Barton Ex. p. 101)

Given that \( G_0^{(2)}(r \mid r') = \frac{1}{2\pi} \ln \left( \frac{a}{a-r} \right) \), we work with \( G(r) \equiv G_0^{(2)}(r \mid 0) = \frac{1}{2\pi} \ln(a/r) \) and the radial Laplacian \( \nabla_r^2 = \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) \). Given that \( r \frac{\partial}{\partial r} G(r) = \left( -\frac{1}{2\pi r} \right) = -\frac{1}{2\pi r} \), it follows that \( \nabla_r^2 G(r) = 0 \) provided that \( r > 0 \). Note that \( \delta(r) = \operatorname{sgn}(r) \) does not show up here since, for \( r' = 0, |r| \equiv r \geq 0 \) always!!! Thus, there is no delta function at this stage. Continuing, \( \int dA \left( \nabla^2 G \right) = \oint d\ell \cdot (\nabla G) \), so we see that \( \int dA \left( -\nabla^2 G(r) \right) = \frac{1}{2\pi} \oint d\ell \cdot \nabla G = \frac{1}{2\pi} \int_0^r r \, dr \, \delta(r) = 1. \) Taken together, this implies that \( -\nabla_r^2 G_0^{(2)}(r \mid 0) = \delta(r) \).